DIVISOR ORIENTATIONS OF POWERS OF PATHS AND POWERS OF CYCLES

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Abstract. In this paper, we prove that for any positive integers \( k, n \) with \( k \geq 2 \), the graph \( P_n^k \) is a divisor graph if and only if \( n \leq 2k + 2 \), where \( P_n^k \) is the \( k^{th} \) power of the path \( P_n \). For powers of cycles we show that \( C_n^k \) is a divisor graph when \( n \leq 2k + 2 \), but is not a divisor graph when \( n \geq 2k + \lfloor \frac{k}{2} \rfloor + 3 \), where \( C_n^k \) is the \( k^{th} \) power of the cycle \( C_n \). Moreover, for odd \( n \) with \( 2k + 2 < n < 2k + \lfloor \frac{k}{2} \rfloor + 3 \), we show that the graph \( C_n^k \) is not a divisor graph.

1. Introduction

A graph \( G \) is a \textit{divisor graph} if there is a bijection \( f : V(G) \to S \), for some finite set \( S \) of positive integers such that \( uv \in E(G) \) if and only if \( \gcd(f(u), f(v)) = \min\{f(u), f(v)\} \), that is either \( f(u) \) divides \( f(v) \) or \( f(v) \) divides \( f(u) \). The function \( f \) is called a \textit{divisor labeling} of \( G \).

The length \( g(n) \) of a longest path in the divisor graph whose divisor labeling has range \( \{1, 2, \ldots, n\} \) was studied in [6], [8], and [9]. The concept of divisor graph involving finite nonempty sets of integers rather than positive integers was introduced in [10]. It is shown in [10] that odd cycles of length greater than 3 are not divisor graphs, while even cycles and caterpillars are. Indeed, not only caterpillars, but also all bipartite graphs are divisor graphs as shown in [5]. Divisor graphs do not contain induced odd cycles of length greater than 3, but they may

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contain triangles, for example the complete graphs are divisor graphs, see [5].

Quite different viewpoints of visualizing divisor graphs were introduced in [5] through providing two characterizations of divisor graphs, one in terms of strongly convex digraphs and the other in terms of extreme vertices. The latter one is stated in the next section, for it will be frequently used in this paper. The present paper will investigate which powers of paths and cycles are divisor graphs. A complete characterization is obtained for powers of paths, while for powers of cycles, there are still for each integer $k \geq 8$ few cases not settled yet, namely $C_n^k$ when $n$ is even and $2k + 2 < n < 2k + \lfloor \frac{k}{2} \rfloor + 3$, where $C_n^k$ is the $k^{th}$ power of the cycle $C_n$, they are likely to be not divisor graphs.

For undefined notions and terminology, the reader is referred to [2] and [7].

2. Preliminaries

The following proposition was proved in [5].

**Proposition 1.** Every induced subgraph of a divisor graph is a divisor graph.

If a component of a disconnected graph $G$ is not a divisor graph, then $G$ is not a divisor graph. But if all components of $G$ are divisor graphs, then $G$ is a divisor graph, as explained in the next proposition.

**Proposition 2.** A graph $G$ is a divisor graph if and only if each component of $G$ is a divisor graph.

**Proof.** Let $G_1, G_2, \ldots, G_r$ be the components of $G$, and let $f_i$ be a divisor labeling of $G_i$ for $i = 1, 2, \ldots, r$. Let $p_1, p_2, \ldots, p_r$ be distinct primes none of which appears in any of the labelings $f_i$ for $i = 1, 2, \ldots, r$. Now define $f$ on $V(G)$ as follows: for $i = 1, 2, \ldots, r$, if $x \in V(G_i)$, then $f(x) = p_i f_i(x)$. Clearly $f$ is a divisor labeling of $G$, and hence $G$ is a divisor graph. The converse follows by Proposition 1. □
In a digraph $D$, a \textit{transmitter} is a vertex having indegree 0, a \textit{receiver} is a vertex having outdegree 0, while a vertex $v$ is a \textit{transitive vertex} if it has both positive outdegree and positive indegree such that $(u,w) \in E(D)$ whenever $(u,v)$ and $(v,w) \in E(D)$. We will call an orientation $D$ of a graph $G$ in which every vertex is a transmitter, a receiver, or a transitive vertex, a \textit{divisor orientation} of $G$. The previous two propositions can be directly deduced from the following characterization of divisor graphs given in [5].

