Conjoining Speeds up Information Diffusion in Overlaying Social-Physical Networks

Osman Yağan, Dajun Qian, Junshan Zhang, and Douglas Cochran
{oyagan, dqian, junshan.zhang, cochran}@asu.edu
School of Electrical, Computer and Energy Engineering
Arizona State University, Tempe, AZ 85287-5706 USA

Abstract—We study information diffusion in an overlaying social-physical network. Specifically, we consider the following set-up: There is a physical information network where information spreads amongst people through conventional communication media (e.g., face-to-face communication, phone calls), and conjoint to this physical network, there are online social networks where information spreads via web sites such as Facebook, Twitter, FriendFeed, YouTube, etc. Capitalizing on the theory of inhomogeneous random graphs, we quantify the size and the critical threshold of information epidemics in this conjoint social-physical network by assuming that information diffuses according to the SIR epidemic model. One interesting finding is that even if there is no percolation in the individual networks, percolation (i.e., information epidemics) can take place in the conjoint social-physical network. We also show, both analytically and experimentally, that the fraction of individuals who receive an item of information (started from an arbitrary node) is significantly larger in the conjoint social-physical network case, as compared to the case where the networks are disjoint. These findings reveal that conjoining the physical network with online social networks can have a dramatic impact on the speed and scale of information diffusion.

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I. INTRODUCTION

A. Motivation and Background

Modern society relies on basic physical network infrastructures, such as power stations, telecommunication networks and transportation systems. Recently, due to advances in communication technologies and cyber-physical systems, these infrastructures have become increasingly dependent on one another and have emerged as interdependent networks [1]. One archetypal example of such coupled systems is the smart grid where the power stations and the communication network controlling them are coupled together. See the pioneering work of Buldyrev et al. [2] (see also [3], [4], [5], [6], [7], [8], [9], [10]) for a diverse set of models on coupled networks.

Apart from physical infrastructure networks, coupling can also be observed between different types of social networks. Traditionally, people are tied together in a physical information network through old-fashioned communication media, such as face-to-face interactions. On the other hand, recent advances of Internet and mobile communication technologies have enabled people to be connected more closely through online social networks. Indeed, people can now interact through e-mail or online chatting, or communicate through a Web 2.0 website such as Facebook, Twitter, FriendFeed, YouTube, etc. Clearly, the physical information network and online social networks are not completely separate since people may participate in two or more of these networks at the same time. For instance, a person can forward a message to his/her online friends via Facebook and Twitter upon receiving it from someone through face-to-face communication. As a result, the information spread in one network may trigger the propagation in another network, and may result in a possible cascade of information. One conjecture is that due to this coupling between the physical and online social networks, today’s breaking news (and information in general) can spread at an unprecedented speed throughout the population, and this is the main subject of the current study.

Information cascades over coupled networks can deeply influence the patterns of social behavior. Indeed, people have become increasingly aware of the fundamental role of the coupled social-physical network\(^1\) as a medium for the spread of not only information, but also ideas and influence. Twitter has emerged as an ultra-fast source of news \(^1\) and Facebook has attracted major businesses and politicians for advertising products or candidates. Several music groups or singers have gained international fame by uploading videos to YouTube. In almost all cases, a new video uploaded to YouTube, a rumor started in Facebook or Twitter, or a political movement advertised through online social networks, either dies out quickly or reaches a significant proportion of the population. In order to fully understand the extent to which these events happen, it is of great interest to consider the combined behavior of the physical information network and several online social networks.

Despite the increasingly important role that coupled networks play in many aspects of modern society, there has been little study on information diffusion across such networks. Most existing works consider information (or disease) propagation only on a single network, and can be roughly classified into two categories. The first type of studies are based on “fully mixing” transmissions, where each individual is equally likely to affect any other individual regardless of network topology. The classical susceptible-infectious-recovered (SIR) and susceptible-infectious-susceptible (SIS) models are formulated\(^1\) Throughout, we sometimes refer to the physical information network simply as the physical network, whereas we refer to online social networks simply as social networks. Hence the term coupled (or overlaying, or conjoint) social-physical network.

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under this assumption \cite{12, 13, 14}. Of course, personal contacts are much more likely to take place between people with existing social relationships, such as friends, colleagues and relatives. In other words, the information propagation does depend on the topology of social relations. With this consideration, several authors have studied the spreading of diseases in small-world networks \cite{15, 16}, scale-free networks \cite{17}, and networks with arbitrary degree distributions \cite{18} using the SIR or SIS model. Among them, Newman \cite{18} has shown that epidemics can be quantified for a wide variety of networks by using percolation theory. In particular, the threshold and the size of epidemics are characterized by studying the phase transition properties of the underlying random graph models of these networks (see \cite{19, 20}).

In this paper, we aim to develop a new theoretic framework towards understanding the characteristics of information diffusion across multiple networks that are coupled. To facilitate exposition, we focus on the case where there exists only one online social network along with the physical information network. We model the physical network and the social network as random graphs with different topological properties. We assume that each individual in the population is a member of the physical network, and becomes a member of the social network independently with a certain probability. It is also assumed that information is transmitted between two nodes (that are connected by a link in any one of the graphs) according to the SIR model; see Section II for precise definitions.

Setting aside the information diffusion problem, there has been some recent interest on various properties of coupled (or interacting or layered) networks (see \cite{4, 21, 22}). However, the models studied in those references have significant differences with the model considered here. For instance, \cite{4} and \cite{21} consider a layered network structure where the networks in distinct layers are composed of identical nodes. On the other hand, in \cite{22}, the authors studied the percolation problem in two interacting networks with completely disjoint vertex sets; their model is similar to interdependent networks introduced in \cite{2}. To the best of our knowledge, there has been no work in the literature that studies the information diffusion in overlaying networks, where the vertex set of one network is a subset of the vertices of the other network.

B. Summary of Main Contributions

We use random graph modeling to transform the information diffusion problem into a percolation problem, which offers interesting insight into the information cascade process (see \cite{15, 16, 18}). The problem under consideration is intricate since the relevant random graph model corresponds to a union of coupled random graphs, and the techniques employed in \cite{18, 23} for single networks fall short of characterizing its phase transition properties. Capitalizing upon very recent progress in inhomogeneous random graphs \cite{24, 25, 26}, we show that the overlaying social-physical network exhibits a “critical point” above which information epidemics are possible; i.e., a single node can spread an item of information (a rumor, an advertisement, a video, etc.) to a positive fraction of individuals in the asymptotic limit. Below the critical point, only small information outbreaks can occur and the fraction of influenced individuals always tends to zero.

Specifically, we consider two different models for the individual networks. First, we assume that both the physical information network and the online social network are Erdős-Rényi (ER) graphs \cite{27}, and then we consider the case where both networks are random graphs with arbitrary degree distributions \cite{23}; i.e., both networks are generated according to the configuration model \cite{27, 28} with specified degree distributions. In each case, we quantify the aforementioned critical point by computing the phase transition threshold of the conjoined random graph model, and show that it depends on both the degree distributions of the networks and the number of individuals that are members of the online social network. Further, we compute the probability that information originating from an arbitrary individual will yield an epidemic along with the resulting fraction of individuals that are influenced; this is done for both cases by computing the giant component size of the corresponding models.

Although we focus on the case where there exists only one online social network conjoint to the physical network, the results established here can easily be extended to the case where there are multiple online social networks (see Appendix A). Indeed, the impact of conjoining the networks on the speed of information diffusion is already pronounced even with only one social network. For instance, consider a physical information network \(W\) and an online social network \(F\) that are ER graphs with respective mean degrees \(\lambda_w\) and \(\lambda_f\), and assume that each node in \(W\) is a member of \(F\) independently with probability \(\alpha\). If \(\lambda_w = 0.6\) and \(\alpha = 0.2\), we show that information epidemics are possible in the overlaying social-physical network \(H = W \cup F\) whenever \(\lambda_f \geq 0.77\). In stark contrast, this happens only if \(\lambda_w > 1\) or \(\lambda_f > 1\) when the two networks are disjoint. Furthermore, in a single ER network \(W\) with \(\lambda_w = 1.5\), an information item originating from an arbitrary individual gives rise to an epidemic with probability 0.58 (i.e., can reach at most 58% of the individuals). However, if the same network \(W\) is conjoined with an ER network \(F\) with \(\alpha = 0.5\) and \(\lambda_f = 1.5\), the probability of an epidemic becomes 0.82 (indicating that up to 82% of the population can be influenced). These results show that the conjoint social-physical network can spread an item of information to a significantly larger fraction of the population as compared to the case where the two networks are disjoint.

To the best of our knowledge, this paper is the first work that characterizes the information diffusion across multiple networks. The techniques (and the model) presented in this paper can also pave the way in studying the influence maximization \cite{29} problem over multiple networks. We believe that our findings along this line shed light on the understanding on information (and influence) propagation across coupled social-physical networks.
C. Notation and Conventions

All limiting statements, including asymptotic equivalences, are understood with \( n \) going to infinity. The random variables (rvs) under consideration are all defined on the same probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \). Probabilistic statements are made with respect to this probability measure \( \mathbb{P} \), and we denote the corresponding expectation operator by \( \mathbb{E} \). The mean value of a random variable \( k \) is denoted by \(< k >\). We use the notation \( a_s \rightarrow b \) to indicate distributional convergence, \( a_s \rightarrow b \) to indicate almost sure convergence and \( E^n \) to indicate convergence in probability. For any discrete set \( S \) we write \(|S|\) for its cardinality. The Kronecker delta \( \delta_{ij} \) is used in the usual manner, with \( \delta_{ij} = 1 \) if \( i = j \) and \( \delta_{ij} = 0 \) otherwise. For a random graph \( \mathcal{G} \) we write \( C_i(\mathcal{G}) \) for the number of nodes in its \( i \)th largest connected component; i.e., \( C_1(\mathcal{G}) \) stands for the size of the largest component, \( C_2(\mathcal{G}) \) for the size of the second largest component, etc.

The indicator function of an event \( E \) is denoted by \( 1 \{ E \} \). We say that an event holds with high probability (whp) if it holds with probability 1 as \( n \rightarrow \infty \). For sequences \( \{ a_n \}, \{ b_n \} : \mathbb{N}_0 \rightarrow \mathbb{R}_+ \), we write \( a_n = O(b_n) \) as a shorthand for the relation \( \lim_{n \rightarrow \infty} \frac{a_n}{b_n} = 0 \), whereas \( a_n = \Theta(b_n) \) means that there exists \( c > 0 \) such that \( a_n \leq cb_n \) for all \( n \) sufficiently large. Also, we have \( a_n = \Omega(b_n) \) if \( b_n = O(a_n) \), or equivalently, if there exists \( c > 0 \) such that \( a_n \geq cb_n \) for all \( n \) sufficiently large. Finally, we write \( a_n = \Theta(b_n) \) if we have \( a_n = O(b_n) \) and \( a_n = \Omega(b_n) \) at the same time.

