GALTON WATSON FRACTAL SIGNALS

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ABSTRACT

Iterated Function Systems (IFS) is a relevant model to produce fractal functions, whether deterministic (with strict self-similarity) or random (self-similar up to probability distribution). The basic idea of such a construction is to start with an initial function and then compress, dilate and translate it such that by doing so over and over again, we end up with a self-similar signal. This construction relies on a construction tree which has always been deterministic in the literature for signals. Here we introduce new fractals, called Galton Watson fractals, as fixed points of IFS with a random underlying construction tree and deterministic operators. We give a proof of the existence and uniqueness of a fixed point at the random and distribution level.

Index Terms— Fractals, Galton Watson Trees, Iterated Function Systems, Random fixed points, Self-Similarity.

1. INTRODUCTION

Since the discovery of the relevancy of fractal and 1/f processes to model natural phenomena, the fractal formalism has received increasing interests during the last 20 years, with applications to a wide variety of fields such as finance, turbulence, meteorology, image compression, network traffic [1, 2, 3]. Iterated Function Systems (IFS) are a simple class of fractal sets, first described rigorously by Hutchinson [4]. IFS were later adapted to functions and measures and were randomised in various ways [5]. In what follows we will define a new class of random IFS for functions, and prove existence and uniqueness using the approach of Hutchinson and Rüschendorf [5].

Roughly speaking, a (deterministic) IFS recursively apply a contractive operator $T$ (random or not) on an initial function $f_0$. Functions considered in signal processing are usually finite energy signals and we will denote this space by $L_2(\mathbb{X})$ for deterministic signals and $L_2(\mathbb{X})$ for random signals so that we can assume $f_0 \in L_2(\mathbb{X})$. The completeness of the metric space where the fractal lives assures the existence and uniqueness of a fixed point $f^*$, thanks to the well known Banach fixed point theorem. In other words, if we denote by $T^n$ the $n$-th iterate of $T$, one has:

$$T^n f_0 \rightarrow f^* \text{ as } n \rightarrow +\infty$$

(1)

where $f^*$ is the only function which satisfies $f = Tf$. At each iteration, functions are stretched, compressed, translated by means of the contractive operator $T$. We assume that we can decompose this operator into a set of $M$ simpler operators $\phi_i : \mathbb{X} \times \mathbb{X} \rightarrow \mathbb{X}$, $i \in \{1, \ldots, M\}$. Each $\phi_i$ will have its own way of deforming the signal, and the resulting signal will lie in a subinterval of $\mathbb{X}$ (hence the compression). Mathematically, this can be written as

$$\left(Tf\right)(x) = \sum_{i=1}^{M} \phi_i\left[f\left(q_i^{-1}(x)\right), q_i^{-1}(x)\right]1_{\varphi_i(x)}(x)$$

(2)

for any $x \in \mathbb{X}$. The $\varphi_i$’s, $\varphi_i : \mathbb{X} \rightarrow \mathbb{X}$, partition the interval $\mathbb{X}$ into disjoint subintervals and $1_{\varphi_i(x)}$ is the indicator function of the interval $\varphi_i(\mathbb{X})$. In [5] this construction is randomised by letting the set of $M$ operators $(\varphi_1, \ldots, \varphi_M)$ be a random variable, but $M$ remains fixed. The function $f$ must also be randomised, and in the right hand side of (2) we replace $\varphi_i\left[f\left(q_i^{-1}(x)\right), q_i^{-1}(x)\right]$ by $\varphi_i\left[f^{(i)}\left(q_i^{-1}(x)\right), q_i^{-1}(x)\right]$ where the $f^{(i)}$ are i.i.d. copies of $f$. If $f \overset{d}{=} Tf$ then we say $f$ is a random fractal function satisfying the random IFS. $\overset{d}{=}$ stands for equality in distribution.

