Scaling of Fractional Delay Filters Based on the Farrow Structure

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Abstract—In this work we consider scaling of fractional delay filters using the Farrow structure. Based on the observation that the subfilters approximate the Taylor expansion of a differentiator, we derive estimates of the $L_2$-norm scaling values at the outputs of each subfilter as well as at the inputs of each delay multiplier. The scaling values can then be used to derive suitable wordlengths in a fixed-point implementation.

I. INTRODUCTION

Many applications require the use of digital filters having adjustable (variable) frequency responses [1]. This paper considers adjustable fractional delay (FD) filters which are required in several contexts [2]–[4]. One example is sampling rate conversion by arbitrary conversion factors where the traditional interpolators and decimators used for integer- and rational-factor conversion fail or imply very high interpolation and decimation factors [3]. An efficient structure for adjustable FD filtering is the so called Farrow structure [3], [5], [6] for which the impulse response values are expressed as polynomials in the delay parameter. This means that the corresponding filter structure is composed of a number of fixed linear-phase FIR subfilters and only one variable delay parameter as seen in Fig. 1. The implementation complexity is therefore much lower for Farrow-based filtering methods than for methods based on either on-line design or storage of a (large) amount of different impulse responses.

During the past decade, numerous papers have appeared that primarily have considered the design of the Farrow filter. In this paper, we are concerned with scaling of the Farrow structure which is important when implementing the filter using fixed-point arithmetic. It appears that scaling issues have not been addressed before in the literature. Specifically, we will in this paper compute the required scaling for the outputs of each subfilter as well as the intermediate nodes before the delay multipliers.

II. ADJUSTABLE FD FIR FILTERS AND THE FARROW STRUCTURE

Let the desired frequency response, $H_{\text{des}}(e^{j\omega T})$, of an adjustable FD filter be

$$H_{\text{des}}(e^{j\omega T}, b) = e^{-j\omega T(b+b)}, \quad |\omega T| \leq \omega_c T < \pi$$

(1)

where $D$ and $b$ are fixed and adjustable real-valued constants, respectively. In this paper, it is assumed that $D$ is either an integer, or an integer plus a half, whereas $b$ takes on values in the interval $[-1/2, 1/2]$. In this way, a whole sampling interval is covered by $b$, and the fractional delay equals $b(b+0.5)$ when $D$ is an integer (an integer plus a half).

The ideal filter response in (1) can be approximated using the Farrow structure depicted in Fig. 1 [5]. The transfer function, $H(z, b)$, of this structure is

$$H(z, b) = \sum_{k=0}^{L} b^k H_k(z)$$

(2)

where $H_k(z)$ are fixed finite-length impulse response (FIR) subfilters approximating $k$:th-order differentiators with frequency responses

$$e^{-j\omega TD} (-j\omega T)^k$$

(3)

which follows from a Taylor series expansion of (1). A filter with a transfer function in the form of (2) can approximate the ideal response in (1) as close as desired by choosing $L$ and designing the subfilters appropriately [3], [6]. Furthermore, it is possible and efficient (due to coefficient symmetry) to impose linear-phase constraints on the subfilters. Hence, we assume that each $H_k(z)$ is an $N$:th-order linear-phase FIR filter with either symmetric (for even $k$) or anti-symmetric (for odd $k$) impulse response, i.e., $h_k(n) = h_k(N - n)$ and $h_k(n) = -h_k(N - n)$, $n = 0, 1, ..., N$, for even and odd values of $k$, respectively. This means that the fixed delay $D$ becomes $D = N/2$ and that the subfilters for an even $N$ are linear-phase FIR filters of Type I and Type III [7] for even and odd values of $k$, respectively. For an odd $N$, the filters are instead alternatingly of Type II and Type IV. Moreover, it has been demonstrated that it is beneficial, in terms of implementation complexity, to make use of subfilters of unequal effective orders instead of equal orders [6]. To keep the notation simple, this is here taken care of by assuming that $N = \max\{N_k\}$, where $N_k$ denote the effective subfilter

Fig. 1. Farrow structure
orders, and introducing additional delays to make all branches have the same delay. This corresponds to appending zeros in the beginning and the end of the impulse response. It is also noted that one can impose relations between the coefficients of the different subfilters in order to reduce the complexity of the Farrow structure even further [8]. However, this does not change the main ideas dealt with in this paper. Even with such constraints imposed, the branches still approximate the ideal differentiators mentioned above. Hence, the analysis provided in this paper still applies.

III. SCALING OF THE FARROW STRUCTURE FOR FD FIR FILTERS

The motivation for scaling is to adjust the signal values such that they utilize the available numerical range as good as possible. An alternative interpretation is to determine the required wordlength for each computation such that overflows are avoided. In this work we will primarily consider this second interpretation. However, the derived results can straightforwardly be applied for “traditional” scaling. A properly scaled filter is crucial if the implementation complexities should be measured beyond the number of multiplications and additions.

