A New Closed-Form Frequency Estimator in the Presence of Fading-Induced Multiplicative Noise

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Abstract—This paper deals with the correlation-based carrier frequency offset estimation (CFO) problem in the presence of fading-induced multiplicative noise. We derive a new closed-form estimator using the nonlinear least-squares principle in conjunction with the summation-by-parts formula. The proposed estimator, as opposed to most of the existing methods, is able to operate at the maximum estimation range of about $\pm 0.5$ the symbol rate without the additional phase unwrapping and robust to the lack of information on the form of the correlation of the multiplicative noise. The performance of the proposed estimator is shown to be efficient in both non-fading and fading scenarios.

I. INTRODUCTION

Frequency estimation of a signal in the presence of multiplicative noise is of fundamental importance in various contexts such as in fading multipath channels, global satellite mobile communication systems, backscatter radar signal processing, and array processing of spatial signals. For such contexts, the frequency estimators developed using the signal model with constant amplitude may not be sufficient. This paper focuses on the correlation-based carrier frequency offset (CFO) estimation of the signals with the (flat) fading-induced multiplicative noise. This is also often the case in orthogonal frequency division multiplexing (OFDM), where the frequency selective channel is converted into parallel flat fading subchannels [2][3]. Several correlation-based estimators have been presented in literature, good examples may be found in [4]-[9]. However, most of them require the knowledge of the form of the multiplicative noise correlation as well as the additional phase unwrapping. The FFT-based CFO estimator proposed in [8] is able to overcome these requirements but at the price of high implementation complexity.

To circumvent these restrictions, this paper presents an estimator that relies on the nonlinear least-squares (NLS) principle in conjunction with the summation-by-parts formula. The proposed estimator is simple and robust to the lack of information on the form of the correlation of the multiplicative noise, moreover it can operate at the maximum estimation range without the need of the additional phase unwrapping algorithm.

The paper is organized as follows. The signal and observation models as well as the theoretical lower bound are defined in section II. In section III, the NLS estimator and its simplifications are derived. Estimation range extension by the summation-by-parts formula is also explained in this section. Simulation results are presented in section IV. Finally, the conclusions are discussed in section V.

II. SIGNAL AND OBSERVATION MODELS

Under the data-aided and perfect symbol timing scenarios, the baseband received signal can be expressed as

$$ y(n) = a(n)e^{j2\pi n\nu} + v(n), \quad n = 0, \ldots, N - 1 $$

where $a(n)$ is the fading-induced multiplicative noise, $\nu$ is the frequency offset normalized to $1/T$, and $v(n)$ is a complex white Gaussian noise with variance $\sigma_v^2$. $a(n)$ and $v(n)$ are mutually independent. The correlation function of $a(n)$ is defined as $r_a(m) = E\{a(n)a^*(n-m)\}$, and is parameterized by a real-valued parameter vector, $r_a = [r_a(0), \ldots, r_a(M-1)]^T$. The normalized Doppler spread ($B_DT$), is typically used to characterize $r_a(m)$. The variance of $a(n)$ is thus $\sigma_a^2 = r_a(0)$. The signal-to-noise ratio (SNR) is defined as $\sigma_a^2/\sigma_v^2$, provided that $a(n)$ has zero mean (Rayleigh case).

The theoretical correlation sequence corresponding to the data model (1) is given by

$$ r(m) = E\{y(n)y^*(n-m)\}, \quad m = 0, 1, \ldots, M - 1 $$

$$ = r_a(m)e^{j2\pi m\nu} + \sigma_v^2 \delta(m). $$

The consistent and unbiased estimate of $r(m)$ [8] can be obtained by

$$ \hat{r}(m) = \frac{1}{N - m} \sum_{n=m}^{N-1} y(n)y^*(n-m) $$

and will be used as the observations in deriving the NLS estimator. The total number of complex multiplications required to compute $\hat{r}(m)$, is $\sum_{m=1}^{M}(N - m) = M(2N - M - 1)/2$.