In this study we randomise (2) in a different manner: instead of fixing $M$ we allow it to be random. We will assume that for each value $j$ of $M$ there is a unique set $\varphi^j$ of $j$ operators $(\varphi_{j,1}, \ldots, \varphi_{j,j})$. We can allow $\varphi^j$ to be random but to simplify our discourse we do not do so here.

IFS for sets and measures allowing random $M$ have been considered by Falconer [6], Mauldin and Williams [7], but not for functions. More recently, Barnsley et. al. have generalised IFS to so called $V$-variable superfractals, however they also keep $M$ fixed [8].

When we iterate $T$ we get a tree-structure describing the recursive application of the $\varphi_i$. If $M$ is fixed then this is an $M$-ary tree. In figure (1) we depict the underlying construction tree of a deterministic IFS with 2 maps. However if $M$ is random then assuming it is chosen independently each time $T$ is applied, the tree is a Galton-Watson tree. In the next section, we will briefly describe Galton Watson trees before describing our new random IFS in section 3.

2. GALTON WATSON TREES

A Galton Watson tree is a tree with a random number of branches at each node where the offspring distribution is independent and identically distributed at each node. A node can be identified by means of its label. If we denote by $\varnothing$ the root node, then the first generation of children will be denoted by $i$ where $1 \leq i \leq \nu_0$ and $\nu_0$ is the number of children at $\varnothing$. Then the second generation will be labelled $ij$, $1 \leq j \leq \nu_i$, and so on. More generally, a node is an element of $U = \bigcup_{n \geq 0} \mathbb{N}^n$ and a branch is a couple $(u, u_j)$ where $u \in U$ and $j \in \mathbb{N}^*$. The length of a node $u = i_1 \ldots i_n$, $|u|$, is $n$.

By definition a tree is a set of nodes, that is each tree $\omega$ is a subset of $U$: $\omega \subset U$. However, a subset of $U$ must meet further requirements in order to be a tree: (i) The root node $\varnothing$ belongs to the tree. (ii) If a node $i = i_1 \ldots i_n$ of $\omega$ has length $n$, then every shorter node $i_1 \ldots i_k$, $k \leq n$ belongs to the tree as well. (iii) If the
node labelled $u$ belongs to $\omega$, then $u^j$ is also in $\omega$ if $j$ is a child of $u$. Formally, these three conditions can be written as follows:

- $\emptyset \in \omega$
- $\forall v \in U \quad uv \in \omega \implies u \in \omega$
- If $u \in \omega$ then $u^j \in \omega \iff 1 \leq j \leq \nu_u(\omega)$ where $\nu_u(\omega)$ represents the number of children at node $u$ for the tree $\omega$.

To illustrate, we present in Figure 2 the tree $\omega = \{0, 1, 2, 3, 11, 12, 21, 22, 211, 311, 312\}$.

Let $\Omega$ be the space of trees, and $u \in U$. Then, define:

$$\Omega_u = \{\nu_u \mid u \in \omega\}$$

(3)

$\Omega_u$ is a subset of $\Omega$ whose elements are trees containing the node $u$. In particular, $\Omega_0 = \emptyset$. Clearly, $\nu_u$ defines a map from $\Omega_u$ to $\mathbb{N}$ noting that $\nu_u$ is not defined over the whole space $\Omega$ and represents the number of children (in $\mathbb{N}$) at a given node $u$ of $\omega \in \Omega_u$. Note that if $j \in \mathbb{N}^*$ then there is no change of notations: $\Omega_u^j$ is the space of trees containing the node $u^j$. Formally,

$$\Omega_u^j = \Omega_u \cap \{\nu_u \geq j\}$$

(4)

We endow $\Omega$ with the $\sigma$-algebra $A$ defined by

$$A = \sigma(\Omega_u \mid u \in U)$$

(5)

Then we endow the subspaces $\Omega_u$ with the $\sigma$-algebras $\Omega_u \cap A$ so that $\nu_u$ are measurable. Next we define another function which will be relevant in the remainder. Let $T_u(\omega)$ be the tree $\{v \mid v \in U \text{ and } uv \in \omega\}$. In other words, if $v \in \Omega$ then $T_u(\omega)$ is the subtree of $\omega$ rooted at $u$. Then $T_u$ is a map from $\Omega_u \rightarrow \Omega$. One can check that $T_u$ are also $\Omega_u$-measurable functions.