\textbf{Theorem 1.} A graph $G$ is a divisor graph if and only if $G$ has a divisor orientation.

A vertex of a digraph $D$ is an \textit{extreme vertex} if it is a transmitter with positive outdegree, a receiver with positive indegree, or a transitive vertex, see [3]. Therefore, a nontrivial connected graph $G$ is a divisor graph if and only if $G$ has an orientation in which every vertex is an extreme vertex. Nontrivial connected divisor graphs where characterized as the graphs of order $n \geq 2$ whose upper orientable hull number equals $n$, see [4].

If $D$ is an orientation of a graph $G$, then every transmitter (receiver) in $D$ is a receiver (transmitter) in the converse of $D$, and every transitive vertex in $D$ is also transitive in the converse of $D$. Thus we have the following result.

\textbf{Theorem 2.} If $D$ is a divisor orientation of a graph $G$, then the converse of $D$ is also a divisor orientation of $G$.

3. Powers of paths

In this section, we characterize which powers of paths are divisor graphs. We start by showing that $P_7^2$ is not a divisor graph.

\textbf{Theorem 3.} Let $P_7$ be the path $12\ldots7$. Then each of the graphs $P_7^2$, $P_7^2 - 12$, $P_7^2 - \{12, 67\}$, $P_7^2 + 17$, $(P_7^2 - 12) + 17$, and $(P_7^2 - \{12, 67\}) + 17$ is not a divisor graph.
Proof. Let $G$ be one of the graphs $P^2_7$, $P^2_7 - 12$, $P^2_7 - \{12, 67\}$, $P^2_7 + 17$, $(P^2_7 - 12) + 17$, or $(P^2_7 - \{12, 67\}) + 17$. Assume to the contrary that $G$ is a divisor graph. Then, by Theorem 1, $G$ has a divisor orientation $D$. In view of Theorem 2, suppose that $(4, 5) \in E(G)$. Then, since $25, 62 \notin E(G)$, we must have $(4, 2), (7, 5) \in E(D)$. Then we get $(3, 5), (4, 6) \in E(D)$, because $73, 62 \notin E(G)$. But also $25, 15, 36 \notin E(G)$, which implies that $(3, 2), (3, 1), (4, 3) \in E(D)$. Now since $(4, 3), (3, 1) \in E(D)$ but $(4, 1) \notin E(D)$, this leads to a contradiction. □

The following theorem determines precisely when a power of a path is a divisor graph.

**Theorem 4.** For any integer $k \geq 2$, the graph $P^k_n$ is a divisor graph if and only if $n \leq 2k + 2$.

**Proof.** Let $P_n$ be the path $12...n$.

**Case 1.** $n > 2k + 2$.

Then $\{1, 2, 3, k + 2, k + 3, k + 4, 2k + 3\}$ induces in $P^k_n$ a graph which is isomorphic to $P^2_7$. Thus, by Proposition 1 and Theorem 3, $P^k_n$ is not a divisor graph.

**Case 2.** $n = 2k + 2$.

Let $D$ be the orientation of $P^k_n$ with $E(D) = A \cup B \cup C$, where

$A = \{(i, j) : 1 \leq i \leq k, i < j \leq k + 1\},$

$B = \{(i, j) : k + 2 \leq i \leq 2k + 1, i < j \leq 2k + 2\},$

and

$C = \{(j, i) : k + 2 \leq j \leq 2k + 1, j - k \leq i \leq k + 1\}.$

Now we will show that $D$ is a divisor orientation of $P^k_n$. Clearly, $1, k + 2$ are transmitters, while $k + 1, 2k + 2$ are receivers. So, let $i \in V(D) - \{1, k + 1, k + 2, 2k + 2\}$ and let $aib$ be a directed path in $D$. Distinguish two subcases.

