Metric-Locating-Dominating Sets in Graphs

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Abstract

If \( u \) and \( v \) are vertices of a graph, then \( d(u, v) \) denotes the distance from \( u \) to \( v \). Let \( S = \{v_1, v_2, \ldots, v_k\} \) be a set of vertices in a connected graph \( G \). For each \( v \in V(G) \), the \( k \)-vector \( c_S(v) \) is defined by \( c_S(v) = (d(v, v_1), d(v, v_2), \ldots, d(v, v_k)) \). A dominating set \( S = \{v_1, v_2, \ldots, v_k\} \) in a connected graph \( G \) is a metric-locating-dominating set, or an MLD-set, if the \( k \)-vectors \( c_S(v) \) for \( v \in V(G) \) are distinct. The metric-location-domination number \( \gamma_M(G) \) of \( G \) is the minimum cardinality of an MLD-set in \( G \). We determine the metric-location-domination number of a tree in terms of its domination number. In particular, we show that \( \gamma(T) = \gamma_M(T) \) if and only if \( T \) contains no vertex that is adjacent to two or more end-vertices. We show that for a tree \( T \) the ratio \( \gamma_L(T)/\gamma_M(T) \) is bounded above by 2, where \( \gamma_L(G) \) is the location-domination number defined by Slater (Dominating and reference sets in graphs, *J. Math. Phys. Sci.* 22 (1988), 445–455). We establish that if \( G \) is a connected graph of order \( n \geq 2 \), then \( \gamma_M(T) = n - 1 \) if and only if \( G = K_{1,n-1} \) or \( G = K_n \).

The connected graphs \( G \) of order \( n \geq 4 \) for which \( \gamma_M(T) = n - 2 \) are characterized in terms of seven families of graphs.

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1 Introduction

Let $S = \{v_1, v_2, \ldots, v_k\}$ be a set of vertices in a connected graph $G$, and let $v \in V(G)$. The $k$-vector (ordered $k$-tuple) $c_S(v)$ of $v$ with respect to $S$ is defined by

$$c_S(v) = (d(v, v_1), d(v, v_2), \ldots, d(v, v_k)),$$

where $d(v, v_i)$ is the distance between $v$ and $v_i$ ($1 \leq i \leq k$). The set $S$ is called a locating set if the $k$-vectors $c_S(v)$, $v \in V(G)$, are distinct. The location number $\text{loc}(G)$ of $G$ is the minimum cardinality of a locating set in $G$. These concepts were studied in [1, 4, 8, 10].

A set $S$ of vertices of a graph $G = (V, E)$ is a dominating set of $G$ if every vertex in $V - S$ is adjacent to a vertex of $S$. The domination number $\gamma(G)$ is the minimum cardinality of a dominating set of $G$. A dominating set of cardinality $\gamma(G)$ is called a $\gamma(G)$-set. Domination and its variations in graphs are now well studied. The literature on this subject has been surveyed and detailed in the two books by Haynes, Hedetniemi, and Slater [5, 6].

In this paper we merge the concepts of a locating set and a dominating set by defining the metric-locating-dominating set, denoted by an MLD-set, in a connected graph $G$ to be a set of vertices of $G$ that is both a dominating set and a locating set in $G$. We define the metric-locating-domination number $\gamma_M(G)$ of $G$ to be the minimum cardinality of an MLD-set in $G$. An MLD-set in $G$ of cardinality $\gamma_M(G)$ is called a $\gamma_M(G)$-set.

Slater [9, 10] defined a locating-dominating set, denoted by an LD-set, in a connected graph $G$ to be a dominating set $D$ of $G$ such that for every two vertices $u$ and $v$ in $V(G) - D$, $N(u) \cap D \neq N(v) \cap D$. The location-dominating number $\gamma_L(G)$ of $G$ is the minimum cardinality of an LD-set for $G$. An LD-set in $G$ of cardinality $\gamma_L(G)$ is called a $\gamma_L(G)$-set. These concepts were studied in [2, 3, 7, 9, 10, 11] and elsewhere. If $N(u) \cap D \neq N(v) \cap D$, then $c_D(u) \neq c_D(v)$. Thus every LD-set is an MLD-set, implying that $\gamma_M(G) \leq \gamma_L(G)$. The graph of Figure 1 illustrates that $\gamma_M(G) \neq \gamma_L(G)$ in general. Note that $\{v_1, v_2, w_1, w_2\}$ is a minimum cardinality MLD-set and that $\{v_1, v_2, w_{11}, w_{12}, w_{21}, w_{22}\}$ is a minimum cardinality LD-set.

