Quadruple Systems with Independent Neighborhoods

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Abstract

A 4-graph is odd if its vertex set can be partitioned into two sets so that every edge intersects both parts in an odd number of points. Let $b(n) = \max_{\alpha} \left\{ \alpha \left( \frac{n-\alpha}{3} \right) + (n-\alpha) \left( \frac{\alpha}{3} \right) \right\} = \left( \frac{1}{2} + o(1) \right) \left( \frac{n}{4} \right)$ denote the maximum number of edges in an $n$-vertex odd 4-graph. Let $n$ be sufficiently large, and let $G$ be an $n$-vertex 4-graph such that for every triple $xyz$ of vertices, the neighborhood $N(xyz) = \{ w : wxyz \in G \}$ is independent. We prove that the number of edges of $G$ is at most $b(n)$. Equality holds only if $G$ is odd with the maximum number of edges. We also prove that there is $\varepsilon > 0$ such that if a 4-graph $G$ has minimum degree at least $(1/2 - \varepsilon) \binom{n}{3}$, then $G$ is 2-colorable.

Our results can be considered as a generalization of Mantel’s theorem about triangle-free graphs, and we pose a conjecture about $k$-graphs for larger $k$ as well.

1 Introduction

Let $G$ be a $k$-uniform hypergraph ($k$-graph for short). The neighborhood of a vertex subset $S \subset V(G)$ of size $k-1$ is $N_G(S) = \{ v : S \cup \{ v \} \in G \}$ (we associate $G$ with its edge set, and will often omit the subscript $G$). Suppose we impose the restriction that all neighborhoods of $G$ are independent sets (that is, span no edges), and $G$ has $n$ vertices. What is the maximum number of edges that $G$ can have? When $k = 2$, the answer is $\lfloor n^2/4 \rfloor$, achieved by the complete bipartite graph $K_{[n/2],[n/2]}$. This result, due originally to Mantel in 1907, was the first result of extremal graph theory. Recently, the same question was answered for $k = 3$, where the unique

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extremal example (for $n$ large) is obtained by partitioning the vertex set into two parts $X, Y$, where $|X| - 2n/3 < 1$, and taking all triples with two points in $X$. This was proved by Füredi, Pikhurko, and Simonovits [3, 4], and settled a conjecture of Mubayi and Rödl [7].

In this paper, we settle the next case, namely $k = 4$. It is noteworthy that determining exact results for extremal problems about $k$-graphs is in general a hard problem. Consequently, our proof is by no means a straightforward generalization of the corresponding proofs for $k = 2$ and $3$, and at present, we do not see how to generalize it to larger $k$.

Let $F^k$ be the $k$-graph with $k + 1$ edges, $k$ of which share a common vertex set of size $k - 1$, and the last edge contains the remaining vertex from each of the first $k$ edges. Writing $[a, b] = \{a, a + 1, \ldots, b - 1, b\}$ (with $[a, b] = \emptyset$ if $a > b$) and $[n] = \{1, \ldots, n\}$, a formal description is

$$F^k = \{[k + i] \setminus [k, k + i - 1] : 0 \leq i \leq k - 1\} \cup ([2k - 1] \setminus [k - 1]).$$

Note that a $k$-graph contains no copy of $F^k$ (as a not necessarily induced subsystem) if and only if each of its neighborhoods is independent.

Call a 4-graph odd if its vertex set can be partitioned into $X \cup Y$, such that every edge intersects $X$ in an odd number of points. Let $B(n)$ be one of at most two odd 4-graphs on $n$ vertices with the maximum number of edges and let $b(n) = |B(n)|$. Note that the vertex partition of $B(n)$ is not into precisely equal parts, but they have sizes $n/2 - t$ and $n/2 + t$, where, as it follows from routine calculations,

$$|t - \frac{1}{2}\sqrt{3n - 4}| < 1.$$  

It is easy to check that an odd 4-graph has independent neighborhoods, and one might believe that among all $n$-vertex 4-graphs with independent neighborhoods, the odd ones have the most edges. Our first result confirms this for large $n$.