The Cramer-Rao lower bound (CRLB) is given as the corresponding diagonal element of the inverse of the Fisher information matrix (FIM), $J_{\omega_o,\omega_o}^{-1}$. The CRLB is then

$$ \text{CRLB}(\hat{\omega}_o) = J_{\omega_o,\omega_o}^{-1} $$

where $\hat{\omega}_o = 2\pi \nu$. The $J_{\omega_o,\omega_o}$ is found to be [7]

$$ J_{\omega_o,\omega_o} = 2\text{tr}\{R^{-1}\text{DRD} - \text{D}^2\} $$

where $R = R_a + \sigma_a^2 I$, $R_a(i,j) = r_a(|i-j|)$ with $i, j = 1, \ldots, N$, and $D = \text{diag}(0, \ldots, N - 1)$.

For high SNR or $\sigma_a^2 \rightarrow 0$, $J_{\omega_o,\omega_o}^{-1} = 2\text{tr}\{R_a^{-1}\text{DRD} - \text{D}^2\}$.

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is a constant which leads to the so called error floor effect.

Note that the shape of the fading correlation \( r_a(m) \) is often
defined either by the Bessel: \( \sigma_n^2 J_0(2\pi BDT|m|) \),
Exponential: \( \sigma_n^2 \exp\{-2\pi|m|BD_T\} \),
Gaussian: \( \sigma_n^2 \exp\{-2\pi(BD_T m/2)^2\} \),
or Sinc: \( \sigma_n^2 \text{sinc}\{2\pi|m|BD_T\} \) functions. However, it is widely
recognized that the actual shape of \( r_a(m) \) has little impact on the
overall system performance. The sole influential parameter
is the Doppler spread.

III. NONLINEAR LEAST-SQUARES ESTIMATOR

A. General NLS estimator and its simplified form

The NLS estimate of \( \nu \) can be obtained by solving the following minimization problem

\[
\{ \hat{\sigma}_n^2, \hat{r}_n, \hat{\nu} \} = \mathcal{J}_N(\sigma_n^2, r_n, \nu)
\]

\[
= \arg\min_{\sigma_n^2, r_n, \nu} \sum_{m=0}^{M-1} |\hat{r}(m) - r_a(m)e^{j2\pi\nu m}|^2
\]

with respect to (w.r.t.) \( \sigma_n^2, r_n \) and \( \nu \), respectively. Since \( \hat{r}(m) \) is a consistent estimate of \( r(m) \), it follows that
\( \lim_{N \to \infty} \mathcal{J}(\sigma_n^2, r_n, \nu) \) achieves a global minimum at the true parameters. Consequently, \( \hat{\nu} \) is a consistent estimate of \( \nu \) for whatever form of \( r_a(m) \). Setting the derivative of the criterion
\( \mathcal{J}(\sigma_n^2, r_n, \nu) \) w.r.t. \( \sigma_n^2 \) to zero, we obtain
\( \hat{\sigma}_n^2 = \hat{r}(0) - r_a(0) \). Substitute this solution into \( \mathcal{J}(\hat{\sigma}_n^2, r_n, \nu) \), yields the reduced criterion

\[
\{ \hat{r}_n, \hat{\nu} \} = \arg\min_{r_n, \nu} \sum_{m=1}^{M-1} |\hat{r}(m) - r_a(m)e^{j2\pi\nu m}|^2
\]

with \( \mathcal{J}(r_n, \nu) \). It will be shown that the criterion in (8) can be concentrated
w.r.t. \( r_n \) so as to leave one-dimensional search over \( \nu \) only.

Using the fact that \( r_a(m) \in \mathbb{R} \), \( \mathcal{J}(r_n, \nu) \) can be rewritten as

\[
\sum_{m=1}^{M-1} \left\{ |\hat{r}(m)|^2 - 2r_a(m)\text{Re}\{\hat{r}(m)e^{-j2\pi\nu m}\} + r_a^2(m) \right\}.
\]

Taking the derivative of (9) w.r.t. \( r_n \) and equating the result to zero, yields

\[
\hat{r}_n(m) = \text{Re}\{\hat{r}(m)e^{-j2\pi\nu m}\}
\]

where the estimate of the frequency offset has yet to be determined. Inserting (10) into (9), the NLS estimate of the parameter \( \nu \) is obtained as

\[
\hat{\nu}_{\text{NLS}} = \arg\max_{\nu} \text{Re}\left\{ \sum_{m=1}^{M-1} \hat{r}^2(m)e^{-j4\pi\nu m} \right\}
\]

which can be realized by FFT with a typically large number of
zero padding to ensure a better localization of the peak. Its
complexity is \( O(N_f \log_2 N_f) \), where \( N_f \) is the Fourier bins.