2 Preliminary Results

In this section, we present a few preliminary results the proof of which are straightforward and are therefore omitted.

Observation 1 Let $S$ be an MLD-set in a connected graph $G$. If $u$ and $v$ are distinct vertices of $G$ such that $d(u, w) = d(v, w)$ for all $w \in V(G) - \{u, v\}$, then $|S \cap \{u, v\}| \geq 1$. In particular, if $u$ and $v$ are vertices of $G$ such that $N(u) - \{v\} = N(v) - \{u\}$, then $|S \cap \{u, v\}| \geq 1$. 

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Observation 2 If $G$ is a connected graph containing a support vertex $v$, then any MLD-set of $G$ contains all the leaves adjacent to $v$ or all but one of the leaves adjacent to $v$ as well as the vertex $v$.

Observation 3 For every connected graph $G$ of order $n \geq 2$,
\[ \gamma(G) \leq \gamma_M(G) \leq n - 1. \]

3 Trees

Our aim in this section is first to characterize the trees $T$ for which $\gamma(T) = \gamma_M(T)$ and secondly to determine the metric-location-domination number of a tree in terms of its domination number.

Lemma 4 If a tree $T$ contains no strong support vertex, then every dominating set is an MLD-set.

Proof. Let $S$ be a dominating set of $T$. Let $u, v \in V(T) - S$. If $N(u) \cap S \neq N(v) \cap S$, then $c_S(u) \neq c_S(v)$. Suppose, then, that $N(u) \cap S = N(v) \cap S$. Then, since $T$ is a tree, there is a unique vertex $w \in S$ such that $N(u) \cap S = N(v) \cap S = \{w\}$. Since $w$ is not a strong support vertex, at least one of $u$ and $v$ cannot be a leaf. We may assume that $\deg v \geq 2$. Let $x \in N(v) - \{w\}$. Since $N(v) \cap S = \{w\}, x \notin S$. Thus, $x$ is adjacent to a vertex $y \in S$. Since $T$ is a tree, $w \neq y$. Thus, $d(v, y) = 2$, while $d(u, y) = 4$. Thus, once again, $c_S(u) \neq c_S(v)$. □

As an immediate consequence of Observation 3 and Lemma 4 we have the following result.

Corollary 5 If $T$ is a tree that contains no strong support vertex, then $\gamma(T) = \gamma_M(T)$. 

Figure 1: A graph $G$ with $\gamma_L(G) = 6$ and $\gamma_M(G) = 4$. 

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Theorem 6 Let \( T \) be a tree. Then, \( \gamma(T) = \gamma_M(T) \) if and only if \( T \) contains no strong support vertex.

Proof. The sufficiency follows from Corollary 5. Next we consider the necessity. Suppose that \( T \) contains a strong support vertex \( v \). Let \( S \) be an MLD-set of \( G \) of cardinality \( \gamma_M(T) \). Then, by Observation 2, \( S \) contains all the end-vertices adjacent to \( v \) or all but one of the end-vertices adjacent to \( v \) as well as the vertex \( v \). Deleting all the leaves adjacent to \( v \) from the set \( S \) and adding the vertex \( v \) to \( S \), produces a dominating set of \( T \) of cardinality less than that of \( S \), and so \( \gamma(T) < |S| = \gamma_M(T) \). This proves the necessity. \( \square \)

We are now in a position to determine the metric-location-domination number of a tree in terms of its domination number. Recall that the set of strong support vertices in a tree \( T \) is denoted by \( S(T) \). For a tree \( T \), we denote the total number of leaves in \( T \) that are adjacent to a strong support vertex by \( \ell(T) \).

Theorem 7 If \( T \) is a tree, then

\[
\gamma_M(T) = \gamma(T) + \ell(T) - |S(T)|.
\]

Proof. If \( T \) contains no strong support vertex, then, by Corollary 5, \( \gamma(T) = \gamma_M(T) \). Thus, since \( \ell(T) = |S(T)| = 0 \) in this case, \( \gamma_M(T) = \gamma(T) + \ell(T) - |S(T)| \). Hence we may assume that \( |S(T)| \geq 1 \).

