Problem 11056. Proposed by John H. Lindsey II, Cambridge, MA. If $I$ is a polynomial of degree at most $n$, and if $I$ is any finite interval then $\sup_{x \in I} |P(x)|$ is at most $2n^2$ times the average on $I$ of $x \mapsto |P(x)|$.

Solution, by Omran Kouba Higher Institute for Applied Sciences And Technology, Damascus, Syria.

We obtain a better estimate than the one in the statement of the problem. That is, if $P$ is a polynomial of degree at most $n$, and if $I$ is any finite interval then $\sup_{x \in I} |P(x)|$ is at most $(n+1)^2$ times the average on $I$ of $x \mapsto |P(x)|$.

We will use Legendre polynomials, defined, for $n \in \mathbb{N}$, by $P_n(X) = \frac{1}{2^n n!} \frac{d^n}{dX^n}((X^2 - 1)^n)$. In fact, we will make use of the following two properties which can be found in [1] for example.

1. Orthogonality: for $(n, m) \in \mathbb{N}^2$, we have $\int_{-1}^1 P_n(x)P_m(x) \, dx = \left\{ \begin{array}{ll} 0 & \text{if } n \neq m \\ \frac{2}{2n+1} & \text{if } n = m \end{array} \right.$

2. For $n \in \mathbb{N}$, we have $\sup_{-1 \leq x \leq 1} |P_n(x)| = P_n(1) = 1$.

First, assume that $I = [-1, 1]$, and let $P$ be a polynomial of degree at most $n$. Using orthogonality we find that

$$P(X) = \sum_{k=0}^n \frac{2k+1}{2} \lambda_k(P) P_k(X), \quad \text{with} \quad \lambda_k(P) = \int_{-1}^1 P(t) P_k(t) \, dt$$

Then, using 2., we have

$$\forall x \in [-1, 1], \quad |P(x)| \leq \sum_{k=0}^n \frac{2k+1}{2} |\lambda_k(P)| |P_k(x)| \leq \sum_{k=0}^n \frac{2k+1}{2} |\lambda_k(P)| \sup_{-1 \leq t \leq 1} |P_k(t)|$$

$$\leq \sum_{k=0}^n \frac{2k+1}{2} |\lambda_k(P)|$$

and using 2. again, we obtain

$$|\lambda_k(P)| \leq \left( \int_{-1}^1 |P(t)| \, dt \right) \cdot \sup_{-1 \leq t \leq 1} |P_k(t)| = \int_{-1}^1 |P(t)| \, dt$$

finally,

$$\forall x \in [-1, 1], \quad |P(x)| \leq \left( \sum_{k=0}^n \frac{2k+1}{2} \right) \int_{-1}^1 |P(t)| \, dt$$

that is

$$\sup_{-1 \leq x \leq 1} |P(x)| \leq (n+1)^2 \cdot \frac{1}{2} \int_{-1}^1 |P(t)| \, dt$$

In general, if $I = [a, b]$, and $P$ is a polynomial of degree $n$, we just apply the preceding result to the polynomial $Q(X) = P \left( \frac{a+b}{2} + \frac{b-a}{2} X \right)$ and we find

$$\sup_{x \in I} |P(x)| \leq (n+1)^2 \cdot \frac{1}{|I|} \int_I |P(t)| \, dt$$

which is the desired result. □

Reference:

My solution was published in AMM Oct 2005 page 752.