D. Organization of the Paper

The rest of the paper is organized as follows. In Section \( \text{III} \) we introduce a model for the overlaying social-physical network. Section \( \text{IV} \) summarizes the main results of the paper. In Appendix \( \text{A} \) we give extensive simulations. The proofs of the main results are provided in Sections \( \text{V} \) and \( \text{VI} \). In Appendix \( \text{B} \) we study information diffusion in an interesting case where only a sublinear fraction of individuals are members of the online social network.

II. System Model

We consider the following model for an overlaying social-physical network. Let \( \mathcal{W} \) stand for the physical information network of human beings on the node set \( \mathcal{N} = \{1, \ldots, n\} \). We assume that the graph \( \mathcal{W} \) characterizes the possible spread of information amongst people through old-fashioned communication media; e.g., face-to-face communication, phone calls, etc. Next, let \( \mathcal{F} \) stand for the network that characterizes the information spread through an online social networking web site, e.g., Facebook. We assume that each node in \( \mathcal{N} \) is a member of this auxiliary network with probability \( \alpha \in (0, 1] \) independently from any other node. In other words, let

\[
\mathbb{P}[i \in \mathcal{N}_F] = \alpha, \quad i = 1, \ldots, n, \tag{1}
\]

with \( \mathcal{N}_F \) denoting the set of human beings that are members of Facebook. With this assumption, it is clear that the vertex set \( \mathcal{N}_F \) of \( \mathcal{F} \) satisfies

\[
\frac{|\mathcal{N}_F|}{n} \overset{a.s.}{\rightarrow} \alpha \tag{2}
\]

by the law of large numbers (we consider the case where \( \mathcal{N}_F = o(n) \) separately in Appendix \( \text{B} \).

In order to study information diffusion amongst human beings, a key step is to characterize an overlaying graph \( \mathcal{H} \) that is constructed by taking the union of \( \mathcal{W} \) and \( \mathcal{F} \). In other words, for any distinct pair of nodes \( i, j \), we say that \( i \) and \( j \) are adjacent in the network \( \mathcal{H} \), denoted \( i \sim \mathcal{W} j \), as long as at least one of the conditions \( \{ i \sim \mathcal{W} j \} \) or \( \{ i \sim \mathcal{F} j \} \) holds. This is intuitive since a node \( i \) can forward information to another node \( j \) either by using old-fashioned communication channels (i.e., links in \( \mathcal{W} \)) or by using Facebook (i.e., links in \( \mathcal{F} \)). Of course, for the latter to be possible, both \( i \) and \( j \) should be Facebook users.

We assume that information spreads among the population according to the SIR model. In this context, an individual is either susceptible meaning that he/she has not yet received a particular item of information, or infectious meaning that he/she is aware of the information and is capable of spreading it to his/her contacts, or recovered meaning that he/she is no longer spreading the information. As in [18], we assume that an infectious individual \( i \) transmits the information to a susceptible contact \( j \) with probability \( T_{ij} \) where

\[
T_{ij} = 1 - e^{-r_{ij} \tau_i}. \tag{3}
\]

Here, \( r_{ij} \) denotes the average rate of being in contact over the link from \( i \) to \( j \), and \( \tau_i \) is the time \( i \) keeps spreading the information; i.e., the time it takes for \( i \) to become recovered.

It is expected that the information propagates over the physical and social networks at different speeds, which manifests from different probabilities \( T_{ij} \) across links in this case. Specifically, let \( T_{ij}^w \) stand for the probability of information transmission over a link (between and \( i \) and \( j \)) in \( \mathcal{W} \) and let \( T_{ij}^f \) denote the probability of information transmission over a link in \( \mathcal{F} \). For simplicity, we assume that \( T_{ij}^w \) and \( T_{ij}^f \) are independent for all distinct pairs \( i, j = 1, \ldots, n \). Furthermore, we assume that the random variables \( r_{ij}^w \) and \( \tau_i^w \) are independent and identically distributed (i.i.d.) with probability densities \( P_w(r) \) and \( P_w(\tau) \), respectively. In that case, it was shown in [18] that the information propagates over \( \mathcal{W} \) as if all transmission probabilities were equal to \( T_w \), where \( T_w \) is the mean value of \( T_{ij}^w \), i.e.,

\[
T_w := \frac{1}{2} - \int_0^\infty \int_0^\infty e^{-r^w} P_w(r)P_w(\tau)drd\tau.
\]

We refer to \( T_w \) as the transmissibility of the information over the physical network \( \mathcal{W} \) and note that \( 0 \leq T_w \leq 1 \). In the same manner, we assume that \( r_{ij}^f \) and \( \tau_i^f \) are i.i.d. with respective densities \( P_f(r) \) and \( P_f(\tau) \) leading to a transmissibility \( T_f \) of information over the online social network \( \mathcal{F} \) (in most practical scenarios we expect to see that \( T_f > T_w \)).
For the case $T_w = T_f = 1$, information diffusion becomes equivalent to the percolation problem in the conjoint network $\mathbb{H} = \mathbb{W} \cup \mathbb{F}$. More specifically, the condition $T_w = T_f = 1$ corresponds to the case where nodes transmit the information to all their contacts automatically upon receiving it; this can be incorporated to the current model by assuming that $\tau_i$ is arbitrarily large for all $i$ so that $T_{ij} \to 1$. Therefore, the threshold and the size of the information epidemic can be obtained by studying the phase transition threshold and the giant component size of $\mathbb{H}$. Yet, as we shall see soon, this is not a trivial problem due to the complicated structure of $\mathbb{H}$.

Needless to say, the case where there is no constraint on the transmissibilities $T_w$ and $T_f$ is more interesting (after all, the assumption that $T_w = T_f = 1$ is hardly in effect in a realistic scenario). To study this model, we assume (as in [13]) that each edge in $\mathbb{W}$ is occupied, meaning that it can be used in spreading the information, with probability $T_f$ independently from all other edges. Similarly, each edge in $\mathbb{F}$ is deemed occupied (independently) with probability $T_f$. This time, the size of the information epidemic in $\mathbb{H}$ is equal to the number of individuals that can be reached from the initial node by using only the occupied links of $\mathbb{H}$. Hence, the threshold and the size of the information epidemic can be computed by studying the phase transition problem in $\mathbb{H}(T_w, T_f)$ where $\mathbb{H}(T_w, T_f)$ is the random graph containing only the occupied edges of $\mathbb{H}$.

### III. Main Results

In what follows, we summarize the main results on information diffusion on overlaying social-physical networks.

#### A. Information Diffusion in coupled ER graphs

We first consider a basic scenario where both the physical information network $\mathbb{W}$ and the online social network $\mathbb{F}$ are Erdős-Rényi graphs $[27]$. More specifically, let $\mathbb{W} = \mathbb{W}(n; \lambda_w/n)$ be an ER network on the vertices $\{1, \ldots, n\}$ such that there exists an edge between any pair of distinct nodes $i,j = 1, \ldots, n$ with probability $\lambda_w/n$; this ensures that mean degree of each node is asymptotically equal to $\lambda_w$. Next, obtain a set of vertices $\mathcal{N}_F$ by picking each node $1, \ldots, n$ independently with probability $\alpha \in (0, 1]$. Now, let $\mathbb{F} = \mathbb{F}(n; \alpha, T_f/(\alpha n))$ be an ER graph on the vertex set $\mathcal{N}_F$ with edge probability given by $\frac{T_f}{\alpha n}$. The mean degree of a node in $\mathbb{F}$ is given (asymptotically) by $\lambda_f$ as seen via [2]. Assume further that each edge in $\mathbb{W}$ (resp. in $\mathbb{F}$) is occupied with probability $T_w$ (resp. with probability $T_f$), independently from all other edges. Under these conditions, the online social network $\mathbb{F}$ is still an ER graph, but with average degree $T_f \lambda_f$, whereas the physical network $\mathbb{W}$ is an ER graph with average degree $T_w \lambda_w$.

The overall system model $\mathbb{H}$ can now be obtained by conjoining the physical information network $\mathbb{W}$ and the online social network $\mathbb{F}$. In other words, $\mathbb{H}$ is constructed on the vertices $1, \ldots, n$ by conjoining the occupied edges of $\mathbb{W}$ and $\mathbb{F}$, i.e., we have $\mathbb{H}(n; \alpha, T_w \lambda_w, T_f \lambda_f) = \mathbb{W}(n; T_w \lambda_w/n) \cup \mathbb{F}(n; \alpha, T_f \lambda_f/(\alpha n))$. Next, we present the first main result that characterizes the critical threshold and the size of the information epidemic in the overlaying social-physical network $\mathbb{H}$.

Let $\lambda^*_w$ be defined by

$$
\lambda^*_w := \frac{1}{2} \left( T_f \lambda_f + T_w \lambda_w \right) - \frac{1}{2} \sqrt{\left( T_f \lambda_f + T_w \lambda_w \right)^2 - 4(1-\alpha)T_f \lambda_f T_w \lambda_w}.
$$

(4)

Also, let $\rho_1$ be the largest solution of the equation

$$(1-\alpha)T_w \lambda_w \left( (1-\alpha)\rho_1 T_f \lambda_f - 1 \right) - \log(1-\rho_1) = \rho_1 (T_f \lambda_f + \alpha T_w \lambda_w)$$

(5)

with $\rho_1 \in [0, 1]$, and let $\rho_2$ be given by

$$\rho_2 = \frac{-\log(1-\rho_1) - \rho_1 (\alpha T_w \lambda_w + T_f \lambda_f)}{(1-\alpha)T_w \lambda_w}.$$

(6)

**Theorem 3.1:** With the above assumptions, we have

(i) If $\lambda^*_w \leq 1$, then with high probability, the size of the largest component satisfies $C_1(\mathbb{H}(n; \alpha, T_w \lambda_w, T_f \lambda_f)) = O(\log n)$; in contrast, if $\lambda^*_w > 1$ we have $C_1(\mathbb{H}(n; \alpha, T_w \lambda_w, T_f \lambda_f)) = \Theta(n)$ w.h.p, while the size of the second largest component satisfies $C_2(\mathbb{H}(n; \alpha, T_w \lambda_w, T_f \lambda_f)) = O(\log n)$.

(ii) Moreover,

$$\frac{1}{n} C_1(\mathbb{H}(n; \alpha, T_w \lambda_w, T_f \lambda_f)) \sim \alpha \rho_1 + (1-\alpha) \rho_2.$$

A proof of Theorem 3.1 is given in Section V.

Theorem 3.1 quantifies the number of individuals in the overlaying social-physical network that are likely to receive an item of information which starts spreading from a single individual. Specifically, the “critical point” of the information epidemic is marked by $\lambda^*_w = 1$, with the critical threshold $\lambda^*_w$ given by (4). We conclude from Theorem 3.1 that for any parameter set that yields $\lambda^*_w \leq 1$ (the subcritical regime), the largest possible number of individuals who receive the information is $O(\log n)$, meaning that only small (non-epidemic) information outbreaks can take place. On the other hand, if $\lambda^*_w > 1$ (the supercritical regime), the information has a positive probability of reaching a linear fraction of the individuals; i.e, information epidemics can occur. In that case, an information item originating from an arbitrary individual gives rise to an information epidemic with probability $\alpha \rho_1 + (1-\alpha) \rho_2$ and reaches a fraction $\alpha \rho_1 + (1-\alpha) \rho_2$ of individuals in the network; here $0 \leq \rho_1 \leq 1$ is obtained by the largest solution of (5) and $\rho_2$ is given by (6).