**Case 2.1:** $1 < i < k + 1$. 
Then $b \in \{i + 1, \ldots, k + 1\}$ and either $a \in \{1, \ldots, i - 1\}$ or $a \in \{k + 2, \ldots, k + i\}$. Thus either $ab \in A \subseteq E(D)$ or $ab \in C \subseteq E(D)$, respectively. Therefore $i$ is a transitive vertex in $D$.

**Case 2.2:** $k + 2 < i < 2k + 2$.

Then $a \in \{k + 2, \ldots, i - 1\}$ and either $b \in \{i - k, \ldots, k + 1\}$ or $b \in \{i + 1, \ldots, 2k + 2\}$. Thus either $ab \in C \subseteq E(D)$ or $ab \in B \subseteq E(D)$, respectively. Therefore $i$ is a transitive vertex in $D$.

Thus $D$ is a divisor orientation of $P^k_n$. Hence $P^k_n$ is a divisor graph.

**Case 3.** $n < 2k + 2$.

Then $P^k_n$ is isomorphic to an induced subgraph of $P^k_{2k + 2}$, and hence by Proposition 1 and Case 2 above, $P^k_n$ is a divisor graph. □

Note that the divisor orientation of $P^k_{2k + 2}$ given in the proof of the previous theorem can be generated from the divisor labeling $f$ of $P^k_{2k + 2}$ defined by:

$$f(i) = \begin{cases} 2i - 13^i & 1 \leq i \leq k + 1 \\ 2i - (k + 1) & k + 2 \leq i \leq 2k + 2 \end{cases},$$

where $P_{2k + 2}$ is the path $12 \cdots (2k + 2)$.

A subgraph $H$ of a graph $G$ is isometric if $d_H(u, v) = d_G(u, v)$ for all $u, v \in V(H)$, see [1]. For any integer $k \geq 2$, if $G$ is a graph with $\text{diam}(G) = d \geq 2k + 2$, then there exist vertices $x_0, x_1, \cdots, x_{d-1}$ and $x_d$ such that $d(x_0, x_d) = d$ and the path $P : x_0x_1\cdots x_d$ is a shortest $x_0$-$x_d$ path. Since $P$ is a shortest path in $G$, it is isometric, and hence every subpath of $P$ is also isometric in $G$. In particular, since $d \geq 2k + 2$, the path $x_0x_1\cdots x_{2k+2}$ is an isometric path of order $2k + 3$ in $G$. Thus $G$ contains an isometric subgraph which is isomorphic to $P_{2k+3}$. Therefore the graph $G^k$ contains an induced subgraph that is isomorphic to $P^k_{2k+3}$, which is not a divisor graph. Thus we have the following result.

**Corollary 1.** For any integer $k \geq 2$, if $G$ is a graph of diameter $d \geq 2k + 2$, then $G^k$ is not a divisor graph.
4. POWERS OF CYCLES

In this section, we investigate which powers of cycles are divisor graphs. In [10], it was shown that any odd cycle of length greater than 3 is not a divisor graph. This fact will be used in the proof of some of the following results.

We start by considering \( n \) modulo \( 2k \).

**Lemma 1.** For any integer \( k \geq 2 \), if \( n > 4k \) and \( n \equiv i \pmod{2k} \), for some \( i \in \{1, 2, \ldots, k\} \), then \( C_n^k \) is not a divisor graph.

**Proof.** Let \( C_n \) be the cycle \( 12 \ldots n1 \). Then the cycle \( C: 1(1+k)(1+2k)\ldots(1+tk)1 \), where \( t = \frac{n-i}{k} \), is an induced cycle of length \( t+1 \) in \( C_n^k \). Clearly, \( t \) is even, so \( C \) is an odd cycle. Since \( n > 4k \), we have \( n-i > 4k-i \geq 4k-k \), thus \( t = \frac{n-i}{k} > 3 \). Hence, \( C \) has length greater than 3. Therefore \( C_n^k \) is not a divisor graph. \( \square \)

**Lemma 2.** For any integer \( k \geq 2 \), if \( n > 2k \) and \( n \equiv i \pmod{2k} \), for some \( i \in \{k+1, k+2, \ldots, 2k\} \), then \( C_n^k \) is not a divisor graph.