What is determined in the scaling process is the number of integer bits required for representation of the data. Without loss of generality we will assume that the input is bounded as \(|x(n)| \leq 1\). Furthermore, we will assume that two’s complement arithmetic is used. Hence, it is only required to scale the inputs to multipliers. By determining the maximum value of a node, say \(q\), the number of integer bits in addition to the sign bit is determined by \(W_f = \lceil \log_2 q \rceil\).

Here, the \(L_2\)-norm is used to estimate the signal values. While, this may lead to rare overflows, using worst-case (or safe) scaling will in general be too pessimistic.

A. Scaling the output values of the subfilters

As already discussed, Taylor series is employed to transform the (1) in form that provide the subfilters of Farrow structure, with each finite-length FIR subfilter approximating a differentiator of some order

\[
e^{-j\omega T(D+b)} = \sum_{i=0}^{\infty} \frac{(-b)^i(j\omega T)^i e^{-j\omega TD}}{i!}
\]

(4)

Since \(b\) is variable or fractional number in the range \([-0.5, 0.5]\), the power function \(b^i\) approaches zero when \(i\) becomes large. Thus, by truncating the higher order terms, the fractional delay filter can be approximated by a finite number of terms \((L+1)\) as

\[
e^{-j\omega T(D+b)} \approx \sum_{i=0}^{L} b^i H_i(\omega T)
\]

(5)

where

\[
H_i(\omega T) = \frac{(-1)^i(j\omega T)^i e^{-j\omega TD}}{i!}
\]

(6)

To approximate the output values of each subfilter \(H_i(\omega T)\) in a Farrow structure, we can calculate different norms at the output of each subfilter.

Generally speaking, for a sequence \(x(n)\), with Fourier transform \(X(e^{j\omega T})\), the \(L_p\)-norm is defined as

\[
\|X(e^{j\omega T})\|_p = \left( \frac{1}{2\pi} \int_{-\pi}^{\pi} |X(e^{j\omega T})|^p d(\omega T) \right)^{1/p}
\]

The \(L_2\)-norm of a continuous-time Fourier transform is related to the power contained in the signal. The \(L_2\)-norm of discrete-time Fourier transform has an analogous interpretation. The \(L_2\)-norm for subfilter \(H_i(\omega T)\) is given by

\[
\|H_i(\omega T)\|^2_2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| (-1)^i(j\omega T)^i e^{-j\omega TD} \right|^2 d(\omega T)
\]

(7)

This leads to the \(L_2\)-norm of subfilter \(i\) can be estimated as

\[
\|H_i(\omega T)\|^2_2 = \frac{(\omega_i T)^{2i+1}}{(i!)^2 \sqrt{\pi(2i+1)}}
\]

(8)

In Fig. 2 the resulting values of the \(L_2\)-norm is plotted for three different values of \(\omega_i\). As can be seen the \(L_2\)-norm, and, hence, the estimated maximum output value varies significantly with subfilter number. Quite expectedly, the estimate decreases with decreasing bandwidth. From this it can be seen that the outputs from the first few subfilters require more than one integer bit to be represented, while for higher order subfilter, the number of integer bits will be negative, i.e., the signals can be represented without the most significant fractional bits.

B. Scaling in the polynomial evaluation

We will also need to scale the inputs to the delay multipliers. Hence, the transfer function from the input to these nodes will be used to compute the corresponding \(L_2\)-norms.

Let \(G_M\) denote the transfer function to the input of the first delay multiplier after filter \(H_M(\omega T)\), i.e.,
The $L_2$-norm for $G_M(\omega T)$ can then be written as

$$\|G_M(\omega T)\|_2^2 = \frac{1}{2\pi} \int_{-\omega_T}^{\omega_T} \left| \sum_{i=M}^{L} \left( bT \right)^{i-M} e^{-j\omega T d} \right|^2 d(\omega T)$$

$$= \frac{1}{2\pi} \int_{-\omega_T}^{\omega_T} \left| \sum_{i=M}^{L} \left( bT \right)^{i-M} (-1)^i \right|^2 d(\omega T)$$

$$= \frac{1}{2\pi} \int_{-\omega_T}^{\omega_T} \left| \Re(G_M(\omega T)) + j\Im(G_M(\omega T)) \right|^2 d(\omega T)$$

$$= \frac{1}{2\pi} \int_{-\omega_T}^{\omega_T} \left[ \Re(G_M(\omega T))^2 + \Im(G_M(\omega T))^2 \right] d(\omega T)$$

where $\Re(G_M(\omega T))$ and $\Im(G_M(\omega T))$ are the real and imaginary parts of $G_M(\omega T)$. The real part can be written as

$$\Re(G_M(\omega T))^2 = \sum_{d=Q_1}^{P_1} \frac{(bT)^{2d-M}(1)^{2d}(-1)^d(\omega T)^{2d}}{(2d)!}$$