**Theorem 1.1 (Exact Result)** Let $n$ be sufficiently large, and let $G$ be an $n$-vertex 4-graph with all neighborhoods being independent sets. Then $|G| \leq b(n)$, and if equality holds, then $G = B(n)$.

We also prove an approximate structure theorem, which states that if $G$ has close to $b(n)$ edges, then the structure of $G$ is close to $B(n)$.

**Theorem 1.2 (Global Stability)** For every $\delta > 0$, there exists $n_0$ such that the following holds for all $n > n_0$. Let $G$ be an $n$-vertex 4-graph with independent neighborhoods, and $|G| > (1/2 - \varepsilon)\binom{n}{4}$, where $\varepsilon = \delta^2/108$. Then $G$ can be made odd by removing at most $\delta\binom{n}{4}$ edges.

One might suspect that Theorem 1.2 can be taken further, by showing that if $G$ has minimum degree at least $(1/2 - \gamma)\binom{n}{3}$ for some $\gamma > 0$, then $G$ is already odd. Such phenomena hold for $k = 2$ and $3$. For example, when $k = 2$, a special case of the theorem of Andrásfai, Erdős, and
Sós [1] states that a triangle-free graph with minimum degree greater than \(2n/5\) is bipartite. For \(k = 3\), a similar result was proved in [4]. The analogous statement is not true for \(k = 4\). Indeed, one can add an edge \(E\) to \(B(n)\) that intersects each part in two vertices, and then delete all edges of \(B(n)\) that intersect \(E\) in three vertices. The resulting 4-graph has independent neighborhoods, and yet its minimum degree is \((1/2)(n^3) - O(n^{5/2})\). Nevertheless, a slightly weaker statement is true. Let us call a \(k\)-graph 2-colorable if its vertex set can be partitioned into two independent sets.

**Theorem 1.3** Let \(G\) be an \(n\)-vertex 4-graph with independent neighborhoods. There exists \(\varepsilon > 0\) such that if \(n\) is sufficiently large and \(G\) has minimum degree greater than \((1/2 - \varepsilon)(n^3)\), then \(G\) is 2-colorable.

Call a \(k\)-graph odd if it has a vertex partition \(X \cup Y\), and all edges intersect \(X\) in an odd number of points less than \(k\). Let \(B^k(n)\) be an odd \(k\)-graph with the maximum number of edges (this may not be unique).

**Conjecture 1.4** Let \(n\) be sufficiently large and let \(G\) be an \(n\)-vertex \(k\)-graph with independent neighborhoods. Then \(|G| \leq |B^k(n)|\), and if equality holds, then \(G = B^k(n)\).

## 2 Asymptotic Result and Stability

In this section we prove Theorem 1.2. Before doing so we first prove an asymptotic result and a stability result under the assumption of large minimum degree.

Let \(\text{ex}(n, F^4)\) denote the maximum number of edges in an \(n\)-vertex 4-graph containing no copy of \(F^4\). The results of Katona, Nemetz, and Simonovits [5] imply that \(\lim_{n \to \infty} \text{ex}(n, F^4)/\binom{n}{3}\) exists. Let the Turán density \(\pi(F^4)\) be the value of the limit. We need the following standard lemma.

**Lemma 2.1** (See Frankl and Füredi [2]) Let \(F\) be a \(k\)-graph with the property that every pair of its vertices lies in an edge. Then

\[
\pi(F) \binom{n}{k} \leq \text{ex}(n, F) \leq \pi(F) \frac{n^k}{k!}.
\]

Observe that \(F^4\) satisfies the hypothesis of Lemma 2.1. Write \(d_{\min}(G)\) for the minimum vertex degree in \(G\).