**Proof.** Let \( C_n \) be the cycle \( 12 \ldots n1 \), and let \( t = \frac{n-i}{k} \). Then the vertex \( 1+(t+1)k \) is adjacent to 1 in \( C_n^k \), because

\[
d_{C_n}(1 + (t + 1)k, 1) = d_{C_n}(1 + (t + 1)k, n) + 1 = (tk + i) - (1 + tk + k) + 1 = i - k \leq 2k - k = k.
\]

Therefore the cycle \( C: 1(1+k)(2+k)(2+2k)(1+3k)(1+4k)\ldots(1+(t+1)k)1 \) is an induced cycle of length \( t+3 \) in \( C_n^k \), note that \( 1+3k \leq n \) and if \( n \leq 4k \), then \( t+1 = 3 \) and hence \( C \) has length 5. But \( t \) is even, so \( C \) is an odd cycle. Finally, since \( n > 2k \), we have \( n-i > 2k-i \geq 2k-2k = 0 \), thus \( t = \frac{n-i}{k} > 0 \). Therefore, \( C_n^k \) contains the induced odd cycle \( C \) of length greater than 3, hence \( C_n^k \) is not a divisor graph. \( \square \)
Combining the previous two lemmas, we obtain the next corollary.

**Corollary 2.** For any integer \( k \geq 2 \), if \( n \geq 3k + 1 \), then \( C_n^k \) is not a divisor graph.

**Proof.** By Lemma 1 and Lemma 2, \( C_n^k \) is not a divisor graph for each \( n > 4k \). Again by Lemma 2, \( C_n^k \) is not a divisor graph for each \( n \in \{2k + (k + 1), 2k + (k + 2), ..., 2k + (2k)\} = \{3k + 1, ..., 4k\} \), hence the result follows immediately. \( \square \)

To improve the result of the previous corollary, we turn to consider \( n \) modulo \( k + 1 \).

**Lemma 3.** For any integer \( k \geq 2 \), if \( n > 2(k + 1) \) and \( n \equiv i \pmod{k + 1} \), for some \( i \in \left\{\left\lfloor \frac{k}{2} \right\rfloor + 1, \left\lfloor \frac{k}{2} \right\rfloor + 2, ..., k\right\} \), then \( C_n^k \) is not a divisor graph.

**Proof.** Let \( C_n \) be the cycle 12...n1, and let \( t = \frac{n-i}{k+1} \). Then the vertex \( 1 + t(k+1) \) is not adjacent to \( (k+1) - (k-i) \), because

\[
d_{C_n}(1 + t(k+1), (k+1) - (k-i)) = d_{C_n}(1 + t(k+1), n) + 1 \\
+ d_{C_n}(1, (k+1) - (k-i)) \\
= (i-1) + 1 + i \\
= 2i \\
> k.
\]

Then the cycle \( C: 1(1(k+1) - (k-i))(1 + 1(k+1))(2(k+1) - (k-i))(1 + 2(k+1))...t(k+1) - (k-i))(1 + t(k+1)) \) is an induced cycle of length \( 2t+1 \) in \( C_n^k \). Since \( n > 2(k+1) \), we have \( n-i > 2k+2-i \geq 2k+2-k = k+2 \), thus \( t = \frac{n-i}{k+1} > \frac{k+2}{k+1} > 1 \). Hence, \( C \) has length greater than 3. Therefore \( C_n^k \) is not a divisor graph. \( \square \)
Using the previous lemma, we obtain the following improvement of Corollary 2.

**Theorem 5.** For any integer \( k \geq 2 \), if \( n \geq 2k + \lfloor \frac{k}{2} \rfloor + 3 \), then \( C_n^k \) is not a divisor graph.

