Problem 11392. Proposed by Omran Kouba, Higher Institute for Applied Sciences and Technology, Damascus, Syria. Let the consecutive vertices of a regular convex \( n \)-gon \( P \) be denoted \( A_0, \ldots, A_{n-1} \), in order, and let \( A_n = A_0 \). Let \( M \) be a point such that for \( 0 \leq k < n \) the perpendicular projections of \( M \) onto each line \( A_kA_{k+1} \) lie interior to the segment \((A_k, A_{k+1})\). Let \( B_k \) be the projection of \( M \) onto \( A_kA_{k+1} \). Show that

\[
\sum_{k=0}^{n-1} \text{Area}(\triangle(MA_kB_k)) = \frac{1}{2} \text{Area}(P).
\]

Solution, by the proposer.

For \( 0 \leq k < n \), let the complex number \( z_k \) represent the point \( A_k \), and let \( z \) represent the point \( M \). Without loss of generality, we may assume that 0 represents the centroid of the polygon, and that the length of the side of \( P \) is equal to 1. Then, clearly we have \( z_k = \omega^k z_0 \) with \( \omega = \exp\left(\frac{2\pi i}{n}\right) \), and \( |z_0| |\omega - 1| = 1 \). Now, the number \( \Re\left((z - z_k)(z_{k+1} - z_k)\right) \) represents the vector \( \overrightarrow{A_kB_k} \), and \( z - z_k \) represents the vector \( \overrightarrow{A_kM} \), so the area of \( \triangle(MA_kB_k) \) is given by

\[
\text{Area}(\triangle(MA_kB_k)) = \frac{1}{2} \Im\left((z - z_k)(z_{k+1} - z_k)\right) \Im\left((z - z_k)(z_{k+1} - z_k)\right) = \frac{1}{4} \Im\left((z - z_k)^2 \omega^k (\omega - 1)^2 z_0^2\right).
\]

But,

\[
\sum_{k=0}^{n-1} (z - z_k)^2 \omega^{2k} = z^2 \sum_{k=0}^{n-1} \omega^{2k} - 2z z_0 \sum_{k=0}^{n-1} \omega^{k} + n z_0^2 = z^2 \left(\frac{\omega^{2n} - 1}{\omega^2 - 1}\right) - 2z z_0 \left(\frac{\omega^n - 1}{\omega - 1}\right) + n z_0^2 = n z_0^2,
\]

so that

\[
\sum_{k=0}^{n-1} \text{Area}(\triangle(MA_kB_k)) = \frac{1}{4} \Im\left((\omega - 1)^2 z_0^2 \sum_{k=0}^{n-1} (z - z_k)^2 \omega^{2k}\right) = \frac{n}{4} \Im\left((\omega - 1)^2 z_0^2 \sum_{k=0}^{n-1} \omega^{2k}\right).
\]

In particular, the right hand side of this equality is independant of \( z \), (in other words, it is independant of the position of \( M \).) And clearly, if \( M \) is the centroid \( O \) of \( P \), then the left hand side of the preceding equality is equal to \( \frac{1}{2} \text{Area}(P) \). This ends the proof.

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