We observe that the threshold function $\lambda^*_w$ is symmetric in $T_f \lambda_f$ and $T_w \lambda_w$, meaning that both networks have identical roles in carrying the conjoined network to the supercritical regime where information can reach a linear fraction of the nodes. To get a more concrete sense, we depict in Figure [1] the minimum $\lambda_f T_f$ required to have a giant component in $\mathbb{H}(n; \alpha, T_w \lambda_w, T_f \lambda_f)$ versus $\lambda_w T_w$ for various $\alpha$ values. Each curve in the figure corresponds to a phase transition boundary above which information epidemics are possible. If
It is well known \cite{28} that for these graphs the critical point of the phase transition is given by

\[
\frac{\mathbb{E}[d_i(d_i - 1)]}{\mathbb{E}[d_i]} = 1
\]

where \(d_i\) is the degree of an arbitrary node. We next show that this condition is not equivalent (and, indeed is not even a good approximation) to \(\lambda_f^* = 1\). First, note that the degree of an arbitrary node \(i\) in \(\mathcal{H}\) follows a Poisson distribution with mean \(T_w\lambda_w\) if \(i \not\in \mathcal{N}_F\) (which happens with probability \(1 - \alpha\)), and it follows a Poisson distribution with mean \(T_f\lambda_f + T_w\lambda_w - \frac{T_f\lambda_f + T_w\lambda_w}{n}\) if \(i \in \mathcal{N}_F\) (which happens with probability \(\alpha\)). When \(n\) becomes large this leads to

\[
\frac{\mathbb{E}[d_i(d_i - 1)]}{\mathbb{E}[d_i]} = \frac{\alpha(T_f\lambda_f + T_w\lambda_w)^2 + (1 - \alpha)(T_w\lambda_w)^2}{\alpha T_f\lambda_f + T_w\lambda_w}
\]

(7)

It can be seen that the above expression is not equal to \(\lambda_f^*\) given by (4). For instance, with \(\alpha = 0.2, T_w\lambda_w = 0.6\) and \(T_f\lambda_f = 0.8\), we have \(\lambda_f^* = 1.03\) while (7) yields 0.89 signaling a significant difference between the exact threshold \(\lambda_f^*\) and the mean field approximation given by (4). We conclude that the results established above go beyond the classical results for random graphs with arbitrary degree distributions.

\subsection{B. Information Diffusion in Coupled Graphs with Arbitrary Degree Distributions}

In the previous section we studied the spread of information over coupled social-physical networks where each individual network is an Erdős-Rényi graph \cite{27}. We now expand these results to a more general and in fact more practically relevant class of graphs usually known as random graphs with arbitrary degree distribution \cite{18, 23}. In particular, we specify a degree distribution that gives the properly normalized probabilities \(\{p_k^w, k = 0, 1, \ldots\}\) that an arbitrary node in \(\mathcal{W}\) has degree \(k\). Namely, we let each node \(i = 1, \ldots, n\) in \(\mathcal{W} = \mathcal{W}(n; \{p_k^w\})\) have a random degree drawn from the distribution \(\{p_k^w\}\) independently from any other node. Similarly, we assume that the degrees of all nodes in \(\mathcal{F}\) are drawn independently from the distribution \(\{p_k^f, k = 0, 1, \ldots\}\); see Section VI-A and \cite{18, 23, 26} for details about the construction of random graphs with given degree distributions. Finally, the vertex set of \(\mathcal{F} = \mathcal{F}(n; \alpha, \{p_k^f\})\) is obtained in the usual manner by picking each node \(1, \ldots, n\) independently with probability \(\alpha\). In what follows, we shall assume that the degree distributions are well-behaved in the sense that all moments of arbitrary order are finite.

As in the previous section, let \(T_w\) be the information transmissibility (i.e., the mean probability of information transfer between any two nodes) in the physical network \(\mathcal{W}\), and let \(T_f\) be the information transmissibility in the online social network \(\mathcal{F}\). In other words, each edge in \(\mathcal{W}\) is deemed occupied, meaning that it can be used in spreading the information, independently with probability \(T_w\). Similarly, we let each edge in \(\mathcal{F}\) be occupied with probability \(T_f\) independently from all the other edges. The overall system model can now be obtained by taking a union of the occupied edges of \(\mathcal{W}\) and \(\mathcal{F}\). That
is, we let $\mathbb{H}(n; \alpha, \{p^w_k\}, T_w, \{p^f_k\}, T_f) = \mathcal{W}(n; \{p^w_k\}, T_w) \cup \mathcal{F}(n; \alpha, \{p^f_k\}, T_f)$ be the corresponding social-physical network over which the information diffuses.

We now present the second main result that characterizes the threshold and the size of the information epidemic in $\mathbb{H}$ by revealing its phase transition properties. First, for notational convenience, let $k_f$ and $k_w$ be random variables independently drawn from the distributions $\{p^f_k\}$ and $\{p^w_k\}$, respectively, and let $< k_f > := \lambda_f$ and $< k_w > := \lambda_w$. Further, assume that $\beta_f$ and $\beta_w$ are given by

$$\beta_f := \frac{-k_f^2 - \lambda_f}{\lambda_f} \quad \text{and} \quad \beta_w := \frac{-k_w^2 - \lambda_w}{\lambda_w}, \quad (8)$$

and define the threshold function $\sigma^*_{fw}$ by

$$\sigma^*_{fw} = T_f \beta_f + T_w \beta_w + \frac{\sqrt{(T_f \beta_f - T_w \beta_w)^2 + 4 \alpha T_f T_w \lambda_f \lambda_w}}{2}. \quad (9)$$

Finally, let $h_1, h_2$ in $(0, 1)$ be given by the pointwise smallest solution of the recursive equations

$$h_1 = \frac{1}{\lambda_f} E \left[ k_f (1 + T_f (h_1 - 1))^{k_f - 1} \right] \times E \left[ (1 + T_w (h_2 - 1))^{k_w} \right] \quad (10)$$

and

$$h_2 = \frac{1}{\lambda_w} E \left[ \alpha (1 + T_f (h_1 - 1))^{k_f} + 1 - \alpha \right] \times E \left[ k_w (1 + T_w (h_2 - 1))^{k_w - 1} \right]. \quad (11)$$

**Theorem 3.2:** Under the assumptions just stated, we have

(i) If $\sigma^*_{fw} \leq 1$ then with high probability the size of the largest component satisfies

$$C_1 \left( \mathbb{H}(n; \alpha, \{p^w_k\}, T_w, \{p^f_k\}, T_f) \right) = o(n).$$

On the other hand, if $\sigma^*_{fw} > 1$, then

$$C_1 \left( \mathbb{H}(n; \alpha, \{p^w_k\}, T_w, \{p^f_k\}, T_f) \right) = \Theta(n) \text{ whp.}$$

(ii) Also,

$$\frac{1}{n} C_1 \left( \mathbb{H}(n; \alpha, \{p^w_k\}, T_w, \{p^f_k\}, T_f) \right) \xrightarrow{p} 1 - \mathbb{E} \left[ \alpha (1 + T_f (h_1 - 1))^{k_f} + 1 - \alpha \right] \times \mathbb{E} \left[ k_w (1 + T_w (h_2 - 1))^{k_w - 1} \right]. \quad (12)$$

A proof of Theorem 3.2 is given in Section VI.

Theorem 3.2 can be viewed as a counterpart of Theorem 3.1. It quantifies the number of individuals in the overlaying social-physical network likely to receive a particular information item when the physical network $\mathcal{W}$ and the social network $\mathcal{F}$ have arbitrary degree distributions $\{p^w_k\}$ and $\{p^f_k\}$, respectively. Specifically, for $\{p^w_k\}$ and $\{p^f_k\}$ with finite moments, Theorem 3.2 shows that the critical point of the information epidemic is marked by $\sigma^*_{fw} = 1$, with the critical threshold $\sigma^*_{fw}$ given by (9). In other words, for any parameter set that yields $\sigma^*_{fw} > 1$ (supercritical regime), an item of information has a positive probability of giving rise to an information epidemic; i.e., reaching a linear fraction of the individuals. In that case, the asymptotic fraction of the individuals who receive the information can be found by first solving the recursive equations (10)-(11) for the smallest $h_1, h_2$ in $(0, 1)$ and then computing the expression given in (12). On the other hand, whenever it holds that $\sigma^*_{fw} \leq 1$ (subcritical regime), we conclude from Theorem 3.2 that the largest number of individuals who receive the information will be $o(n)$ with high probability, meaning that all outbreaks are non-epidemic.

We have some further remarks on the applicability of Theorem 3.2 and Theorem 3.1. Consider the case where both $\mathcal{W}$ and $\mathcal{F}$ are ER graphs; i.e., let $p^w_k = e^{-\lambda_w} \lambda_w^k / k!$ and $p^f_k = e^{-\lambda_f} \lambda_f^k / k!$. We have that $\beta_f = \lambda_f$, $\beta_w = \lambda_w$, and it is easy to check that $\sigma^*_{fw} = \lambda_f \lambda_w$ so that part (i) of Theorem 3.2 is compatible with part (i) of Theorem 3.1. Also, we find (numerically) that the second parts of Theorems 3.1 and 3.2 yield the same asymptotic giant component size. Nevertheless, it is worth noting that, although ER graphs constitute a special case of random graphs with arbitrary degree distributions, Theorem 3.1 is not a corollary of Theorem 3.2. This is because, through a different technique used in the proofs, Theorem 3.1 provides sharper bounds $C_1(\mathbb{H}(n; \alpha, T_w \lambda_w, T_f \lambda_f)) = O(\log n)$ (subcritical case) and $C_2(\mathbb{H}(n; \alpha, T_w \lambda_w, T_f \lambda_f)) = O(\log n)$ (supercritical case) that go beyond Theorem 3.2.

Theorem 3.2 is also compatible with the existing results [28], [18] regarding the phase transition of a single random graph with arbitrary degree distribution. For instance, it was shown [18] that when they are disjoint, the random graphs $\mathcal{W}$ and $\mathcal{F}$ have a giant component if and only if $T_w \beta_w > 1$ and $T_f \beta_f > 1$, respectively. Here, it is easy to see that $\sigma^*_{fw} \geq \max\{T_w \beta_w, T_f \beta_f\}$ so that $\mathbb{H}$ has a giant component whenever $\mathcal{W}$ or $\mathcal{F}$ has a giant component.

**IV. Numerical Results**

We next present numerical results to illustrate the findings of Theorem 3.1 and Theorem 3.2.