Next, we endow the space $(\Omega, A)$ with a probability measure. We do this so that given the tree up to generation $n$, the number of children of each generation node are i.i.d. One have the following result [9]

**Proposition 1** For each probability $q = (q_j, j \in \mathbb{N})$ on $\mathbb{N}$, there exists a unique probability measure $P_q$ on $(\Omega, A)$ which gives to the random variable $\nu_0$ the law $q$ and for which, conditionally on the event $\nu_0 = j$, the random variables $T_u$, $1 \leq i \leq j$ are independent and identically distributed with distribution $P_q$.

$(\Omega, A, P_q)$ is the space of Galton Watson trees.

### 3. GALTON WATSON SIGNALS

#### 3.1. Definition

We are concerned with the definition of a new fractal construction when the underlying tree structure is no longer deterministic. Indeed, we cannot apply the same random operator at each node of the tree as the number of offspring is random. Instead, the operator applied depends (only) on the number of offspring at a given node and is always the same after conditioning on the number of children. The randomness in the construction comes therefore from the non-deterministic tree structure. Consider the space of $p$-integrable functions on a compact subset $\mathcal{X}$ of the real line:

$$L_p(\mathcal{X}) = \{f : \mathbb{X} \rightarrow \mathbb{R} \mid \int_{\mathbb{X}} |f(t)|^p dt < +\infty\}$$

(6)

Let $(\Sigma, \mathcal{F}, P)$ be any probability space and consider the more general space of $p$-integrable random functions:

$$L_p = \{f : \Sigma \rightarrow L_p(\mathcal{X}), t \in \mathbb{X} \mid E\left[\int_{\mathbb{X}} |f(t)|^p dt\right] < +\infty\}$$

(7)

endowed with the metric

$$d_p^\ast : \forall (f,g) \in L_p, d_p^\ast(f,g) = E\left[\frac{1}{2} (d_p(f,g))^2\right]$$

(8)

where $d_p(f,g) = (\int_{\mathbb{X}} |f(t) - g(t)|^p dt)^{1/p}$ is the usual $L_p$ metric. Set $p = 2$ to work with finite energy signals. We will write $f_\sigma$ for the value of $f$ at some $\sigma \in \Sigma$. That is $f_\sigma \in L_p(\mathcal{X})$. In the remainder we will be particularly interested in the probability space $(\Sigma, \mathcal{F}, P) = (\Omega, A, P_q)$. The operator $T$ is then defined on $L_p(\mathcal{X})$ by:

$$(Tf)(x) = \sum_{j=1}^{\nu_0} \phi_{q_j,j}(f(\theta_{q_j,j}^{-1}(x)), \theta_{q_j,j}^{-1}(x))1_{\nu_0}^{-1}(\mathcal{X})(x)$$

(9)

where $f(\theta_{q_j,j}^{-1}(x))$ are i.i.d. copies of $f$ and the $\theta_{q_j,j}$ partition $\mathcal{X}$ into disjoint subintervals. We can consider for example uniform partitions $\theta_{q_j,j}(t) = \frac{1}{q_j} + \frac{1}{q_j} t$, $t \in \mathbb{X}$, $1 \leq j \leq q_0$. The contraction factor of $\theta_{q_j,j}$ is denoted by $r_{q_j,j}$. $\phi_{q_j,j}$ are $2$-variable maps Lipschitz in their first variable, with Lipschitz constant $K_{q_j,j}$. Also note that and that $\nu_0$ is a random variable with probability distribution $q$.