We show first that \( \gamma_M(T) \leq \gamma(T) + \ell(T) - |S(T)| \). For each vertex \( v \in S(T) \), let \( L_v \) denote the set of leaves adjacent to \( v \) and let \( v' \in L_v \). Let

\[
T' = T - \sum_{v \in S(T)} (L_v - \{v'\}).
\]

Hence, \( T' \) is the tree obtained from \( T \) by deleting all but one leaf adjacent to each strong support vertex of \( T \). Then, \( T' \) is a tree with no strong support vertex. Let \( S' \) be a \( \gamma(T') \)-set of \( T' \) that contains all the support vertices of \( T' \). By Lemma 4, \( S' \) is an MLD-set of \( T' \) (and a dominating set of \( T \)). Hence,

\[
S' \cup \left( \bigcup_{v \in S(T)} (L_v - \{v'\}) \right)
\]

is an MLD-set of \( T \), and so

\[
\gamma_M(T) \leq |S'| + \sum_{v \in S(T)} (|L_v| - 1) = \gamma(T) + \ell(T) - |S(T)|.
\]
We show next that $\gamma_M(T) \geq \gamma(T) + \ell(T) - |S(T)|$. Let $D$ be an MLD-set of $T$ of cardinality $\gamma_M(T)$ that contains as few leaves as possible. It follows from Observation 2, that for each $v \in S(T)$, $D$ contains all except one leaf adjacent to $v$ as well as the vertex $v$. Let $v'$ be the leaf adjacent to $v$ that does not belong to $D$. Then,

$$D' = D - \left( \bigcup_{v \in S(T)} (L_v - \{v'\}) \right)$$

is a dominating set of $T$. Thus,

$$\gamma(T) \leq |D| - \sum_{v \in S(T)} (|L_v| - 1) = \gamma_M(T) - \ell(T) + |S(T)|.$$

The desired result now follows.

Since the domination number of a tree can be computed in linear time, it follows from Theorem 7 that so too can the metric-location-domination number of a tree be computed in linear time.

4 \hspace{1cm} \gamma_L(G) \text{ versus } \gamma_M(G)

Our aim in this section is to show that the ratio $\gamma_L(G)/\gamma_M(G)$ can be made arbitrarily large for general connected graphs $G$ but is bounded above by 2 when $G$ is a tree.

Since every LD-set is also an MLD-set, $\gamma_M(G) \leq \gamma_L(G)$ for all graphs $G$. However, for graphs in general there is no constant $c$ such that

$$\frac{\gamma_L(G)}{\gamma_M(G)} \leq c.$$

To see this consider the following construction. For $k \geq 2$, take $\ell$ disjoint stars $K_{1,k}$ and subdivide each edge twice. Let these subdivided stars be $T_1, T_2, \ldots, T_\ell$ with centers $w_1, w_2, \ldots, w_\ell$, respectively. Let $v_{i1}, v_{i2}, \ldots, v_{ik}$ be the leaves of $T_i$ ($1 \leq i \leq \ell$). Let $G$ be the graph obtained from $T_1, T_2, \ldots, T_\ell$ by identifying for each $j$ ($1 \leq j \leq k$) the $\ell$ vertices $v_{1j}, v_{2j}, \ldots, v_{\ell j}$ in a new vertex $v_j$ and then adding a new vertex $u_j$ and the edge $u_jv_j$. (For example, when $k = 2$ and $\ell = 3$ the graph $G$ is illustrated in Figure 2.) Then, $\gamma_L(G) = k(\ell + 1)$ (for example, the set

$$\left( \bigcup_{i=1}^\ell N(w_i) \right) \cup \left( \bigcup_{j=1}^k \{v_j\} \right)$$
is a $\gamma_L(G)$-set) and $\gamma_M(G) = k + \ell$ (for example, the set $\{v_1, v_2, \ldots, v_k\} \cup \{w_1, w_2, \ldots, w_\ell\}$ is a $\gamma_M(G)$-set). Thus,

$$\frac{\gamma_L(G)}{\gamma_M(G)} = \frac{k + k/\ell}{1 + k/\ell} \rightarrow k$$

as $\ell \to \infty$. By choosing $k$ sufficiently large, this ratio can be made arbitrarily large. We show, however, that for trees this ratio is bounded by 2.

![Figure 2: A graph $G$ with $\gamma_L(G) = 8$ and $\gamma_M(G) = 5.$](image)

For ease of presentation, we mostly consider rooted trees. For a vertex $v$ in a (rooted) tree $T$, we let $C(v)$ and $D(v)$ denote the set of children and descendants, respectively, of $v$, and we define $D[v] = D(v) \cup \{v\}$. The maximal subtree at $v$ is the subtree of $T$ induced by $D[v]$, and is denoted by $T_v$.