**Proof.** By Corollary 2, the statement is true for \( n \geq 3k + 1 \). Now by Lemma 3, the statement is true for \( n \in \{2(k + 1) + \lfloor \frac{k}{2} \rfloor, 2(k + 1) + \lfloor \frac{k}{2} \rfloor + 2, ..., 2(k + 1) + k\} = \{2k + \lfloor \frac{k}{2} \rfloor + 3, 2k + \lfloor \frac{k}{2} \rfloor + 4, ..., 3k + 1\} \), hence the result follows immediately. \( \square \)

It is worth mentioning that the result of Theorem 5 can be shown, in view of Theorem 3, by producing an induced subgraph \( H \) of \( C_n^k \) which is isomorphic to one of the two graphs \( P_2^2, P_2^2 + 17 \). Take \( H \) namely to be the subgraph of \( C_n^k \) induced by \( \{1, k + 2 + \lfloor \frac{k}{2} \rfloor + 1\} \) if \( n \) is even and by \( \{1, k + 3 + \lfloor \frac{k}{2} \rfloor + 1\} \) if \( n \) is odd, where in both cases \( C_n \) is the cycle \( 12 \ldots n1 \). This proof is shorter, but it does not include the fact that \( C_n^k \), where \( n \geq 2k + \lfloor \frac{k}{2} \rfloor + 3 \), contains an induced odd cycle of length greater than 3, which is shown in the proofs of Lemmas 1, 2, and 3 above. The following theorem assures that \( 2k + \lfloor \frac{k}{2} \rfloor + 3 \) is the minimum value of \( n \) for which \( C_n^k \) contains an induced odd cycle of length greater than 3.

**Theorem 6.** For any integer \( k \geq 2 \), the graph \( C_n^k \) contains an induced odd cycle of length greater than 3 if and only if \( n \geq 2k + \lfloor \frac{k}{2} \rfloor + 3 \).

**Proof.** If \( n \geq 2k + \lfloor \frac{k}{2} \rfloor + 3 \), then, by the proofs of Lemmas 1, 2, and 3, the graph \( C_n^k \) contains an induced odd cycle of length greater than 3. Suppose \( n < 2k + \lfloor \frac{k}{2} \rfloor + 3 \), and let \( C_n \) be the cycle \( 12 \ldots n1 \). Assume to the contrary that \( C_n^k \) contains an induced odd cycle \( C' \): \( a_1b_1a_2b_2\ldots a_tb_t1 \), for some \( t \geq 2 \). Then \( b_1 \geq k + 2 \), and hence \( b_2 \geq 2k + 3 \). Then, since \( 3k + 4 > 2k + \lfloor \frac{k}{2} \rfloor + 3 > n \), we must have \( t = 2 \). But \( d_{C_n}(b_2, n) < (2k + \lfloor \frac{k}{2} \rfloor + 3) - (2k + 3) = \lfloor \frac{k}{2} \rfloor \), thus \( d_{C_n}(b_2, 1) = d_{C_n}(b_2, n) + 1 \leq \lfloor \frac{k}{2} \rfloor \). Then, since \( a_1b_2 \notin E(C_n^k) \), we have \( a_1 \geq \lfloor \frac{k}{2} \rfloor + 2 \). But also \( a_1a_2 \notin E(C_n^k) \), which implies that \( a_2 \geq \lfloor \frac{k}{2} \rfloor + k + 3 \). Then \( d_{C_n}(a_2, 1) = \)}
\[ d_{C_n}(a_2, n) + 1 < (2k + \lfloor \frac{k}{2} \rfloor + 3) - (\lfloor \frac{k}{2} \rfloor + k + 3) + 1 = k + 1, \] which means that \( 1a_2 \in E(C_k^n) \), a contradiction. \[ \square \]

Although for \( n < 2k + \lfloor \frac{k}{2} \rfloor + 3 \), the graph \( C_k^n \) does not contain an induced odd cycle of length greater than 3, \( C_k^n \) need not be a divisor graph as we will see in the sequel.

Now we consider cases of \( n \) for which \( C_k^n \) is a divisor graph.