**Definition 1** For $i \geq 1$ and $1 \leq j \leq i$ is a random function scaling system.
We say that a function is statistically self-similar for an IFS if it satisfies $f = Tf$ in distribution. The major result in the next part is to show the existence and uniqueness at random (a.s.) convergence and distribution level of a function $f$ which satisfies the random function scaling system.

3.2. Existence and Uniqueness of a fixed point

**Theorem 1** Let $(Ω, A, P_q)$ be the space of Galton Watson trees. Consider $(p_{i,j}, q_{i,j}; y)$ a random function scaling system, $i \geq 1$ and $1 \leq j \leq i$. Let $1 < p < +\infty$. If $λ = \sum_{i \geq 1} \sum_{j=1}^i q_{i,j} K_{i,j}^p < 1$ and $\sum_{i \geq 1} \sum_{j=1}^i r_{i,j} \int | \phi_{i,j}(0,x) |^p dx < +\infty$ then for any $f_0 \in L_p(\mathcal{X})$, there exists a unique random function $f^*$ which satisfies $f^* = Tf^*$ and such that

$$d_p^p(T^n f_0, f^*) \leq \frac{λ^n}{1 - λ^p} d_p^p(f_0, T f_0) \quad (10)$$

which tends to $0$ as $n \to +\infty$.

This theorem states that the IFS converges to a random fixed point starting from any initial function $f_0$ under certain conditions. The fixed point exhibits self-similarity up to probability distribution. The proof is in two steps. The first thing to check is that the operator $T : L_p \to L_p$. Secondly, we need to show that $T$ is contractive in the complete metric space $(L_p, d_p)$. The Banach fixed point will assure the existence and uniqueness of a limit function at the random level.

We first define $f^{(1)}$ and $g^{(1)}$. To prove theorem 1 we need to construct i.i.d. copies of the random function $f$. This can be achieved using the homogeneity of Galton Watson trees: $f^{(1)} = f_T(\omega)$. Since by proposition 1 the variables $T_i$ are independent and identically distributed with distribution $P_q$, the $f^{(i)}$ functions are also i.i.d.

**Step 1:** Let $f \in L_p$. We show that $E \int_X | (T f)(x) |^p dx < +\infty$, that is $T f \in L_p$. To do so, first notice that in the expression of $T f$, the indicator function partitions $\mathcal{X}$ into disjoint subintervals, so that the absolute value of the sum equals the sum of absolute values. Furthermore, using $E(\cdot) = E[E(\cdot | \nu_q)]$ (tower property of expectation) and contractive properties of $q_{i,j}$, it is straightforward to check that $E \int_X | (T f)(x) |^p dx$ is always smaller than

$$\sum_{j=1}^{r_{q,j}} E \int_X | \phi_{i,j}^{(1)}(y) |^p dy \nu_q \quad (11)$$

On the right hand side, we have set $y = \phi_{i,j}^{-1}(x)$ and we have majorized the Jacobian of the transformation by $r_{q,j}$, the Lipschitz factor of $\phi_{i,j}$. Note that $E \int_X | \phi_{i,j}^{(1)}(y) |^p dy$ can also be written $d_p^p(\phi_{i,j}^{(1)}, I d_I, 0)$ where $I d_I$ stands for the identity function and 0 the zero function. The combination of triangle inequality of distance and the fact that for any positive $x$ and $y$: $(x + y)^p \leq 2^p (x^p + y^p)$, (11) is majorized by:

$$2^p \sum_{j=1}^{r_{q,j}} d_p^p(\phi_{i,j}^{(1)}, I d_I, 0) + 2^p E \sum_{j=1}^{r_{q,j}} d_p^p(\phi_{i,j}^{(1)}, 0, I d_I, 0) \quad (12)$$

The first part is bounded since $f \in L_p$. The second part can be written $\sum_{i \geq 1} \sum_{j=1}^i q_{i,j} \int | \phi_{i,j}(0,x) |^p dx$ and is bounded by assumption.