**A. ER Networks**

We first study the case where both the physical information network $\mathcal{W}$ and the online social network $\mathcal{F}$ are Erdős-Rényi graphs. As in Section III-A let $\mathbb{H}(n; \alpha, T_w \lambda_w, T_f \lambda_f) = \mathcal{W}(n; T_w \lambda_w / n) \cup \mathcal{F}(n; \alpha, T_f \lambda_f / (\alpha n))$ be the conjoint social-physical network, where $\mathcal{W}$ is defined on the vertices $\{1, \ldots, n\}$, whereas the vertex set of $\mathcal{F}$ is obtained by picking each node $1, \ldots, n$ independently with probability $\alpha$. The mean degrees of $\mathcal{W}$ and $\mathcal{F}$ are given (asymptotically) by $\lambda_w$ and $\lambda_f$, while the information transmissibilities are equal to $T_w$ and $T_f$, respectively.

We plot in Figure 2 the fractional size of the giant component in $\mathbb{H}(n; \alpha, T_w \lambda_w, T_f \lambda_f)$ versus $T_f \lambda_f = T_w \lambda_w$ for various $\alpha$ values. In other words, the plots illustrate the largest fraction of individuals that a particular item of information can reach. In this figure, the curves stand for the analytical results obtained by Theorem 3.1 whereas marked points stand for the experimental results obtained with $n = 20,000$ nodes by averaging 200 experiments for each parameter set. Clearly, there is a good match between the theoretical and experimental results.
where $m$ from Theorem 3.1, whereas marked points stand for the experimental results

an information epidemic) is given by

critical threshold for the existence of a giant component (i.e.,

clearly due to the finite size effect. It is also seen that the

Fig. 2. The fractional size of the giant component in $\mathbb{V}(n; \alpha, T_w \lambda_w, T_f \lambda_f)$ versus $T_f \lambda_f = T_w \lambda_w$. The curves correspond to analytical results obtained from Theorem 3.2 whereas marked points stand for the experimental results obtained with $n = 20,000$ by averaging 200 experiments for each parameter set. There is good agreement between the theoretical and experimental results; the small discrepancy in the subcritical regime is clearly due to the finite size effect.

results; the small discrepancy near the phase transition is
clearly due to the finite size effect. It is also seen that the

threshold for the existence of a giant component (i.e.,
an information epidemic) is given by $T_f \lambda_f = T_w \lambda_w = 0.760$

when $\alpha = 0.1$, $T_f \lambda_f = T_w \lambda_w = 0.586$ when $\alpha = 0.5$, and

$T_f \lambda_f = T_w \lambda_w = 0.514$ when $\alpha = 0.9$. It is easy to check

that these values are in perfect agreement with the theoretically

obtained critical threshold $\lambda_w^*$ given by (4).

B. Networks with Power Degree Distributions

We now turn to the case where the networks $\mathbb{W}$ and $\mathbb{F}$ are random graphs with arbitrary degree distributions. In order to
gain more insight about the consequences of Theorem 3.2, we
consider a specific example of information diffusion over the
physical information network $\mathbb{W}$ and the online social network $\mathbb{F}$
degree distributions $\{p_k^w\}$ and $\{p_k^f\}$, respectively.

Here, we use power-law distributions with exponential cutoff. Specifically, we let

\[
p_k^w = \begin{cases} 
0 & \text{if } k = 0 \\
\left(\text{Li}_{\gamma_w}(e^{-1/T_w})\right)^{-1} k^{-\gamma_w} e^{-k/T_w} & \text{if } k = 1, 2, \ldots 
\end{cases} 
\tag{13}
\]

and

\[
p_k^f = \begin{cases} 
0 & \text{if } k = 0 \\
\left(\text{Li}_{\gamma_f}(e^{-1/T_f})\right)^{-1} k^{-\gamma_f} e^{-k/T_f} & \text{if } k = 1, 2, \ldots , 
\end{cases} 
\tag{14}
\]

where $\gamma_w$, $\gamma_f$, $T_w$ and $T_f$ are positive constants and the

normalizing constant $\text{Li}_m(z)$ is the $m$th polylogarithm of $z$; i.e.,

\[
\text{Li}_m(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^m} .
\]

It is easy to verify that both distributions satisfy the

normalizing condition $\sum_{k=0}^{\infty} p_k = 1$. Power law distributions

with exponential cutoff are chosen here because they are

applied to a variety of real-world networks (e.g., see [30] for a
detailed empirical study on the degree distributions of real-world networks). Moreover, the distributions (13)-(14) ensure that all moments of arbitrary order are finite as required by Theorem 3.2

To apply Theorem 3.2, we first compute the epidemic threshold given by (9). Under (13)-(14) we find that

\[
\lambda_f = \frac{\text{Li}_{\gamma_f-1}(e^{-1/T_f})}{\text{Li}_{\gamma_f}(e^{-1/T_f})}, \quad \lambda_w = \frac{\text{Li}_{\gamma_w-1}(e^{-1/T_w})}{\text{Li}_{\gamma_w}(e^{-1/T_w})}
\]

and

\[
\beta_f = \frac{\text{Li}_{\gamma_f-2}(e^{-1/T_f}) - \text{Li}_{\gamma_f-1}(e^{-1/T_f})}{\text{Li}_{\gamma_f}(e^{-1/T_f})},
\]

\[
\beta_w = \frac{\text{Li}_{\gamma_w-2}(e^{-1/T_w}) - \text{Li}_{\gamma_w-1}(e^{-1/T_w})}{\text{Li}_{\gamma_w}(e^{-1/T_w})} .
\]

It is now a simple matter to compute the critical threshold $\sigma^*_w$ from (9) using the above relations. Then, we can use Theorem 3.2(i) to check whether or not an item of information can reach a linear fraction of individuals in the conjoint social-physical network $\mathbb{W}(n; \alpha, \{p_k^w\}, T_w, \{p_k^f\}, T_f) = \mathbb{W}(n; \{p_k^w\}, T_w) \cup \mathbb{F}(n; \alpha, \{p_k^f\}, T_f)$.

To that end, we depict in Figure 3 the minimum

$T_f$ value required to have a giant component in

$\mathbb{W}(n; \alpha, \{p_k^w\}, T_w, \{p_k^f\}, T_f)$ versus $T_w$, for various $\alpha$

values. In other words, each curve corresponds to a phase transition boundary above which information epidemics are

possible, in the sense that an information has a positive probability of reaching out to a linear fraction of individuals

in the overlaying social-physical network. In all plots, we

set $\gamma_f = \gamma_w = 2.5$ and $T_f = T_w = 10$. The $T_f$ and $T_w$

values are multiplied by the corresponding $\beta_f$ and $\beta_w$

values to make a fair comparison with the disjoint network case

where it is required (13) to have $\beta_w T_w > 1$ (or $\beta_f T_f > 1$)

for the existence of an epidemic; under the current setting we

have $\beta_f = \beta_w = 1.545$. Figure 3 illustrates (in the arbitrary
distribution case) how conjoining two networks can speed up

the information diffusion. It can be seen that even for small

$\alpha$ values, two networks, albeit having no giant component

individually, can yield an information epidemic when they are

conjoined. As an example, we see that for $\alpha = 0.1$, it suffices

to have that $\beta_f T_f = \beta_w T_w = 0.774$ for the existence of an

information epidemic in the conjoint network $\mathbb{W}$, whereas if

the networks $\mathbb{W}$ and $\mathbb{F}$ are disjoint, an information epidemic

can occur only if $\beta_w T_w > 1$ or $\beta_f T_f > 1$.

Next, we turn to computation of the giant component size.

We start by noting that

\[
\mathbb{E}[(1 + T_w(h_2 - 1))^{k_w}] = \frac{\text{Li}_{\gamma_w}((1 + T_w(h_2 - 1))e^{-1/T_w})}{\text{Li}_{\gamma_w}(e^{-1/T_w})} .
\]

\[
\mathbb{E}[(1 + T_f(h_1 - 1))^{k_f}] = \frac{\text{Li}_{\gamma_f}((1 + T_f(h_1 - 1))e^{-1/T_f})}{\text{Li}_{\gamma_f}(e^{-1/T_f})} .
\]
We see that there is good agreement between theory and experiment even for such a small number of vertices; the small discrepancy in the subcritical regime is attributed to the finite size effect.

V. PROOF OF THEOREM 3.1

In this section, we give a proof Theorem 3.1. First, we summarize the technical tools that will be used.

A. Inhomogeneous Random Graphs

Recently, Bollobás, Janson and Riordan [24] have developed a new theory of inhomogeneous random graphs that would allow studying a very broad class of complex networks rigorously. In particular, their theory can be applied to many real-world graphs, such as models with power-law degree distributions, scale-free networks based on preferential attachment, dynamical random graphs, etc. The authors in [24] established very general results for various properties of these models, including the critical point of their phase transition, as well as the size of their giant component. Here, we summarize these tools with focus on the results used in this paper.

At the outset, assume that a graph is defined on vertices \( \{1, \ldots, n\} \), where each vertex \( i \) is assigned randomly or deterministically a point \( x_i \) in a metric space \( S \). Assume that the metric space \( S \) is equipped with a Borel probability measure \( \mu \) such that for any \( \mu \)-continuity set \( A \subseteq S \) (see [24]), we have

\[
\frac{1}{n} \sum_{i=1}^{n} 1_{[x_i \in A]} \overset{p}{\to} \mu(A).
\]
A vertex space $\mathcal{V}$ is then defined as a triple $(S, \mu, \{x_1, \ldots, x_n\})$ where $\{x_1, \ldots, x_n\}$ is a sequence of points in $S$ satisfying (15).

Next, let a kernel $\kappa$ on the space $(S, \mu)$ define a symmetric, non-negative, measurable function on $S \times S$. The random graph $G^V(n, \kappa)$ on the vertices $\{1, \ldots, n\}$ is then constructed by assigning an edge between $i$ and $j$ (i.e., $i < j$) with probability $\kappa(x_i, x_j)/n$, independently of all the other edges in the graph.

Consider random graphs $G^V(n, \kappa)$ for which the kernel $\kappa$ is bounded and continuous a.e. on $S \times S$. In fact, in this study it suffices to consider only the cases where the metric space $S$ consists of finitely many points, i.e., $S = \{1, \ldots, r\}$; this special case is equivalent to the model studied by Söderberg [25]. Under these assumptions, the kernel $\kappa$ reduces to an $r \times r$ matrix, and $G^V(n, \kappa)$ becomes a random graph with vertices of $r$ different types; e.g., vertices with/without Facebook membership, etc. Two nodes (in $G^V(n, \kappa)$) of type $i$ and $j$ are joined by an edge with probability $n^{-1}\kappa(i, j)$ and the condition (15) reduces to

$$\frac{n_i}{n} \to \mu_i, \quad i = 1, \ldots, r,$$

where $n_i$ stands for the number of nodes of type $i$ and $\mu_i$ is equal to $\mu(\{i\})$.