**Lemma 8** If $T$ is a tree that contains no strong support vertex, then $\gamma_L(T) < 2\gamma(T)$.

**Proof.** We proceed by induction on the order $n$ of a tree $T$ with no strong support vertex. If $n = 1$ or $n = 2$, then $\gamma_L(T) = \gamma(T) = 1$, and so the result holds in this case. This establishes the base case.

Suppose that if $T'$ is a tree of order less than $n$, where $n \geq 3$, with no strong support vertex, then $\gamma_L(T') < 2\gamma(T')$, and let $T$ be a tree of order $n$ with no strong support vertex. We now root the tree $T$ at a vertex $r$ and let $u$ be a vertex at maximum distance from $r$. Then, $u$ is a leaf in $T$. Let $v$ be the parent of $u$. Since $T$ has no strong support vertex, $\text{deg} v = 2$. Let $w$ denote the parent of $v$.

Suppose $w$ is adjacent with a leaf $v'$. Let $T' = T - \{u, v\}$. Then, $\gamma(T) = \gamma(T') + 1$ and $\gamma_L(T) \leq \gamma_L(T') + 1$. Applying the inductive hypothesis to the tree $T'$, $\gamma_L(T') < 2\gamma(T')$. Thus, $\gamma_L(T) - 1 \leq \gamma_L(T') < 2\gamma(T') = 2\gamma(T) - 2$, and so $\gamma_L(T) < \gamma_L(T) + 1 < 2\gamma(T)$.

On the other hand, suppose that $w$ is not adjacent to any leaf. Then each child of $w$ has degree 2 and is adjacent to a leaf. Suppose $|C(w)| = k \geq 1$. 

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Let $T' = T - D[w]$. Then, $\gamma(T) = \gamma(T') + k$. Furthermore, any LD-set of $T'$ can be extended to an LD-set of $T$ by adding the set $C(w) \cup \{w\}$, and so $\gamma_L(T) \leq \gamma_L(T') + k + 1$. Applying the inductive hypothesis to the tree $T'$, $\gamma_L(T') < 2\gamma(T')$. Thus, $\gamma_L(T) - k - 1 \leq \gamma_M(T') < 2\gamma(T') = 2\gamma(T) - 2k$, and so $\gamma_L(T) \leq \gamma_L(T) + k - 1 < 2\gamma(T)$ since $k - 1 \geq 0$. This completes the proof of the lemma. \qed

As an immediate consequence of Corollary 5 and Lemma 8, we have the following result.

**Corollary 9** If $T$ is a tree that contains no strong support vertex, then $\gamma_L(T) < 2\gamma_M(T)$.

**Lemma 10** If $T$ is a tree, then $\gamma_L(T) < 2\gamma_M(T)$.

**Proof.** We proceed by induction on the order $n$ of a tree $T$. If $T$ is a star, then $\gamma_L(T) = \gamma_M(T)$ and the result follows. In particular, the result holds for $n = 1, 2, 3$. Suppose thus that diam $T \geq 3$.

Suppose that if $T'$ is a tree of order less than $n$, where $n \geq 4$, then $\gamma_L(T') < 2\gamma_M(T')$, and let $T$ be a tree of order $n$. If $T$ has no strong support vertex, then the result follows from Corollary 9. Suppose thus that $T$ has a strong support vertex $v$. Let $u$ be a leaf adjacent with $v$ and let $T' = T - u$. Then, $\gamma_L(T) = \gamma_L(T') + 1$ and $\gamma_M(T) = \gamma_M(T') + 1$. Applying the inductive hypothesis to the tree $T'$, gives $\gamma_L(T') < 2\gamma_M(T')$. Thus, $\gamma_L(T) - 1 = \gamma_L(T') < 2\gamma_M(T') = 2\gamma_M(T) - 2$, and so $\gamma_L(T) < \gamma_L(T) + 1 < 2\gamma_M(T)$. \qed

That the bound of Corollary 9 is asymptotically best possible may be seen as follows. For $k \geq 2$, let $T$ be the tree obtained from a star $K_{1,k}$ by subdividing every edge three times. Then, $\gamma_L(T) = 2k$ and $\gamma_M(T) = k + 1$. Thus,

$$\frac{\gamma_L(T)}{\gamma_M(T)} = \frac{2}{1 + 1/k} \to 2$$

as $k \to \infty$. As an immediate consequence of Lemma 10, we have the following result.

**Corollary 11** For any tree $T$,

$$\frac{1}{2} \gamma_L(T) < \gamma_M(T) \leq \gamma_L(T).$$

Furthermore, as an immediate consequence of Theorem 7 and Lemma 10, we have the following result.
Corollary 12 For any tree $T$,
$$
\gamma_L(T) < 2(\gamma(T) + \ell(T) - |S(T)|),
$$
where $S(T)$ is the set of strong support vertices of $T$ and $\ell(T)$ is the number of leaves in $T$ that are adjacent to a strong support vertex.