**Step 2:** Let $f$ and $g$ in $L_p$. Then,

$$d_p^p(T f, T g) = E d_p^p(T f, T g) \leq E \int | (T f)(x) - (T g)(x) |^p dx \quad (13)$$

By replacing $(T f)(x)$ and $(T g)(x)$ by their own expression, the distance becomes:

$$\sum_{j=1}^{r_{q,j}} E \int_X | (\phi_{i,j}^{\nu_{q,j}}(y), y - \phi_{i,j}^{\nu_{q,j}}(y) |^p dy | \nu_q | \quad (16)$$

where we have made the same change of variable $y = \phi_{i,j}^{-1}(x)$. By furthermore exploiting the Lipschitz property of $\phi_{i,j}$ and the i.i.d. distributions of $f^{(i)}(y)$ and $g^{(i)}(y)$, we obtain the inequality

$$d_p^p(T f, T g) \leq λ d_p^p(f, g) \quad (13)$$

Next, using similar arguments as in Step 1 (tower property of expectation and Lipschitz property of $\phi_{i,j}$), the distance between $T f$ and $T g$ is smaller than

$$\sum_{j=1}^{r_{q,j}} d_p^p(T f_0, f^*) \leq \lambda d_p^p(f_0, f^*) \quad (14)$$

which leads to

$$d_p^p(T^n f_0, f^*) \leq λ^n d_p^p(f_0, f^*) \quad (15)$$

Now using triangle inequality:

$$d_p^p(f_0, f^*) \leq d_p^p(f_0, T f_0) + \lambda \frac{1}{1 - λ^p} d_p^p(f_0, f^*) \quad (16)$$

so that:

$$d_p^p(T^n f_0, f^*) \leq \frac{λ^n}{1 - λ^p} d_p^p(f_0, T f_0) \quad (17)$$

which concludes the proof of the theorem.

Remark: It follows directly from this result that for all $f$ in $L_p(\mathcal{X})$, $T^n f \to f^*$ almost surely as $n \to +\infty$. To see this, suppose that the converse is true, that is, suppose that for some $f \in L_p(\mathcal{X})$ there exists $A \in \mathcal{A}$ such that $P_q(A) > 0$ and for all $\omega \in A$, $T^n f_\omega \not\to f^*$. Given $A$ we can find $\epsilon > 0$ and $B \subset A$ such that $P_q(B) > 0$ and lim inf $d_p^p(T^n f_\omega, f^*) \geq \epsilon$ for all $\omega \in B$. Then it follows that lim inf $d_p^p(T^n f, f^*) \geq \epsilon P_q(B) > 0$, a contradiction.

Next, note that this equality at the random level is also true at the distribution level. However, equality in distribution does not implies equality at the random level. The following result can be proven in the same way as Hutchinson and Ruschendorf [5].
where we have only considered functions $\phi$ with only 1 variable for simplicity. From this formal solution, one clearly see that the randomness directly appers in the IFS parameters.

4. CONCLUSION AND PERSPECTIVES

This new approach generalizes the IFS based construction of fractal signals proposed by Hutchison and Ruschendorf in [5]. We proved the existence and uniqueness of a self-similar function when we allow a random construction tree and deterministic operators. Moreover, this construction does not force the number of offsprings to be bounded.

The fixed points obtained all have a very erratic behaviour and we speculate that they might not have a density. An efficient tool to characterize such irregular objects is their multifractal spectrum introduced first by Frisch and Parisi [10] in the context of turbulence, and adapted to random processes and functions. The motivation to think of a multifractal spectrum for such fractal signals is due to its cascade construction. Cascade processes are indeed known to exhibit multifractal properties and results for random measures are known [11].

Finally, his construction can be further extended when we consider a random construction tree with random operators. In this case the space of Galton Watson trees need to be extended in order to endow each branch of a tree with a new operator.

5. REFERENCES