As usual, the phase transition properties of $G^V(n, \kappa)$ can be studied by exploiting the connection between the component structure of the graph and the survival probability of a related branching process. In particular, consider a branching process that starts with an arbitrary vertex and recursively reveals the largest component reached by exploring its neighbors. For each $i = 1, \ldots, r$, we let $\rho(\kappa; i)$ denote the probability that the branching process produces infinite trees when it starts with a node of type $i$. The survival probability $\rho(\kappa)$ of the branching process is then given by

$$\rho(\kappa) = \sum_{i=1}^{r} \rho(\kappa; i) \mu_i.$$  

In analogy with the classical results for ER graphs [27], it can be shown [24, 25] that $\rho(\kappa; i), i = 1, \ldots, r$ satisfy the recursive equations

$$\rho(\kappa; i) = 1 - \exp \left\{ - \sum_{j=1}^{r} \kappa(i, j) \mu_j \cdot \rho(\kappa; j) \right\}, \quad i = 1, \ldots, r.$$  

The value of $\rho(\kappa)$ can be computed via (17) by characterizing the stable fixed point of (13) reached from the starting point $\rho(\kappa; 1) = \cdots = \rho(\kappa; r) = 0$. It is a simple matter to check that, with $M$ denoting an $r \times r$ matrix given by $M(i, j) = \kappa(i, j) \cdot \mu_j$, the iterated map (18) has a non-trivial solution (i.e., a solution other than $\rho(\kappa; 1) = \cdots = \rho(\kappa; r) = 0$) if and only if

$$\sigma(M) := \max(|\lambda_i| : \lambda_i \text{ is an eigenvalue of } M) > 1.$$  

For a square matrix $M$, its largest eigenvalue in absolute value, $\sigma(M)$, defines its spectral radius. Thus, we see that if the spectral radius of $M$ is less than or equal to one, the branching process is subcritical with $\rho(\kappa) = 0$ and the graph $G^V(n, \kappa)$ has no giant component; i.e., we have that $C_1(G^V(n, \kappa)) = o(n)$ whp.

On the other hand, if $\sigma(M) > 1$, then the branching process is supercritical and there is a non-trivial solution $\rho(\kappa; i) > 0, i = 1, \ldots, r$ that corresponds to a stable fixed point of (13). In that case, $\rho(\kappa) > 0$ corresponds to the probability that an arbitrary node belongs to the giant component, which asymptotically contains a fraction $\rho(\kappa)$ of the vertices. In other words, if $\sigma(M) > 1$, we have that $C_1(G^V(n, \kappa)) = \Omega(n)$ whp, and $\frac{1}{n} C_1(G^V(n, \kappa)) \sim \rho(\kappa)$.

Bollobas et al. [24, Theorem 3.12] have shown that the bound $C_1(G^V(n, \kappa)) = o(n)$ in the subcritical case can be improved under some additional conditions: They established that whenever $\sup_{i,j} \kappa(i, j) < \infty$ and $\sigma(M) \leq 1$, then we have $C_1(G^V(n, \kappa)) = O(\log n)$ whp in the case of ER graphs. They have also shown that if either $\sup_{i,j} \kappa(i, j) < \infty$ or $\inf_{i,j} \kappa(i, j) > 0$, then in the supercritical regime (i.e., when $\sigma(M) > 1$) the second largest component satisfies $C_2(G^V(n, \kappa)) = O(\log n)$ whp.

**B. A Proof of Theorem 3.1**

We start by studying the information spread over the network $H$ when information transmissibilities $T_w$ and $T_f$ are both equal to 1. Clearly, this corresponds to studying the phase transition in $H = H(n; \alpha, \lambda_w, \lambda_f)$, and we will do so by using the techniques summarized in the previous section. Let $S = \{1, 2\}$ stand for the space of vertex types, where vertices with Facebook membership are referred to as type 1 while vertices without Facebook membership are said to be of type 2. In other words, we let

$$x_i = \begin{cases} 1 & \text{if } i \in N_F \\ 2 & \text{if } i \notin N_F \end{cases}$$

for each $i = 1, \ldots, n$. Assume that the metric space $S$ is equipped with a probability measure $\mu$ that satisfies the condition (16); i.e., $\mu(\{1\}) := \mu_1 = \alpha$ and $\mu(\{2\}) := \mu_2 = 1 - \alpha$.

Finally, we compute the appropriate kernel $\kappa$ such that, for each $i, j = \{1, 2\}$, $\kappa(i, j)/n$ gives the probability that two vertices of type $i$ and $j$ are connected. Clearly, we have

$$\kappa(1, 1) = n \left( 1 - \left( 1 - \frac{\lambda_w}{n} \right) \left( 1 - \frac{\lambda_f}{\alpha n} \right) \right) = \lambda_w + \frac{\lambda_f}{\alpha} - \frac{\lambda_w \lambda_f}{\alpha n},$$

whereas

$$\kappa(1, 2) = \kappa(2, 1) = \kappa(2, 2) = \lambda_w.$$  

We are now in a position to derive the critical point of the phase transition in $H(n; \alpha, \lambda_w, \lambda_f)$ as well as the giant component size $C_1(H(n; \alpha, \lambda_w, \lambda_f))$. First, we compute the matrix $M(i, j) = \kappa(i, j) \mu_j$ and get

$$M = \begin{pmatrix} \alpha \lambda_w + \lambda_f - \frac{\lambda_w \lambda_f}{n} & (1 - \alpha) \lambda_w \\ \alpha \lambda_w & (1 - \alpha) \lambda_w \end{pmatrix}$$
It is a simple matter to check that the spectral radius of $M$ is given by

$$\sigma(M) = \lambda_f + \lambda_w - \frac{\lambda_f \lambda_w}{n}$$

$$+ \frac{1}{2\sqrt{\left(\lambda_f + \lambda_w - \frac{\lambda_f \lambda_w}{n}\right)^2 - 4(1 - \alpha)\lambda_f \lambda_w}}$$

In fact, it is clear that the term $\frac{\lambda_w \lambda_f}{n}$ has no effect on the results as we eventually let $n$ go to infinity. This leads to the conclusion that the random graph $\mathbb{H}(n; \alpha, \lambda_w, \lambda_f)$ has a giant component if and only if

$$\lambda_f + \lambda_w + \sqrt{(\lambda_f + \lambda_w)^2 - 4(1 - \alpha)\lambda_f \lambda_w} > 1$$

as we recall [19]. If condition (21) is not satisfied, then we have $C_1(\mathbb{H}(n; \alpha, \lambda_w, \lambda_f)) = O(\log n)$ as we note that $\sup_{i,j} \kappa(i,j) < \infty$. From [24] Theorem 3.12), we also get that $C_2(\mathbb{H}(n; \alpha, \lambda_w, \lambda_f)) = O(\log n)$ whenever (21) is satisfied.

Next, we compute the size of the giant component whenever it exists. Let $\rho(\kappa; 2) = \rho_1$ and $\rho(\kappa; 3) = \rho_2$. In view of (17) and the arguments presented in the previous section, the asymptotic fraction of nodes in the giant component is given by

$$\rho(\kappa) = \alpha \rho_1 + (1 - \alpha) \rho_2,$$  
(22)

where $\rho_1$ and $\rho_2$ constitute a stable simultaneous solution of the transcendental equations

$$\rho_1 = 1 - \exp\left\{-\rho_1 (\alpha \lambda_w + \lambda_f) - \rho_2 (1 - \alpha) \lambda_w\right\}$$

$$\rho_2 = 1 - \exp\left\{-\rho_1 \alpha \lambda_w - \rho_2 (1 - \alpha) \lambda_w\right\}$$

By easy algebra, we see that $\rho_1$ is given by the largest solution of the equation

$$(1 - \alpha) \lambda_w (1 - \rho_1) e^{\rho_1 \lambda_f} - 1 - \log(1 - \rho_1)$$

$$= \rho_1 (\lambda_f + \alpha \lambda_w)$$

on the interval $0 \leq \rho_1 \leq 1$, while $\rho_2$ can be computed via

$$\rho_2 = \frac{-\log(1 - \rho_1) - \rho_1 \alpha \lambda_w + \lambda_f}{(1 - \alpha) \lambda_w}.$$  
(25)

So far, we have established the epidemic threshold and the size of the information epidemic when $T_w = T_f = 1$. In the more general case where there is no constraint on the transmissibilities, we see that the online social network $\mathbb{F}$ becomes an ER graph with average degree $T_f \lambda_f$, whereas the physical network $\mathbb{W}$ becomes an ER graph with average degree $T_w \lambda_w$. Therefore, the critical threshold and the size of the information epidemic can be found by substituting $T_f \lambda_f$ for $\lambda_f$ and $T_w \lambda_w$ for $\lambda_w$ in the relations (21), (22), (24), and (25). This establishes Theorem 3.1.

**VI. PROOF OF THEOREM 3.2**

In this section we give a proof of Theorem 3.2. Again, we first introduce the necessary technical tools.

A. Colored Random Graphs with Arbitrary Degree Distributions

The approach outlined in Section V-A allows one to differentiate between different types of vertices in the random graph $\mathbb{H}$ (e.g., vertices with and without Facebook memberships), and assign probabilities to edges according to the types of vertices that they are connecting together. This leads to a mixed Poisson distribution for the degree of a randomly chosen vertex in $\mathbb{H}$. However, in the case where the underlying random graphs $\mathbb{W}$ and $\mathbb{P}$ have arbitrary degree distributions, it is more useful to differentiate between different types of edges, i.e., edges in Facebook and edges in the physical network. In fact, Söderberg has followed this approach and studied the phase transition in the so-called “colored degree-driven random graphs” [31], [26]. Here, we give a simplified version of their model and summarize the main results.

First, let $\{1, \ldots, r\}$ be the possible types of edges in the graph. The colored degree of a node $i$ is then represented by an integer vector $d_i = [d_{i1}, \ldots, d_{ir}]$, where $d_{ij}$ stands for the number of edges of type $j$ that are incident on node $i$. It is assumed that the colored degrees (i.e., $d_{i1}, \ldots, d_{ir}$) are drawn independently from a colored degree distribution $\{p_m\}$ such that for any $i$

$$\mathbb{P} \left[d_{ij} = m_j, \ j = 1, \ldots, r \right] = p_m$$  
(26)

whenever $m = (m_1, \ldots, m_r)$. Given that the colored degrees are picked such that $\sum_{i=1}^{n} d_{ij}^m$ is even for each $j = 1, \ldots, r$, we construct the graph as in [13]. To do this, each node $i = 1, \ldots, n$ is first given the appropriate number $d_{ij}$ of stubs of type $j$ for each $j = 1, \ldots, r$. Then, pairs of these stubs that are of the same type are picked randomly and connected together to form complete edges; in this paper we assume that two stubs can be connected together only if they are of the same type. Pairing of stubs continues until none are left. It is always assumed that the colored degree distribution $\{p_m\}$ is well behaved such that all moments of arbitrary order are defined.