$$= \sum_{d=Q_1}^{P_1} \frac{(bT)^{2d-M}(1)^{2d}(-1)^d(\omega T)^{2d}}{(2d)!}$$

$$= \sum_{d=Q_1}^{P_1} C_{d\omega}(\omega T)^{2d}$$

where $P_1 = \lfloor L/2 \rfloor$, $Q_1 = \lfloor M/2 \rfloor$, and

$$C_{d\omega} = \frac{(bT)^{2d-M}(1)^d}{(2d)!}$$

The expression in (10) is a multinomial that can be evaluated using the multinomial theorem

$$\left( \sum_{i=1}^{m} x_i \right)^n = \sum_{k_1, k_2, \ldots, k_m \geq 0} \left( \sum_{i=1}^{m} k_i \right)! x_1^{k_1} x_2^{k_2} \cdots x_m^{k_m}$$

where the sum is taken over all combinations of $k_i$ such that $k_1, k_2, \ldots, k_m \geq 0$ and $\sum k_i = n$. For our case we have $n = 2$.

With the help of above multinomial theorem, the contribution from the real part of $G_M(\omega T)$ to the $L_2$-norm can be written as in (13).

For the imaginary part we have

$$\Im(G_M(j\omega))^2 = \left( \sum_{d=Q_2}^{P_2} \frac{(bT)^{2d-M}(1)^d(\omega T)^{2d}}{(2d)!} \right)^2$$

where $P_2 = \lfloor (L + 1)/2 \rfloor$ and $Q_2 = \lfloor (M + 1)/2 \rfloor$.

Following a similar approach we get the contribution from the imaginary part as shown in (15) where

$$C_{d\omega} = \frac{(bT)^{2d-M}(1)^d}{(2d)!}$$

IV. DESIGN EXAMPLE

To show the viability of the derived estimates a fractional-delay Farrow filter with a bandwidth of $0.9\pi$ rad was designed with $L = 6$. The magnitude responses for the different subfilters are shown in Fig. 3 along with the ones obtained using the Taylor expansion in (6). As can be seen there is a close correspondence for the magnitude responses to that of the theoretical differentiator.

Computing the $L_2$-norms for the example filter and using (8) gives the results shown in Table I. It can be seen that the difference is very small for the first subfilters, while it increases for the latter ones. However, in this context a
\[ \left\| \Re(G_M(\omega T)) \right\|_2^2 = \frac{1}{\pi} \sum_{k_{Q_1},k_{Q_1+1},\ldots,k_{P_1}} \frac{2!}{k_{Q_1}!k_{(Q_1+1)}!\ldots k_{P_1}!} \left( \prod_{d=Q_1}^{P_1} C_{d}^{k_d} \right) \frac{(\omega c T)}{\left( 1 + \sum_{d=Q_1}^{P_1} (2d)k_d \right)} \] (13)

\[ \left\| \Im(G_M(j\omega)) \right\|_2^2 = \frac{1}{\pi} \sum_{k_{Q_2},k_{Q_2+1},\ldots,k_{P_2}} \frac{2!}{k_{Q_2}!k_{(Q_2+1)}!\ldots k_{P_2}!} \left( \prod_{d=Q_2}^{P_2} C_{d}^{k_d} \right) \frac{(\omega c T)}{\left( 1 + \sum_{d=Q_2}^{P_2} (2d-1)k_d \right)} \] (15)

TABLE I
ESTIMATED AND COMPUTED L2-NORMS FOR THE SUBFILTERS, RELATIVE ERRORS, AND RESULTING INTEGER BITS.

<table>
<thead>
<tr>
<th>Subfilter</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Estimated</td>
<td>1.4886</td>
<td>1.6959</td>
<td>1.3508</td>
<td>0.8421</td>
<td>0.4307</td>
<td>0.1867</td>
</tr>
<tr>
<td>Simulated</td>
<td>1.5485</td>
<td>1.6971</td>
<td>1.3469</td>
<td>0.8460</td>
<td>0.4027</td>
<td>0.1939</td>
</tr>
<tr>
<td>Error, %</td>
<td>0.039</td>
<td>-0.072</td>
<td>0.288</td>
<td>-0.468</td>
<td>6.969</td>
<td>-3.729</td>
</tr>
<tr>
<td>Int. bits</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>-1</td>
<td>-2</td>
</tr>
</tbody>
</table>

deviation of 7% is still not a major issue as this corresponds to only fractions of a bit.

Finally, the results on scaling the delay multiplier inputs are summarized in Table II. For these computations the worse case value of \( b \), i.e., \( b = 0.5 \) was used. Here, a slightly larger error is visible for some of the nodes. However, it does still not affect the required number of integer bits.

V. CONCLUSION

Based on the observation that the subfilters in a Farrow-based fractional delay filter approximate the Taylor expansion of a differentiator we have derived estimates of the \( L_2 \)-norm scaling values at the outputs of each subfilter as well as at the inputs of each delay multiplier. The scaling values can then be used to derive suitable wordlengths in a fixed-point implementation. The results show a close correspondence between the estimated values and those obtained in an actual filter design.

REFERENCES