A much simpler form of (11) can be obtained by assuming
that the phase of the correlation estimate, \( \phi(m) = \text{arg}\{\hat{r}(m)\} \),
is available, and \( 2\phi(m) \approx 4\pi \nu m \). The NLS criterion in (11)
can be rewritten as

\[
\sum_{m=0}^{M-1} |\hat{r}_y(m)|^2 \cos(2\phi(m) - 4\pi \nu m).
\]

The value of \( \nu \) that maximizes the above expression can be obtained
by setting the derivative of (12) w.r.t. \( \nu \) to zero, we then have
\( \sum_{m=0}^{M-1} m|\hat{r}_y(m)|^2 \sin(2\phi(m) - 4\pi \nu m) = 0 \),
under the small error approximation, i.e., \( \phi(m) \approx 2\pi \nu m \)
and \( \sin(2\phi(m) - 4\pi \nu m) \approx (2\phi(m) - 4\pi \nu m) \), the Simplified-NLS
(SNLS) estimator is obtained as

\[
\hat{\nu}_{\text{SNLS}} = \frac{1}{2\pi} \sum_{m=1}^{M-1} \frac{m|\hat{r}(m)|^2\phi(m)}{m^2|\hat{r}(m)|^2}.
\]

Without applying an additional phase unwrapping algorithm to \( \phi(m) \), its estimation range is limited to about \( \pm 1/2M \) the
symbol rate.

B. Absolute phase vs. phase difference

To explain the limitation of using the absolute phase and
the benefit of using the phase difference of the estimated cor-
relation (3), the noiseless condition is assumed. The absolute
phase of \( \hat{r}(m) \) then reads

\[
\phi(m) = \text{arg}\{\hat{r}(m)\} = [2\pi \nu m]_{-\pi}^{\pi}
\]

where \([x]_{-\pi}^{\pi} \) means that \( x \) is confined within the interval
\( (-\pi, \pi) \). As \( m \) varies from 1 to \( M-1 \), eq. (14) establishes
a linear relation between \( \nu \) and \( \phi(m) \), only when \( \nu \) lies
within the interval about \( \pm \frac{1}{2M} \). If \( \nu \) lies outside the interval,
eq. (14) becomes a highly nonlinear function and introduces
ambiguities.

On the other hand, the phase difference of one lag reads

\[
\Delta\phi(m) = \text{arg}\{\hat{r}(m)\hat{r}^*(m-1)\} = [2\pi \nu m]_{-\pi}^{\pi}
\]

Eq. (15) is independent from \( m \) which implies that, as long
as \( \nu \) lies within \( \pm \frac{1}{2M} \), \( \Delta\phi(m) \) keeps equal to \( 2\pi \nu \) for \( 1 \leq m \leq M-1 \). Clearly, (15) relates \( \nu \) and \( \Delta\phi(m) \) in a linear
fashion and the problem is to estimate a constant, \( \nu \), from
the noisy measurements, \( \Delta\phi(m) \). It can be seen that the
estimation range is enlarged roughly by a factor of \( M \) using
phase differences as the observations.

C. Estimation range extension by summation-by-parts formula

It is known that by exploiting the phase differences,
\( \Delta\phi(m) = \text{arg}\{\hat{r}(m)\hat{r}^*(m-1)\} \), the \textit{wrapped around} effect
can be overcome. For this reason, we apply the well known
summation-by-parts formula (SBP) to (13) in order to trans-
form the absolute phase-based into the phase difference-based
estimator. The SBP formula is given as

\[
\sum_{m=0}^{M-1} \Delta b_m a_m = b_{M-1}a_{M-1} - b_0 a_0 - \sum_{m=1}^{M-1} b_{m-1}\Delta a_m
\]

where \( \Delta b_m = b_m - b_{m-1} \), and \( \Delta a_m = a_m - a_{m-1} \).
Inspecting (16), and the nominator term in (13) yields
\[ \sum_{m=1}^{M-1} m|\hat{r}(m)|^2 \cdot \text{arg}\{\hat{r}(m)\} / \Delta b(m) \]
\[ = b_{M-1} \text{arg}\{\hat{r}(M-1)\} - \sum_{m=1}^{M-1} b_{m-1} \Delta \phi(m) \]
\[ = \sum_{m=1}^{M-1} (b_{M-1} - b_{m-1}) \Delta \phi(m) \]  
(17)