This random stub-pairing mechanism will be encoded through an $r \times r$ color preference matrix $C$ where $C(i, j)/n$ is equal to the asymptotic probability that two arbitrary stubs of types $i$ and $j$ are connected. Here, this amounts to setting

$$C(i, j) = \begin{cases} 0 & \text{if } i \neq j \\ \frac{1}{<d_j>} & \text{if } i = j \end{cases}$$

(27)

where $<d_j>$ is the expected number of edges of type $j$ for an arbitrary node. In other words, we have

$$<d_j> = \sum_{\ell=0}^{\infty} \ell \cdot \mathbb{P} \left[d_{ij} = \ell \right] = \sum_{\ell=0}^{\infty} \ell \cdot \sum_{m: m_j = \ell} p_m.$$  

Let $G(n, p_m, C)$ define the random graph constructed in the manner outlined above. In order to study the phase transition in $G(n, p_m, C)$, we use generating function of the colored degree distribution $\{p_m\}$. This distribution can be represented by a multivariate generating function, $H(x) = \sum_m p_m x^m$.  


where $a^m = \prod_{i=1}^m a_i^{m_i}$; the normalizing condition $H(1) = 1$ is clearly satisfied.

Now, let $P_k$ ($0 \leq k < \infty$) denote the size distribution of the largest connected component that can be reached from a randomly chosen initial vertex in $G(n, p_m, C)$. Also, let $g(z)$ denote the generating function of $P_k$; i.e., $g(z) = \sum_k P_k z^k$. It is useful to express $g(z)$ in terms of $h(z) = [h_1(z) \cdots h_r(z)]$, where $h_i(z)$ stands for the generating function of the size distribution of the component reached by following a stub of type $i$. In fact, it was shown [31, 26] that $g(z)$ and $h(z)$ satisfy the recursive relations

$$g(z) = z \sum_m p_m \prod_{i=1}^r h_i(z)^{m_i} = z H(h(z)), \quad (28)$$

and

$$h_i(z) = z \sum_{j=1}^r C(i,j) \partial_i \partial_j H(h(z)), \quad i = 1, \ldots, r. \quad (29)$$

We are interested in the solution of the recursive relations (29) for the case $z = 1$. This case exhibits a trivial fixed point $h(1) = 1$ which yields $g(1) = 1$ meaning that the underlying branching process is in the supercritical regime and that all components have finite size as understood from the conservation of probability. However, the fixed point $h(1) = 1$ corresponds to the physical solution only if it is an attractor; i.e., a stable solution to the recursion (29). The stability of this fixed point can be checked via linearization of (29) around $h(1) = 1$, which yields the Jacobian $J$ given by

$$J(i,j) = \sum_{\ell=1}^r C(i,\ell) \partial_i \partial_j H(h)|_{h=1}, \quad i, j = 1, \ldots, r.$$  

This is equivalent to having

$$J = CE \quad (30)$$

where $E$ is given by

$$E(i,j) = \partial_i \partial_j H(h)|_{h=1} := < d_i d_j - d_i d_j > . \quad (31)$$

If all the eigenvalues of $J$ are less than one in absolute value (i.e., if the spectral radius of $J$ is less than one), then the solution $h(1) = 1$ is an attractor and $g(1) = 1$ becomes the physical solution, meaning that $G(n, p_m, C)$ does not possess a giant component whp. On the other hand, if the spectral radius of $J$ is larger than one, then the fixed point $h(1) = 1$ is unstable pointing out that the asymptotic branching process is supercritical, with a positive probability of producing infinite trees. In that case, a nontrivial fixed point exists and becomes the attractor of the recursions (29), yielding a solution with $h_i(1) < 1$, $i = 1, \ldots, r$. In view of (28) this implies $g(1) < 1$ and the corresponding probability deficit $1 - g(1)$ is attributed to the existence of a giant component. In fact, the quantity $1 - g(1)$ is equal to the probability that a randomly chosen vertex belongs to the giant component, which contains asymptotically a fraction $1 - g(1)$ of the vertices.

B. A Proof of Theorem 3.2

Consider random graphs $\mathbb{W}(n, \{p_{ij}^w\})$ and $\mathbb{F}(n; \alpha, \{p_{ik}^f\})$ as in Section III-B in order to study the phase transition in $\mathbb{H} = \mathbb{W} \cup \mathbb{F}$, we use the tools outlined in Section V. Let us first start with the case where $T_f = T_w = 1$. To adopt the notation presented in Section VI-A let $\{1, 2\}$ be the set of edge types in $\mathbb{H}$ where type 1 stands for the edges of $\mathbb{F}$ and type 2 refers to the edges in $\mathbb{W}$. The colored degree of node $i$ is given by $d_i = [d_i^f d_i^w]$, where $d_i^f$ and $d_i^w$ are the number of edges incident upon $i$ in the graphs $\mathbb{F}$ and $\mathbb{W}$, respectively. Of course, if $i$ is not a member of the vertex set $N_F$ of $\mathbb{F}$ (which happens with probability $1 - \alpha$), $d_i^f$ becomes automatically zero. When considering the colored degrees of an arbitrary node in $\mathbb{H}$, we write $d_f$ (for the number of edges in $\mathbb{F}$) and $d_w$ (for the number of edges in $\mathbb{W}$) for convenience. Throughout, we shall differentiate between the random variables $k_f$ and $d_f$, where $k_f$ is the random degree of an arbitrary node given that it is a member of $N_F$; i.e., $k_f$ is a random variable that follows the distribution $\{p_{ik}^f\}$. It is clear that $d_f$ is a random variable that is statistically equivalent to $k_f$ with probability $\alpha$, and equal to zero otherwise. For notational convenience, we also define $k_w$ in an analogous manner, but note that $k_w = d_w$ since each node in $\mathbb{H}$ is automatically a member of the vertex set of $\mathbb{W}$.

We now determine the appropriate colored degree distribution $\{p_{jm}\}$. First, observe that

$$p_{i,j} := \mathbb{P}[d_f = i, d_w = j] = \mathbb{P}[d_f = i] \mathbb{P}[d_w = j]$$

due to independence of $\mathbb{W}$ and $\mathbb{F}$. This yields that

$$p_{i,j} = (\alpha p_i^f + (1 - \alpha)1_{i = 0}) \cdot p_j^w \quad (32)$$

as we recall (11) and the fact that for a node to have $i > 0$ edges in $\mathbb{F}$ it should be a member of $N_F$.

Let $< d_f >$ be the mean number of the online social network (i.e., type 1) edges for an arbitrary node; i.e.,

$$< d_f > = \sum_{i=0}^\infty i \mathbb{P}[d_f = i].$$

Similarly, let $< d_w >$ be the mean number of physical connections (i.e., type 2 edges) for an arbitrary node. In view of (32), we have

$$< d_f > = \alpha \lambda_f \quad \text{and} \quad < d_w > = \lambda_w,$$

where $\lambda_f := \sum_{k=0}^\infty k p_k^f$ and $\lambda_w := \sum_{k=0}^\infty k p_k^w$ are mean node degrees in $\mathbb{F}$ and $\mathbb{W}$ respectively. Recalling (27), this yields a color preference matrix given by

$$C = \begin{bmatrix}
\frac{1}{\alpha \lambda_f} & 0 \\
0 & \frac{1}{\lambda_w}
\end{bmatrix} \quad (33)$$

Next, let $< d_f^1 >$ and $< d_f^2 >$ denote the second moments of the number of Facebook and physical connections of an arbitrary node, respectively; i.e., let $< d_f^1 > = \sum_{i=0}^\infty i^2 \mathbb{P}[d_f = i]$ and $< d_f^2 > = \sum_{i=0}^\infty i^2 \mathbb{P}[d_f = i]$. We find

$$E = \begin{bmatrix}
< d_f^2 - d_f > & < d_f d_w > \\
< d_f d_w > & < d_w^2 - d_w >
\end{bmatrix}.$$
as we recall (31) and the Jacobian $J = CE$ is given by

$$J = \begin{bmatrix} \frac{\lambda_u}{\alpha} & \lambda_w \\ \frac{\lambda_u}{\alpha} & \lambda_w \end{bmatrix}$$

where $< k_f^2 > = \sum_{k=0}^\infty k^2 p_f^k$ and $< k_w^2 > = \sum_{k=0}^\infty k^2 p_w^k$. With

$$\beta_f := \frac{< k_f^2 > - \lambda_f}{\lambda_f} \quad \text{and} \quad \beta_w := \frac{< k_w^2 > - \lambda_w}{\lambda_w},$$

the spectral radius of $J$ is given by

$$\sigma(J) = \frac{1}{2} \left( \beta_f + \beta_w + \sqrt{(\beta_f - \beta_w)^2 + 4\alpha \lambda_f \lambda_w} \right) \quad (34)$$

The critical point of the phase transition is now within easy reach by the arguments outlined in Section VI-A. If $\sigma(J)$ given by (34) is less than unity, then with high probability the size of the largest connected component of $\mathbb{H}(n; \alpha, \{ p_w^k \}, \{ p_f^k \})$ is $o(n)$. If, however, $\sigma(J) > 1$, then with probability there exists a giant component in $\mathbb{H}(n; \alpha, \{ p_w^k \}, \{ p_f^k \})$ in that $C_1(\mathbb{H}(n; \alpha, \{ p_w^k \}, \{ p_f^k \})) = \Theta(n)$.

Next, we compute the size of the giant component. Under the above assumptions, we see from (28) and (29) that

$$g(1) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} p(i,j) h_1(1)^i h_2(1)^j = \mathbb{E} \left[ h_1(1)^d \right] \mathbb{E} \left[ h_2(1)^d \right]$$

where $h_1(1)$ and $h_2(1)$ are given by the stable solution to

$$h_1(1) = \frac{1}{\alpha x_f} \mathbb{E} \left[ d_f h_1(1)^{d_f - 1} \right] \mathbb{E} \left[ h_2(1)^d \right]$$

$$h_2(1) = \frac{1}{\lambda_w} \mathbb{E} \left[ h_1(1)^{d_f} \right] \mathbb{E} \left[ d_w h_2(1)^{d_w - 1} \right]$$

upon noting (33). Recalling the arguments of Section VI-A we see that the asymptotic fraction of nodes that appear in the giant component is given by $1 - g(1)$. To summarize, we see by simple algebra that with $h_1, h_2$ in $(0, 1)$ being the pointwise smallest solution of the recursive equations

$$h_1 = \frac{1}{\lambda_f} \mathbb{E} \left[ k_f h_1^{k_f - 1} \right] \mathbb{E} \left[ h_2^k \right]$$

$$h_2 = \frac{1}{\lambda_w} \mathbb{E} \left[ \alpha h_1^{k_f} + 1 - \alpha \right] \mathbb{E} \left[ k_w h_2^{k_w - 1} \right], \quad (35)$$

we have

$$\frac{1}{n} C_1 \left( \mathbb{H}(n; \alpha, \{ p_w^k \}, \{ p_f^k \}) \right) \xrightarrow{d} 1 - \mathbb{E} \left[ \alpha h_1^{k_f} + 1 - \alpha \right] \mathbb{E} \left[ h_2^k \right]. \quad (36)$$

The above results reveal the epidemic threshold and the size of the information epidemic when nodes transmit the information to each of their contacts automatically upon receiving it. In the more general case where there is no such constraint (i.e., when $T_f, T_w < 1$), we can still use the above findings with slight modifications and obtain the corresponding threshold and size of the information epidemic by the tools outlined in Section VI-A.