5 Graphs $G$ with $\gamma_M(G) = n - 1$

Our aim in this section is to characterize connected graphs $G$ of order $n \geq 2$ for which $\gamma_M(G) = n - 1$.

Theorem 13 Let $G$ be a connected graph of order $n \geq 2$. Then, $\gamma_M(G) = n - 1$ if and only if $G = K_{1,n-1}$ or $G = K_n$.

Proof. If $G = K_{1,n-1}$, then, by Observation 2, $\gamma_M(G) = n - 1$, while if $G = K_n$, then, by Observation 1, $\gamma_M(G) = n - 1$. This establishes the sufficiency.

To prove the necessity, suppose that $\gamma_M(G) = n - 1$. If $\text{diam } G \geq 3$, then let $u$ and $v$ be two vertices distance at least 3 apart in $G$. Then, $V(G) - \{u,v\}$ is an MLD-set of $G$, and so $\gamma_M(G) \leq n - 2$, a contradiction. Thus, $\text{diam } G \leq 2$.

Suppose that $\delta(G) = 1$. Let $u$ be an end-vertex of $G$ and let $N(u) = \{v\}$. Since $\text{diam } G \leq 2$, $v$ is adjacent to every other vertex of $G$. If two vertices $x$ and $y$ in $N(v)$ are adjacent, then $V(G) - \{u,x\}$ is an MLD-set of $G$, and so $\gamma_M(G) \leq n - 2$, a contradiction. Thus, $N(v)$ is an independent set, i.e., $G = K_{1,n-1}$.

Suppose then that $\delta(G) \geq 2$. If $u$ and $v$ are distinct vertices of $G$ such that $d(u,w) \neq d(v,w)$ for some vertex $w \in V(G) - \{u,v\}$, then $V(G) - \{u,v\}$ is an MLD-set of $G$, and so $\gamma_M(G) \leq n - 2$, a contradiction. Thus, for every two distinct vertices $u$ and $v$ of $G$ we must have $d(u,w) = d(v,w)$ for every vertex $w \in V(G) - \{u,v\}$. If $u$ and $v$ are two vertices that are not adjacent, then for $x \in N(u)$ we have $d(x,v) = 1$, while $d(u,v) = 2$. So again $V(G) - \{u,v\}$ is an MLD-set, a contradiction. Hence, every two vertices of $G$ must be adjacent, i.e., $G = K_n$. \qed

6 Graphs $G$ with $\gamma_M(G) = n - 2$

In order to characterize those connected graphs $G$ of order $n \geq 4$ with $\gamma_M(T) = n - 2$ we begin by defining seven families of graphs.

Let $\mathcal{F}_1$ be the family of double stars $S(m,k)$, $m,k \geq 1$, of order $m + k + 2 \geq 4$, where a double star $S(m,k)$ is obtained by appending $m$ leaves to one of the vertices in a $K_2$ and $k$ leaves to the other vertex of the $K_2$. 


Let $F_2$ be the family of graphs obtained from a complete graph of order at least 3 by appending any positive number of leaves to exactly one of its vertices.

Let $F_3$ be the family of graphs obtained from $K_2 + K_m$, $m \geq 2$, by appending any positive number of leaves to exactly one of the vertices of degree $m$.

Let $F_4$ be the family of graphs obtained from $K_2 + K_m$, $m \geq 2$, by appending any positive number of leaves to exactly one of the vertices of degree $m + 1$.

Let $F_5$ be the family of all complete bipartite graphs $K_{m,k}$ where $m,k \geq 2$.

Let $F_6$ be the family of all complete multipartite graphs $K_m + K_k$ where $m,k \geq 2$.

Let $F_7$ be the family of graphs obtained from a complete graph $K_m$ by deleting $k$ edges incident with some vertex $u$ where $2 \leq k \leq m - 3$.

Let $F = \cup_{i=1}^7 F_i$.

**Theorem 14** Let $G = (V,E)$ be a connected graph of order $n \geq 4$. Then, $\gamma_M(G) = n - 2$ if and only if $G \in F$.