**Theorem 7.** For any integer \( k \geq 2 \), if \( n \leq 2k + 2 \), then \( C_k^n \) is a divisor graph.

**Proof.** For \( n < 2k + 2 \), the graph \( C_k^n \) is a complete graph and hence is a divisor graph. Suppose \( n = 2k + 2 \). Then \( C_k^n \) is isomorphic to the graph \( (k + 1)K_2 \), so let \( V(C_k^n) = \{1, \bar{1}, 2, \bar{2}, \ldots, k + 1, k + 1\} \) where \( i\bar{i} \notin E(C_k^n) \) for \( i \in \{1, 2, \ldots, k + 1\} \). Define the orientation \( D \) of \( C_k^n \) as follows: \( E(D) = \{ij : i < j\} \cup \{i\bar{j} : i < j\} \cup \{i\bar{i} : i < j\} \cup \{i\bar{j} : i < j\} \cup \{i\bar{j} : i < j\} \). Clearly, the vertices 1, \( \bar{1} \) are transmitters, while the vertices \( k + 1, k + 1 \) are receivers. Let \( v \in V(C_k^n) - \{1, \bar{1}, k + 1, k + 1\} \). Say \( v = i \), for some \( 1 < i < k + 1 \) (the case \( v = \bar{i} \) is similar). Let \( ab \) be a directed path in \( D \). Then \( a = i_1 \) or \( \bar{i}_1 \), for some \( 1 \leq i_1 < i \), and \( b = i_2 \) or \( \bar{i}_2 \), for some \( i < i_2 < k + 1 \). Thus \( ab \in E(D) \), hence \( v \) is a transitive vertex. Therefore \( D \) is a divisor orientation of \( C_k^n \). \[ \square \]

Next, we will investigate when \( C_k^n \) is a divisor graph for some cases of \( n \) between \( 2k + 2 \) and \( 2k + \lfloor \frac{k}{2} \rfloor + 3 \). We start by the following result.

**Theorem 8.** Let \( C \) be an odd cycle in a graph \( G \) such that for each vertex \( v \in V(C) \), the two neighbors of \( v \) in \( C \) are not adjacent in \( G \). Then \( G \) is not a divisor graph.

**Proof.** Let \( C \) be the cycle \( 12\ldots(2k + 1)1 \). Assume to the contrary that \( G \) is a divisor graph. Let \( D \) be a divisor orientation of \( G \). In view of Theorem 2, we can suppose that \( (1, 2) \in E(D) \). Since \( 13 \notin E(G) \), we must have \( (3, 2) \in E(D) \), and hence 2 is a receiver in the corresponding orientation \( D_C \) of \( C \). By the same argument, we have 4, 6, ..., \( 2k \) are
receivers and $3, 5, \ldots, 2k + 1$ are transmitters in $D_C$, a contradiction since $(2k + 1, 1)$ and $(1, 2) \in E(D)$ but $(2k + 1, 2) \notin E(D)$.

We use the previous theorem to show that for any odd integer $n$ with $n > 2k + 2$, the graph $C_n^k$ is not a divisor graph.

**Theorem 9.** For any integer $k \geq 2$, if $n$ is an odd integer with $n > 2k + 2$, then $C_n^k$ is not a divisor graph.

**Proof.** Since $n$ is odd, the element $a = \lfloor \frac{k}{2} \rfloor + 1$ has an odd order in the cyclic group $\mathbb{Z}_n$. But $3a = 3\lfloor \frac{k}{2} \rfloor + 3 < 2k + 3 \leq n$, which implies that $|a| > 3$, where $|a|$ denotes the order of $a$ in $\mathbb{Z}_n$. But $n \geq 2k + 3$ and $a > \frac{k}{2}$, which implies that for each $1 \leq i \leq |a| - 1$, the two vertices $ia$ and $(i + 2)a$ are not adjacent (note that for $j = 1, 2, \ldots, |a| + 1$, the numbers $ja$ are taken modulo $n$). Thus the cycle $C: (a)(2a)\ldots(|a|a)(a)$ is an odd cycle in $C_n^k$ such that the two neighbors of any vertex of $C$ are not adjacent in $C_n^k$. Therefore, by the previous theorem, $C_n^k$ is not a divisor graph.

By Theorems 7, 5 and 9, we obtain the following characterizations for the cases when $k = 2, 3$.