Now, let $\{ \tilde{p}_f^k \}$ and $\{ \tilde{p}_w^k \}$ be the occupied degree distributions obtained from the original distributions $\{ p_w^k \}$ and $\{ p_f^k \}$ by deleting each edge with probability $1 - T_w$ and $1 - T_f$, respectively. For each $k = 0, 1, \ldots$, we can compute $\tilde{p}_f^k$ and $\tilde{p}_w^k$ from the generating functions of the distributions $\{ p_f^k \}$ and $\{ p_w^k \}$. First let $G_w(x)$ and $G_f(x)$ be the respective generating functions of $\{ \tilde{p}_f^k \}$ and $\{ \tilde{p}_w^k \}$, i.e.,

$$G_w(x) = \sum_{k=0}^\infty \tilde{p}_w^k x^k$$

$$G_f(x) = \sum_{k=0}^\infty \tilde{p}_f^k x^k$$

Similarly, let $\tilde{G}_w(x)$ and $\tilde{G}_f(x)$ be defined by

$$\tilde{G}_w(x) = \sum_{k=0}^\infty \tilde{p}_w^k x^k$$

$$\tilde{G}_f(x) = \sum_{k=0}^\infty \tilde{p}_f^k x^k$$

It is a simple matter to check that (18), we have

$$\tilde{G}_w(x) = G_w \left( 1 + (x - 1)T_w \right) \quad (37)$$

$$\tilde{G}_f(x) = G_f \left( 1 + (x - 1)T_f \right) \quad (38)$$

The occupied degree distributions can now be computed using the standard relations

$$\tilde{p}_w^k = \frac{1}{k!} \frac{d^k \tilde{G}_w(x)}{dx^k} \bigg|_{x=0} \quad \text{and} \quad \tilde{p}_f^k = \frac{1}{k!} \frac{d^k \tilde{G}_f(x)}{dx^k} \bigg|_{x=0}$$

for each $k = 0, 1, \ldots$.

In view of (37)-(38), we now derive the critical point of the phase transition for arbitrary $0 \leq T_f, T_w \leq 1$. Let $\tilde{k}_f, \tilde{k}_w$ be random variables drawn from the distributions $\{ \tilde{p}_w^k \}$ and $\{ \tilde{p}_f^k \}$, respectively. Furthermore, we let $\tilde{\lambda}_w$ and $\tilde{\lambda}_f$ define the corresponding mean values, while as before we use

$$\tilde{\beta}_f := \frac{< \tilde{k}_f^2 > - \tilde{\lambda}_f}{\tilde{\lambda}_f} \quad \text{and} \quad \tilde{\beta}_w := \frac{< \tilde{k}_w^2 > - \tilde{\lambda}_w}{\tilde{\lambda}_w}. $$

We find

$$\tilde{\lambda}_f = \tilde{G}_f'(1) = T_f \tilde{G}_f'(1) = T_f \tilde{\lambda}_f$$

and

$$< \tilde{k}_f^2 > - \tilde{\lambda}_f = \tilde{G}_w'(1) = T_w \tilde{G}_w'(1) = T_w(\lambda_w)$$

so that

$$\tilde{\beta}_f = T_f \beta_f \tilde{\lambda}_f.$$
and similarly
\[
\mathbb{E}\left[k_f h_1^{k_f-1}\right] = G_f'(h_1) = T_f G_f'(1 + (h_1 - 1)T_f) = T_f \mathbb{E}\left[k_f (1 + T_f(h_1 - 1))^{k_f-1}\right]
\]

In the same manner, we can compute \(\mathbb{E}\left[h_i^{k_i}\right]\) and \(\mathbb{E}\left[k_w h_i^{k_i-1}\right]\) in terms of \(k_w\) and \(T_w\). Reporting these expressions into (35)-(36), we see that, with \(h_1, h_2 \in (0, 1]\) corresponding to the pointwise smallest solution of the recursive equations

\[
h_1 = \frac{1}{\lambda_f} \mathbb{E}\left[k_f (1 + T_f(h_1 - 1))^{k_f-1}\right] \times \mathbb{E}\left[(1 + T_w h_2 - 1)^k_w\right]
\]

\[
h_2 = \frac{1}{\lambda_w} \mathbb{E}\left[\alpha(1 + T_f(h_1 - 1))^{k_f} + 1 - \alpha\right] \times \mathbb{E}\left[k_w (1 + T_w h_2 - 1)^{k_w-1}\right],
\]

we have

\[
\frac{1}{n} C_{\alpha} \left(\mathbb{H}(n; \alpha, \{\tilde{p}_f\}, T_w, \{\tilde{p}_f\}, T_f)\right) \rightarrow 1 - \mathbb{E}\left[\alpha(1 + T_f(h_1 - 1))^{k_f} + 1 - \alpha\right] \times \mathbb{E}\left[(1 + T_w h_2 - 1)^{k_w}\right].
\]

This establishes part (ii) of Theorem 3.2 and the proof is now complete.

VII. CONCLUSION

In this paper, we characterize the critical threshold and the asymptotic size of information epidemics in an overlaying social-physical network. To capture the spread of information, we consider a physical information network that characterizes the face-to-face interactions of human beings, and some overlaying online social networks (e.g., Facebook, Twitter, etc.) that are defined on a subset of the population. Assuming that information is transmitted between individuals according to the SIR model, we show that the critical point and the size of information epidemics on this overlaying social-physical network can be precisely determined by employing the approaches on inhomogeneous random graphs.

To the best of our knowledge, this study marks the first work on the phase transition properties of conjoint networks where the vertex sets are neither identical (as in \(\mathbb{H}, \mathbb{W}\)) nor disjoint (as in \(\mathbb{F}\)). We believe that our findings here shed light on the further studies on information (and influence) propagation across social-physical networks.

APPENDIX A
INFORMATION DIFFUSION WITH MULTIPLE ONLINE SOCIAL NETWORKS

So far, we have assumed that information diffuses amongst human beings via only a physical information network \(\mathbb{W}\) and an online social network \(\mathbb{F}\). To be more general, one can extend this model to the case where there are multiple online social networks. For instance assume that there is an additional online social network, say Twitter, denoted by \(\mathbb{T}(n; \alpha_T)\) whose members are selected by picking each node \(1, \ldots, n\) independently with probability \(\alpha_T\). In other words, with \(\mathcal{N}_T\) denoting the set vertices of \(\mathbb{T}\), we have

\[
P[i \in \mathcal{N}_T] = \alpha_T, \quad i = 1, \ldots, n.
\]

To be consistent with this notation, we assume that the members of the online social network \(\mathbb{F}\) (i.e., Facebook) are selected by picking each node \(1, \ldots, n\) independently with probability \(\alpha_F\).

The overlaying social-physical network now consists of a network formed by conjoining \(\mathbb{W}, \mathbb{F}\) and \(\mathbb{T}\); i.e., we have \(\mathbb{H} = \mathbb{W} \cup \mathbb{F} \cup \mathbb{T}\). Using the techniques presented in Section V-A and Section VI-A one can analyze the phase transition properties of \(\mathbb{H}\) and determine the critical threshold as well as the size of an information epidemic in this overlaying social-physical network. Since both techniques admit such a generalization, the phase transition analysis can be carried out in both of the cases considered in Section \(\mathbb{H}\) i.e., when the networks \(\mathbb{W}, \mathbb{F}\) and \(\mathbb{T}\) are ER graphs or when they are random graphs with arbitrary degree distributions.

To demonstrate the applicability of the techniques for this general set-up, we now consider a simple example where \(\mathbb{W}, \mathbb{F}\) and \(\mathbb{T}\) are all ER graphs with edge probabilities given by \(\lambda_w, \lambda_f, \lambda_t\) respectively. This yields an asymptotic mean degree of \(\lambda_w, \lambda_f, \lambda_t\) in the networks \(\mathbb{W}, \mathbb{F}\) and \(\mathbb{T}\), respectively. For the time being, assume that the transmissibilities \(T_w, T_f\) and \(T_t\) are all equal to one.

Now, recall the concept of inhomogeneous random graphs presented in Section V-A. Let \(S = \{1, 2, 3, 4\}\) stand for the space of vertex types, where vertices with Facebook and Twitter membership are referred to as type 1, vertices with Facebook membership but without Twitter membership are referred to as type 2, vertices with Twitter membership but without Facebook membership are referred to as type 3, and finally vertices with neither Facebook nor Twitter membership are said to be of type 4. That is, we set

\[
x_i = \begin{cases} 1 & \text{if } i \in \mathcal{N}_F\text{ and } i \in \mathcal{N}_T \\ 2 & \text{if } i \in \mathcal{N}_F\text{ and } i \notin \mathcal{N}_T \\ 3 & \text{if } i \notin \mathcal{N}_F\text{ and } i \in \mathcal{N}_T \\ 4 & \text{if } i \notin \mathcal{N}_F\text{ and } i \notin \mathcal{N}_T \end{cases}
\]

for each \(i = 1, \ldots, n\). Assume that the metric space \(S\) is equipped with a probability measure \(\mu\) that satisfies condition \(\text{\ref{eq:condition}}\); i.e., \(\mu_1 = \alpha_f \alpha_t, \mu_2 = \alpha_f(1 - \alpha_t), \mu_3 = (1 - \alpha_f) \alpha_t, \text{ and } \mu_4 = (1 - \alpha_f)(1 - \alpha_t)\). The next step is to compute the appropriate kernel \(\kappa\) such that, for each \(i, j = \{1, 2, 3, 4\}, \kappa(i, j)/n\) gives the probability that two vertices of type \(i\) and \(j\) are connected. For \(n\) large, it is not difficult to see that we have

\[
\kappa = \begin{bmatrix} \lambda_w + \frac{\lambda_f}{\alpha_f} + \frac{\lambda_t}{\alpha_t} & \lambda_w & \lambda_w + \frac{\lambda_f}{\alpha_f} & \lambda_w \\ \lambda_w + \frac{\lambda_f}{\alpha_f} & \lambda_w & \lambda_w & \lambda_w \\ \lambda_w + \frac{\lambda_t}{\alpha_t} & \lambda_w & \lambda_w & \lambda_w \\ \lambda_w & \lambda_w & \lambda_w & \lambda_w \end{bmatrix}
\]
The matrix $M(i, j) = \kappa(i, j)\mu_j$ is now given by

$$M = \begin{bmatrix}
\lambda_w \alpha f \alpha t + \lambda_f \alpha t + \lambda_t \alpha f & \lambda_w \alpha f \alpha t + \lambda_f \alpha t \\
\lambda_w \alpha f \alpha t + \lambda_f \alpha t & \lambda_w \alpha f \alpha t + \lambda_f \alpha t \\
\lambda_w \alpha f \alpha t + \lambda_f \alpha t & \lambda_w \alpha f \alpha t + \lambda_f \alpha t \\
\lambda_w \alpha f \alpha t & \lambda_w \alpha f \alpha t \\
\lambda_w \alpha f \alpha t & \lambda_w \alpha f \alpha t \\
\lambda_w \alpha f \alpha t & \lambda_w \alpha f \alpha t \\
\lambda_w \alpha f \alpha t & \lambda_w \alpha f \alpha t \\
\lambda_w \alpha f \alpha t & \lambda_w \alpha f \alpha t
\end{bmatrix} \quad (A.1)$$

and the critical point of the phase transition as well as the giant component size of $\mathbb{H}(n; \alpha_f, \alpha_t, \lambda_w, \lambda_f, \lambda_t)$ can now be obtained by using the arguments of Section V-A. An item of information originating from a single node in $\mathbb{H} = \mathbb{W} \cup \mathbb{F} \cup \mathbb{T}$ can reach a positive fraction of the individuals only if the spectral radius of $M$ is greater than unity. If it is the case that $\sigma(M) < 1$, then there is no information epidemic and all information outbreaks have size $O(\log n)$.