**Proof.** If $G \in F$, then it is a straightforward task to show that $\gamma_M(G) = n - 2$.

Suppose now that $\gamma_M(G) = n - 2$. We begin by showing that $\text{diam} \ G \leq 3$. Suppose there are vertices $u$ and $v$ such that $d(u,v) = 4$. Let $u,x,y,z,v$ be a shortest $u$-$v$ path. Then, $V - \{u,y,v\}$ is an MLD-set of $G$, and so $\gamma_M(G) \leq n - 3$, a contradiction. Thus, $\text{diam} \ G \leq 3$.

We proceed by induction on $n \geq 4$ to show that if $G$ is a connected graph of order $n$ and $\gamma_M(G) = n - 2$, then $G \in F$. If $n = 4$, then $G \cong P_4 \in F_1$ or $G$ is the graph in $F_2$ obtained from $K_4$ by appending a leaf or $G \cong C_4 \in F_5$ or $G \cong K_4 - e \in F_6$ (where $e$ is an edge of $K_4$). Hence, $G \in F$.

Assume for any connected graph $G'$ of order $n'$, where $4 \leq n' < n$ and $\gamma_M(G') = n' - 2$, that $G' \in F$. Suppose $G$ is a connected graph of order $n$ with $\gamma_M(G) = n - 2$. Before proceeding further, we prove the following claim.

**Claim 1** If $\delta(G) = 1$, then $G \in \cup_{i=1}^4 F_i$.

**Proof.** Suppose $\delta(G) = 1$. Let $v$ be a support vertex of $G$ and let $L_v$ be the collection of leaves adjacent to $v$. Let $G' = G - v$ and let $n'$ be its order. By Observation 3, $\gamma_M(G') \leq n' - 1$. By construction, $v$ is not adjacent to any leaf in $G'$.

If $\gamma_M(G') \leq n' - 3$, let $S'$ be a $\gamma_M(G')$-set. Then, $S' \cup L_v$ is an MLD-set of $G$ and hence $\gamma_M(G) \leq n - 3$, a contradiction. So, $\gamma_M(G') = n' - 2$ or $n' - 1$. 

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Suppose $\gamma_M(G') = n' - 1$. Then, by Theorem 13, $G'$ is a star or a complete graph. $G'$ is a star, then $v$ must be a leaf of $G'$ and hence $G \in \mathcal{F}_1$. If $G'$ is a complete graph, then $G \in \mathcal{F}_2$.

Suppose now that $\gamma_M(G') = n' - 2$. By the inductive hypothesis, $G' \in \mathcal{F}$. If $G' \in \mathcal{F}_1$, then $v$ must be a leaf. However, then diam $G = 4$, a contradiction. So, $G' \notin \mathcal{F}_1$.

Suppose $G' \in \mathcal{F}_2$. Let $w$ be the vertex of maximum degree in $G'$ and let $x$ be a non-leaf neighbor of $w$. Then, $v \neq w$. Suppose first that $v$ is a leaf of $G'$. Then, $V - \{v, w, x\}$ is an MLD-set of $G$, and so $\gamma_M(G) \leq n - 3$, a contradiction. Hence, $v$ is a non-leaf adjacent with $w$. We may assume $v \neq x$. Let $v' \in L_v$ and let $w'$ be a leaf adjacent with $w$. Then, $V - \{v', w, x\}$ is an MLD-set of $G$, a contradiction. So, $G' \notin \mathcal{F}_2$.

Suppose $G' \in \mathcal{F}_3$. Let $w$ be the vertex of maximum degree in $G'$, $w'$ a leaf adjacent with $w$, $y$ a vertex of degree 2 adjacent with $w$, and $x$ the vertex not adjacent with $w$ in $G'$. Then, $v \neq w$. If $v$ is a leaf of $G'$ or if $v = x$, then diam $G = 4$, a contradiction. So we may assume $v = y$. Let $v' \in L_v$. Then, $V - \{v', w', x\}$ is an MLD-set of $G$, a contradiction. So, $G' \notin \mathcal{F}_3$.

Suppose $G' \in \mathcal{F}_4$. Let $w$ be the vertex of maximum degree in $G'$, $w'$ a leaf adjacent with $w$, $y$ a vertex of degree 2 adjacent with $w$, and $x$ the vertex of degree exceeding 2 adjacent with $w$ in $G'$. Then, $v \neq w$. Let $v' \in L_v$. If $v$ is a leaf of $G'$, then $V - \{v', x, y\}$ is an MLD-set of $G$, a contradiction. If $v = x$, then $V - \{v', w', y\}$ is an MLD-set of $G$, a contradiction. If $v$ is a vertex of degree 2 in $G'$ adjacent with $w$, then we may assume $v \neq y$. Then, $V - \{v', w', y\}$ is an MLD-set of $G$, a contradiction. So, $G' \notin \mathcal{F}_4$.