**Corollary 3.** $C_n^2$ is a divisor graph if and only if $n \leq 6$.

**Corollary 4.** $C_n^3$ is a divisor graph if and only if $n \leq 8$.

For each value of $k \in \{4, 5, 6, 7\}$ there is exactly one missing case for $n$ to obtain a complete characterization of all divisor graphs $C_n^k$. We investigate these cases in the following lemma. Note that for each $k \geq 8$, there are more than one missing case.

**Lemma 4.** Let $k \in \{4, 5, 6, 7\}$. Then $C_{2k+4}^k$ is not a divisor graph.

**Proof.** Let $C_{2k+4}$ be the cycle $12\ldots(2k + 4)1$. Assume to the contrary that $C_{2k+4}^k$ is a divisor graph. By Theorems 1 and 2, let $(1, 2) \in E(D)$, for some divisor orientation $D$ of $C_{2k+4}^k$. Since $1(k + 2) \notin E(C_{2k+4}^k)$,
we must have \((k + 2, 2) \in E(D)\). But also \((k + 2)(2k + 4) \notin E(C_{2k+4}^k)\), which implies that \((2k + 4, 2) \in E(D)\). Since \((2k + 4)(k + 1), (k + 5)2 \notin E(C_{2k+4}^k)\), we get \((k+1, 2), (2k+4, k+5) \in E(D)\). Now since \((k+1, 2) \in E(D)\) but \((k+1)(2k+2) \notin E(C_{2k+4}^k)\), we must have \((2k+2, 2) \in E(D)\). Hence \((k - 1, 2) \in E(D)\), because \((2k + 2)(k - 1) \notin E(C_{2k+4}^k)\). Thus, since \((k + 3)2 \notin E(C_{2k+4}^k)\), we have \((k - 1, k + 3) \in E(D)\). Now we distinguish two cases:

**Case 1.** \(k \in \{4, 5\}\).

Since \((k - 1, k + 3) \in E(D)\) but \((k - 1)(k + 5) \notin E(C_{2k+4}^k)\), we get \((k + 5, k + 3) \in E(D)\), a contradiction since we now have \((2k + 4, k + 5), (k + 5, k + 3) \in E(D)\) but \((2k + 4, k + 3) \notin E(D)\).

**Case 2.** \(k \in \{6, 7\}\).

Since \((k-1, k+3) \in E(D)\) but \((k-1)(2k) \notin E(C_{2k+4}^k)\), we get \((2k, k + 3) \in E(D)\). Then, since \(2(k + 3) \notin E(C_{2k+4}^k)\), we have \((2k, 2) \in E(D)\). Thus, since \((2k)(k - 3) \notin E(C_{2k+4}^k)\), we must have \((k - 3, 2) \in E(D)\). But \((k + 3)2 \notin E(C_{2k+4}^k)\), which implies that \((k - 3, k + 3) \in E(D)\). Then, since \((k - 3)(k + 5) \notin E(C_{2k+4}^k)\), we have \((k + 5, k + 3) \in E(D)\). Now we have \((2k + 4, k + 5), (k + 5, k + 3) \in E(D)\) but \((2k + 4, k + 3) \notin E(D)\), a contradiction.

For \(k \in \{4, 5, 6, 7\}\), the graph \(C_n^k\) is (by Theorem 7) a divisor graph when \(n \leq 2k + 2\), and is (by Theorem 5) not a divisor graph when \(n \geq 2k + \lfloor \frac{k}{2} \rfloor + 3\). By Theorem 9 and Lemma 4, the graph \(C_n^k\) is not a divisor graph when \(2k+2 < n < 2k + \lfloor \frac{k}{2} \rfloor + 3\). So we have the following result.

**Corollary 5.** Let \(k \in \{4, 5, 6, 7\}\). Then \(C_n^k\) is a divisor graph if and only if \(n \leq 2k + 2\).

For an integer \(k \geq 8\), the case when \(n\) is even and \(2k + 3 < n < 2k + \lfloor \frac{k}{2} \rfloor + 3\) is not yet settled. It is shown in Lemma 4 that \(C_{12}^4, C_{14}^5, C_{16}^6\), and \(C_{18}^7\) are not divisor graphs, we conjecture this would be the case also when \(k \geq 8\).
References


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