The fractional size of the giant component (i.e., information epidemic) can also be found. Recalling (17) and (18), we see that

$$\frac{1}{n} C_1(\mathbb{H}(n; \alpha_f, \alpha_t, \lambda_w, \lambda_f, \lambda_t)) \to \alpha_f \rho_1 \alpha_t + \alpha_f (1 - \alpha_t) \rho_2 + (1 - \alpha_f) \alpha_t \rho_3 + (1 - \alpha_f) (1 - \alpha_t) \rho_4,$$  \quad (A.2)

where $0 \leq \rho_1, \rho_2, \rho_3, \rho_4 \leq 1$ are given by the largest solution to the recursive relations

$$\rho_i = 1 - \exp \left\{-\sum_{j=1}^{4} M(i, j)\rho_j \right\}, \quad i = 1, 2, 3, 4. \quad (A.3)$$

In the case where there is no constraint on the transmissibilities $T_w, T_f$ and $T_t$, the conclusions (A.1), (A.2) and (A.3) still apply if we substitute $T_w \lambda_w$ for $\lambda_w$, $T_f \lambda_f$ for $\lambda_f$ and $T_t \lambda_t$ for $\lambda_t$.

APPENDIX B

ONLINE SOCIAL NETWORKS WITH $o(n)$ NODES

Until now, we have assumed that apart from the physical network $\mathbb{W}$ on $n$ nodes, information can spread over a number of online social networks each of which has $\Omega(n)$ members. However, one may also wonder as to what would happen if the number of nodes in these online networks is a sub-linear fraction of $n$. For instance, consider an online social network $\mathbb{F}$ whose vertices are selected by picking each node $1, \ldots, n$ with probability $n^{\gamma-1}$ where $0 < \gamma < 1$. This would yield a vertex set $\mathcal{N}_F$ that satisfies

$$|\mathcal{N}_F| \leq n^\gamma (1 + \epsilon) \quad (B.4)$$

with high probability for any $\epsilon > 0$. We now show that, asymptotically, social networks with $n^\gamma$ nodes have almost no effect in spreading information. We start by establishing an upper bound on the size of the giant component in $\mathbb{H} = \mathbb{W} \cup \mathbb{F}$.

**Proposition B.1**: Let $\mathbb{W}$ be a graph on vertices $1, \ldots, n$, and $\mathbb{F}$ be a graph on the vertex set $\mathcal{N}_F \subset \{1, \ldots, n\}$. With $\mathbb{H} = \mathbb{W} \cup \mathbb{F}$, we have

$$C_1(\mathbb{H}) \leq C_1(\mathbb{W}) + C_2(\mathbb{W}) (|\mathcal{N}_F| - 1), \quad (B.5)$$

where $C_1(\mathbb{W})$ and $C_2(\mathbb{W})$ are sizes of the first and second largest components of $\mathbb{W}$, respectively.

**Proof**: It is clear that $C_1(\mathbb{H})$ will take its largest value if $\mathbb{F}$ is a fully connected graph; i.e., a graph with edges between every pair of vertices. In that case the largest component of $\mathbb{H}$ can be obtained by taking a union of the largest components of $\mathbb{W}$ that can be reached from the nodes in $\mathcal{N}_F$. With $C_1(\mathbb{W})$ denoting set of nodes in the largest component of $\mathbb{W}$ that can be reached from node $i$, we have

$$C_1(\mathbb{H}) = \left| \bigcup_{i \in \mathcal{N}_F} C_1(\mathbb{W}) \right| \leq C_1(\mathbb{W}) + C_2(\mathbb{W}) + \ldots + C_{|\mathcal{N}_F|}(\mathbb{W}) \quad (B.6)$$

where $C_j(\mathbb{W})$ stands for the $j$th largest component of $\mathbb{W}$. The inequality (B.6) is easy to see once we write

$$\left| \bigcup_{i \in \mathcal{N}_F} C_1^{\gamma(i)}(\mathbb{W}) \right| = \sum_{i=1}^{\gamma} C_n^{\gamma(i)}(\mathbb{W}) - \sum_{j=1}^{\gamma} C_n^{\gamma(j)}(\mathbb{W})$$

where $C_n^{\gamma}(i)$ is the $i$th element of the desired conclusion (B.5) is now immediate as we note that $C_2(\mathbb{W}) \geq C_j(\mathbb{W})$ for all $j = 3, \ldots, |\mathcal{N}_F|$, and classical results for ER graphs.

**Corollary B.1**: Let $\mathbb{W}$ be an ER graph on the vertices $1, \ldots, n$ and let $\mathbb{F}$ be a graph whose vertex set $\mathcal{N}_F$ satisfies whp. The following hold for $\mathbb{H} = \mathbb{W} \cup \mathbb{F}$:

(i) If $\mathbb{W}$ is in the subcritical regime (i.e., if $C_1(\mathbb{W}) = o(n)$), then whp we have $C_1(\mathbb{H}) = o(n)$.

(ii) If $C_1(\mathbb{W}) = \Theta(n)$, then we have

$$C_1(\mathbb{H}) = (1 + o(1)) C_1(\mathbb{W}).$$

**Proof**: It is known that for an ER graph $\mathbb{W}$, it either holds that $C_1(\mathbb{W}) = O(\log n)$ (subcritical regime) or it is the case that $C_1(\mathbb{W}) = \Theta(n)$ while $C_2(\mathbb{W}) = O(\log n)$ (supercritical regime). Under condition (B.4), we see from (B.5) that whenever $C_1(\mathbb{W}) = o(n)$ we have $C_1(\mathbb{H}) \leq \gamma \log n \cdot n^{\gamma}$ for some $\gamma > 0$ and part (i) follows immediately. Next, assume that we have $C_1(\mathbb{W}) = \Theta(n)$. The claim (ii) follows from (B.6) as we note that

$$\lim_{n \to \infty} \frac{C_2(\mathbb{W}) \cdot n^{\gamma}}{C_1(\mathbb{W})} = 0$$

since $C_2(\mathbb{W}) = O(\log n)$.

**Corollary B.1** shows that in the case where only a sub-linear fraction of the population use online social networks, an information item originating at a particular node can reach a positive fraction of individuals if and only if information epidemics are already possible in the physical information
network. Moreover, we see from Corollary \ref{B.1} that the fractional size of a possible information epidemic in the joint social-physical network is the same as that of the physical information network alone. Combining these, we conclude that online social networks with $n^\gamma$ \((0 \leq \gamma < 1)\) members have no effect on the (asymptotic) fraction of individuals that can be influenced by an information item in the joint social-physical network.

Analogs of Corollary \ref{B.1} can be obtained for random graphs $\mathbb{W}$ with arbitrary degree distribution \cite{Bollobas}. This time we rely on the results by Molloy and Reed \cite{Molloy}, Theorem 1 who have shown that if there exists some $\epsilon > 0$ such that

$$\max\{d_i, i = 1, \ldots, n\} \leq n^{\frac{1}{2} - \epsilon}$$

(B.7)

then in the supercritical regime (i.e., when $C_1(\mathbb{W}) = \Theta(n)$) we have $C_2(\mathbb{W}) = O(\log n)$. It was also shown \cite{Molloy} that in the subcritical regime of the phase transition, we have $C_1(\mathbb{W}) = \Theta(w(n)^2 \log n)$ whenever

$$\max\{d_i, i = 1, \ldots, n\} \leq w(n)$$

(B.8)

with $w(n) \leq n^{\frac{1}{2} - \epsilon}$ for some $\epsilon > 0$.

Now, consider a graph $\mathbb{F}$ whose vertex set $\mathbb{N}_F$ satisfies (B.4) whp and let $\mathbb{W}$ be a random graph with a given degree distribution \{d_i\}_{i=1}^n satisfying (B.7). In view of (B.5), it is easy to see that

$$C_1(\mathbb{H}) = (1 + o(1))C_1(\mathbb{W})$$

where $\mathbb{H} = \mathbb{W} \cup \mathbb{F}$ and this provides an analog of Corollary \ref{B.1}(i). It is also immediate that in the subcritical regime, we have

$$C_1(\mathbb{H}) = o(n)$$

as long as (B.8) is satisfied and (B.4) holds for some $\gamma < \frac{1}{2}$; this establishes an analog of Corollary \ref{B.1}(i) for graphs with arbitrary degree distributions.

The above result takes a simpler form for classes of random graphs $\mathbb{W}$ studied in Section \ref{X-B}; i.e., random graphs where the degrees follow a power-law distribution with exponential cutoff. In particular, let the degrees of $\mathbb{W}$ be distributed according to (13). It is easy to see that

$$\max\{d_i, i = 1, \ldots, n\} = O(\log n)$$

with high probability so that conditions (B.7) and (B.8) are readily satisfied. For the latter condition, it suffices to take $w_n = O(\log n)$ so that in the subcritical regime, we have

$$C_1(\mathbb{W}) = O\left((\log n)^3\right).$$

The next corollary is now an immediate consequence of Proposition \ref{B.7}.

Corollary B.2: Let $\mathbb{W}$ be a random graph whose degrees follow the distribution specified in (13) and let $\mathbb{F}$ be a graph whose vertex set $\mathbb{N}_F$ satisfies (B.4) whp. The following hold for $\mathbb{H} = \mathbb{W} \cup \mathbb{F}$:

(i) If $C_1(\mathbb{W}) = o(n)$, then whp we have $C_1(\mathbb{H}) = o(n)$.

(ii) If $C_1(\mathbb{W}) = \Theta(n)$, then we have $C_1(\mathbb{H}) = (1 + o(1))C_1(\mathbb{W})$.

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