Suppose $G' \in \mathcal{F}_5$. Then, $G' \cong K_{m,k}$ where $m, k \geq 2$. Suppose $v$ belongs to the partite set of cardinality $k$ in $G'$. We show first that $k = 2$. If $k > 2$, let $u$ be a vertex in the same partite set as $v$, $v' \in L_u$, and $w$ a vertex adjacent with $v$ in $G'$. Then, $V - \{v', u, w\}$ is an MLD-set of $G$, a contradiction. Hence $k = 2$ and so $G \in \mathcal{F}_3$.

Suppose $G' \in \mathcal{F}_6$. Suppose first that $v$ has degree $n - 1$ in $G$. Let $u$ be a vertex of degree $n' - 1$ adjacent with $v$ in $G'$. Let $w$ be a vertex of degree less than $n' - 1$ in $G'$, and let $v' \in L_v$. Then, $V - \{v', u, w\}$ is an MLD-set of $G$ unless $G'$ has exactly three partite sets, in which case $G \in \mathcal{F}_4$. On the other hand, suppose $v$ has degree less than $n' - 1$ in $G'$. Let $w$ be in the same partite set of $G$ as $v$, $u$ a vertex of degree $n' - 1$ in $G'$, and $v' \in L_u$. Then, $V - \{v', u, w\}$ is an MLD-set of $G$, a contradiction. So, $v$ cannot have degree less than $n' - 1$ in $G'$.

Suppose $G' \in \mathcal{F}_7$. Suppose $v$ is the vertex of degree at most $n' - 3$ in $G'$. Let $v' \in L_v$, $u$ a vertex adjacent to $v$ in $G'$ and $w$ a vertex not adjacent to $v$ in $G'$. Then, $V - \{v', u, w\}$ is an MLD-set of $G$, a contradiction. Suppose $v$ has degree at least $n' - 2$ in $G'$. Let $x$ and $y$ be two nonadjacent vertices
in \( G' \) distinct from \( v \) and let \( v' \in L_v \). Then, \( V - \{v', x, y\} \) is an MLD-set of \( G \), a contradiction. So, \( G' \notin \mathcal{F}_7 \). This completes the proof of Claim 1. \( \square \)

We now return to the proof of Theorem 14. By Claim 1, if \( \delta(G) = 1 \), then \( G \in \mathcal{F} \). Hence we may assume that \( \delta(G) \geq 2 \). Let \( S \) be a maximum independent set of \( G \). Since \( G \) is not a complete graph, \( |S| \geq 2 \).

**Claim 2** If \( G - S \) is a complete graph, then \( G \in \mathcal{F}_6 \cup \mathcal{F}_7 \).

**Proof.** Since \( \delta(G) \geq 2 \), \( V - S \) contains at least two vertices. If all edges between \( S \) and \( V - S \) are present, then \( G \in \mathcal{F}_6 \). Hence we may assume that some vertex \( u \in S \) is nonadjacent to some \( v \in V - S \). Since \( S \) is a maximum independent set, \( v \) is adjacent with some \( w \in S \).

Suppose now that \( z \in S - \{u, w\} \). Since \( \delta(G) \geq 2 \), \( z \) is adjacent with some \( y \neq v \). Then, \( V - \{u, w, y\} \) is an MLD-set of \( G \), a contradiction. Hence, \( |S| = 2 \). Suppose \( w \) is nonadjacent to some vertex \( y \in V - S \). If there is a common neighbor \( z \) of \( u \) and \( w \), then \( V - \{u, w, z\} \) is an MLD-set of \( G \), a contradiction. It follows that \( V - S \) can be partitioned into two sets \( U \) and \( W \) such that \( u \) is adjacent to every vertex of \( U \) and to no vertex of \( W \), while \( w \) is adjacent to every vertex of \( W \) and to no vertex of \( U \). Since \( \delta(G) \geq 2 \), \( |U| \geq 2 \) and \( |V| \geq 2 \). So, \( V - \{v, w, y\} \) is an MLD-set of \( G \), a contradiction. Thus, \( w \) is adjacent with every vertex in \( V - S \). So \( G \in \mathcal{F}_7 \). \( \square \)

By Claim 2, if \( G - S \) is a complete graph, then \( G \in \mathcal{F} \). Hence we may assume that \( G - S \) is not complete. Let \( W \) be a maximum independent set in \( G - S \). Then, \( |W| \geq 2 \).

**Claim 3** If \( V - (S \cup W) \neq \emptyset \), then every vertex in \( V - (S \cup W) \) is adjacent with either every vertex of \( S \) or every vertex of \( W \).