where \( \Delta b_0 = 0, b_1 = 0, \text{arg}\{\hat{r}(M-1)\} = \sum_{m=1}^{M-1} \Delta \phi(m) \). Since \( b_1 = 0 \) and \( \Delta b_0 = 0 \), one obtains
\[ b_m = \sum_{k=1}^{m} \Delta b_k = \sum_{k=1}^{m} k|\hat{r}(k)|^2 \]  
(18)

Inserting (18) into (17) and (13), yields the Approximated-NLS (ANLS) of \( \nu \)
\[ \hat{\nu}_{\text{ANLS}} = \frac{1}{2\pi} \cdot \sum_{m=1}^{M-1} \left[ \sum_{k=1}^{m-1} w_k - \sum_{k=1}^{m-1} w_k \right] \Delta \phi(m) \]
\[ = \sum_{m=1}^{M-1} M(m-1) m(M-1) \Delta \phi(m) \]  
(19)

where \( w_k = k|\hat{r}(k)|^2 \). In very slow fading, \( |\hat{r}(k)| \) is approximately a constant, and hence Eq. (19) further reduces to
\[ \hat{\nu}_{\text{AWGN}} = \frac{3}{2\pi} \sum_{m=1}^{M-1} \left( (M-1) M - (m-1) m \right) M(2M-1)(M-1) \Delta \phi(m). \]  
(20)

Note that the additional \( M-2 \) complex multiplications are required to compute \( \Delta \phi(m) \) in (19), (20).

Table I shows the number of complex multiplications (main contribution to the estimator’s complexity) required for different versions of NLS estimators derived in this paper.

<table>
<thead>
<tr>
<th>Estimators</th>
<th>complex multiplications</th>
</tr>
</thead>
<tbody>
<tr>
<td>FFT-based NLS</td>
<td>( N_f \log_2 N_f + \frac{M(2N-M-1)^2}{2M(2N-M-1)^2} )</td>
</tr>
<tr>
<td>SNLS</td>
<td>( \frac{M(2N-M-1)^2}{2M(2N-M-1)^2} )</td>
</tr>
<tr>
<td>ANLS</td>
<td>( \frac{M(2N-M-1)^2}{2M(2N-M-1)^2} )</td>
</tr>
<tr>
<td>AWGN</td>
<td>( \frac{M(2N-M-1)^2}{2M(2N-M-1)^2} )</td>
</tr>
</tbody>
</table>

TABLE I

**Complexity of different versions of NLS estimators**

### IV. SIMULATION RESULTS

The data sequence of length \( N = 128 \) were generated according to (1) with the correlation of the multiplicative noise that obeys an exponential-shaped correlation, i.e., \( r_n(m) = \sigma_n^2 \exp\{-2\pi|m|B_D T\} \). However, it must be noted that the estimators presented in this paper are independent from this assumption. We compare the performance of the proposed estimator to an efficient estimator proposed in [5][9] which is known as the Single-Lag (SL) estimator and given as \( \hat{\nu}_{\text{SL}} = \text{arg}\{\hat{r}(m_{\text{opt}})\}/m_{\text{opt}} \), for \( m_{\text{opt}} \neq 1 \) the ambiguities of the frequency estimates are introduced and if not resolved the estimation range is limited to about \( \pm 1/2m_{\text{opt}} \). We also compare our proposed ANLS estimator with the well-known CFO estimator introduced by Mengali and Morelli [1] (MM\(_{\text{AWGN}}\)) which was originally designed to be optimum for the AWGN channel. The mean-square-error (MSE) is obtained by 50000 Monte Carlo trials. In all simulations, the performances between ANLS and SNLS are similar, but the estimation range of SNLS is limited to \( \pm 1/2M \) the symbol rate while ANLS can operate at the maximum possible range of about \( \pm 0.5 \) the symbol rate.

Fig. 1 shows that in the non-fading channels the performance of the proposed ANLS and MM\(_{\text{AWGN}}\) estimators using \( M = N/2 = 64 \) attain the CRLB at SNR as low as 0 dB, while the SL estimator, with the optimal lag of \( m_{\text{opt}} = 2N/3 = 85 \) (see [5][9]) is only close to the CRLB. The performance of the ANLS estimator is only slightly better than MM\(_{\text{AWGN}}\) in a very low SNR region. The estimation range of SL estimator is limited to only about \( \pm 1/2M \) the symbol rate at high SNR, as can be seen in Fig. 2. On the other hand, the estimation ranges of ANLS and MM\(_{\text{AWGN}}\) are shown to be independent of the lag number and covers about \( \pm 0.5 \) the symbol rate at high SNR.

Fig. 3 compares the proposed ANLS and MM\(_{\text{AWGN}}\) estimators in a slow fading channel \( (B_D T = 0.0001) \). The lag number (\( M \)) are set to 16 and 32 for both estimators. For both lag values, the proposed ANLS estimator performs better than the MM\(_{\text{AWGN}}\) estimator, especially in the low SNR region. It should be noted that their performances in the AWGN channel are similar as can be seen in Fig. 1.

Fig. 4 shows the optimal choice of the correlation lag for the SL, SNLS, and ANLS estimators in fast and moderate fading. In moderate fading, the same optimal lag of 5 is observed for SL, SNLS, and ANLS. In fast fading, the optimal lag for SL is 2 and for SNLS and ANLS is 3. It is seen that the proposed ANLS estimator is less sensitive to the wrong choice of the lag numbers than the SL estimator.

Fig. 5 shows the estimation ranges of the SL, SNLS, ANLS estimators in moderate fading, however, we observed the same results also in fast fading. It is confirmed again that the estimation range of the proposed ANLS is independent of number of lag and covers the maximum possible range of about 50% the symbol rate, which is not achievable for SL, and SNLS without applying the additional phase unwrapping.

Fig. 6 shows the performances of the SL, SNLS, and ANLS estimators in moderate and fast fading. The performances of ANLS and SNLS are slightly better than that of the SL for both scenarios. All attain the CRLB at high SNR. It can be observed that the estimation variance does not vanish as \( \sigma_n^2 \to 0 \). This behavior is known as the error floor effect which can be explained by (6). As the exponential model for the autocorrelation of the multiplicative noise is adopted in this paper, the closed-form expression of CRLB at high SNR [7] can be expressed as

\[ \text{CRLB}(\hat{\sigma}, \sigma_n^2 \to 0) = \frac{1 - \exp\{-2\pi B_D T\}}{2(N - 1) \exp\{-4\pi B_D T\}}. \]  
(21)
Thus the potential accuracy of frequency estimation decreases with increasing Doppler spread.

V. CONCLUSIONS

In this paper, we present a new closed-form correlation-based frequency estimator for signals with fading-induced multiplicative noise. We show that the proposed estimator can operate at the maximum estimation range of about $\pm 0.5$ the symbol rate without sacrificing its accuracy and does not rely on any assumed form of the fading correlation as needed for most of the existing estimators. The features of the proposed ANLS estimator, as compared to the previously proposed estimators, are summarized in Table II.

REFERENCES


Fig. 5. Estimation ranges of the SL, SNLS, and ANLS estimators for $B_DT = 0.001$ with $M = 5$, SNR = 10 dB, $N = 128$.

Fig. 6. MSE versus SNR of the ANLS and SL estimators for $B_DT = 0.01$ and $B_DT = 0.001$ with the optimal lag values.

<table>
<thead>
<tr>
<th>Est.</th>
<th>Corr. form</th>
<th>Doppler spread</th>
<th>Unwrap</th>
<th>Complexity</th>
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<tr>
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<td>assumed</td>
<td>required</td>
<td>high</td>
</tr>
<tr>
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<td>not needed</td>
<td>high</td>
</tr>
<tr>
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<td>not needed</td>
<td>required</td>
<td>low</td>
</tr>
<tr>
<td>BS [8]</td>
<td>not needed</td>
<td>not needed</td>
<td>not needed</td>
<td>high</td>
</tr>
<tr>
<td>ANLS</td>
<td>not needed</td>
<td>not needed</td>
<td>not needed</td>
<td>low</td>
</tr>
</tbody>
</table>

TABLE II
FEATURES OF DIFFERENT CFO ESTIMATORS