**Proof.** Suppose, to the contrary, that there exists a vertex \( z \in V - (S \cup W) \) that is nonadjacent to some vertex \( u \in S \) and nonadjacent to some vertex \( v \in W \). Since \( S \) and \( W \) are maximum independent sets in \( G \) and \( G - S \), respectively, \( z \) must be adjacent with some vertex \( x \in S \) and some \( y \in W \). Let \( D = V - \{u, v, z\} \). Since \( D \) is not an MLD-set of \( G \), \( c_D(u) = c_D(v) \). So, \( N(u) - \{v\} = N(v) - \{u\} \). Therefore, \( N(v) \cap S \subseteq \{u\} \) and \( N(u) \cap W \subseteq \{z\} \). Since \( S \) is a maximum independent set in \( G \), \( N(v) \cap S = \{u\} \).

If \( |S| \geq 3 \), let \( w \in S - \{u, x\} \). Then, \( V - \{w, u, z\} \) is an MLD-set of \( G \), a contradiction. Therefore, \( |S| = |W| = 2 \).

Let \( w \in V - \{u, v, x, y, z\} \). Since \( S \) is a maximum independent set, \( w \) is adjacent with \( u \) or \( x \). If \( w \) is adjacent with both \( u \) and \( x \), then \( V - \{v, w, z\} \) is an MLD-set of \( G \), a contradiction. Hence either \( uw \in E(G) \) or \( wx \in E(G) \) (but not both). Suppose \( uw \in E(G) \). Then, \( wx \notin E(G) \). Similarly, \( wy \notin E(G) \). Since \( W \) is a maximum independent set in \( G - S \),
vw ∈ E(G). So if a vertex w ∈ V − (S ∪ W) is adjacent with u or v, then \(N(w) \cap (S ∪ W) = \{u, v\}\). Similarly, if a vertex w ∈ V − (S ∪ W) is adjacent with x or y, then \(N(w) \cap (S ∪ W) = \{x, y\}\). Moreover, if \(w_1\) and \(w_2\) are two vertices in \(V - (S ∪ W)\) for which \(N(w_1) \cap (S ∪ W) = \{u, v\}\) and \(N(w_2) \cap (S ∪ W) = \{x, y\}\), then \(w_1w_2 ∉ E(G)\), for otherwise \(V - \{u, y, w_2\}\) is an MLD-set of \(G\). It follows that \(G\) is disconnected, contrary to our assumption that \(G\) is connected. Hence, every vertex in \(V - (S ∪ W)\) is adjacent with either every vertex of \(S\) or every vertex of \(W\). □

Claim 4 \(V = S ∪ W\).

Proof. Suppose, to the contrary, that \(V - (S ∪ W) ≠ \emptyset\). Then, by Claim 3, every vertex in \(V - (S ∪ W)\) is adjacent with every vertex of \(S\) or every vertex of \(W\). By our earlier assumptions, \(|S| ≥ 2\) and \(|W| ≥ 2\). Suppose some vertex \(z ∈ V - (S ∪ W)\) is adjacent with every vertex of \(W\). Let \(x ∈ W\). Since \(S\) is a maximum independent set of \(G\), there is a vertex \(y ∈ S\) adjacent with \(x\). Let \(u ∈ S - \{y\}\). Then, \(V - \{u, x, z\}\) is an MLD-set of \(G\) unless \(N(u) - \{x, z\} = N(z) - \{u, x\}\). In particular, \(N(z) ∩ S ⊆ \{u\}\). Since \(S\) is a maximum independent set of \(G\), we must have \(N(z) ∩ S = \{u\}\). However, then \(V - \{u, x, y\}\) is an MLD-set of \(G\), a contradiction. Similarly, if some vertex of \(V - (S ∪ W)\) is adjacent with every vertex of \(S\), then we obtain a contradiction. It follows that \(V = S ∪ W\). □

We now return to the proof of Theorem 14. By Claim 3, \(V = S ∪ W\). We show now that \(G ∈ F_5\). Suppose that some \(u ∈ S\) is nonadjacent to some \(v ∈ W\). Let \(y\) and \(w\) be two distinct neighbors of \(u\) and let \(z\) be a neighbor of \(w\) distinct from \(u\). Then, \(V - \{v, y, z\}\) is an MLD-set of \(G\), a contradiction. Hence, every vertex of \(S\) is joined to every vertex of \(V - S\), and so \(G ∈ F_5\). This completes the proof of Theorem 14. □

References