**Theorem 2.2** (Asymptotic Result and Minimum Degree Stability) For every \(\delta > 0\), there exists \(n_1\) such that the following holds for all \(n > n_1\). Let \(G\) be an \(n\)-vertex 4-graph with independent neighborhoods and \(d_{\min}(G) > (\pi(F^4) - \delta/24)(n^3)\). Then \(G\) can be made odd by deleting at most \(\delta(n^4)\) edges. Also, \(\pi(F^4) = 1/2\).
Proof. Suppose $\delta > 0$ is given, and set $\gamma = \delta/24 < 1/24$. Let $\pi = \pi(F^4)$. Note that $B(n)$ shows that $\pi \geq 1/2$. Let $A$ be a maximum size neighborhood in $G$. By hypothesis, $A$ is an independent set. Put $B = V \setminus A$, and $\mu = |A|$. Since $d_{\min}(G) > (\pi - \gamma)(\binom{n}{3})$, we have $|G| > (\pi - \gamma)(\binom{n}{3})(n/4)$, and therefore $\mu > (\pi - \gamma)n$. Let $H_i$ be the set of edges in $G$ with precisely $i$ vertices in $B$, and $h_i = |H_i|$. Observe that $h_0 = 0$ since $A$ is an independent set. Recalling that $|G| \leq \pi n^4/24$ by Lemma 2.1, we have

$$\sum_{i=1}^{4} i \cdot h_i = \sum_{x \in B} \deg(x) = 4|G| - \sum_{x \in A} \deg(x) < 3|G| + \pi \frac{n^4}{24} - \mu(\pi - \gamma) \binom{n}{3}.$$  \hspace{1cm} (1)

Let $\sum_{AAB}$ denote the summation of $|N_G(S)|$ over all sets $S = \{u, v, w\}$, with $u, v \in A$ and $w \in B$. By the definition of $A$, each of these terms is at most $\mu$. Consequently,

$$3h_1 + 2h_2 = \sum_{AAB} \leq \mu(n - \mu) \binom{\mu}{2}. \hspace{1cm} (2)$$

Now we add (1) and 2/3 times (2). Using $|G| = \sum_{i=1}^{4} h_i$, we obtain

$$\frac{h_2}{3} + h_4 < \gamma \mu \frac{n^3}{6} + \frac{1}{3} \mu^3(n - \mu) + \frac{\pi}{24} (n - 4\mu)n^3 + O(n^2).$$

The right hand side simplifies to

$$\gamma \mu \frac{n^3}{6} + \frac{1}{48}(2\mu + n)(n - 2\mu)^3 + \frac{\pi - 1/2}{24} (n - 4\mu)n^3 + O(n^2).$$

Since $2n > 2\mu > 2(\pi - \gamma)n \geq (1 - 2\gamma)n$, the second summand above is at most $(\gamma^3/2)n^4$. If $\pi \geq 1/2 + 3\gamma$, then $\mu > n/2$ and

$$\gamma \mu \frac{n^3}{6} + \frac{\pi - 1/2}{24} (n - 4\mu)n^3 \leq -\frac{\gamma n^4}{24}.$$

This implies that $h_2/3 + h_4$ is negative, which is a contradiction. Consequently, $\pi < 1/2 + 3\gamma$, and since $\gamma$ can be arbitrarily close to 0, we conclude that $\pi = 1/2$.

Using $\pi = 1/2$ and $n > n_1$ now yields $h_2/3 + h_4 < (\gamma/6 + \gamma^3/2)n^4 < 8\gamma \binom{n}{4}$. Therefore $h_2 + h_4 < 24\gamma \binom{n}{4} = \delta \binom{n}{4}$. Since we have already argued that $h_0 = 0$, the vertex partition $A, B$ satisfies the requirements of the theorem, and the proof is complete. \hfill $\square$

Proof of Theorem 1.2. The proof is a standard reduction to Theorem 2.2. Let $\delta > 0$ be given. We can assume that $\delta < 1$. Suppose that $n_1$ is the output of Theorem 2.2 with input $\delta/2$. Set $\gamma = \delta/48$, and let $n > n_1/(1 - \delta)$ be sufficiently large. Let $G_n = G$ be the given 4-graph $G$ with the properties in Theorem 1.2.

If the current 4-graph $G_i$ with $i$ vertices has a vertex $x$ of degree at most $(1/2 - \gamma) \binom{i}{3}$, then remove $x$ obtaining the new 4-graph $G_{i-1}$, and repeat; otherwise we terminate the procedure. Let $G_m$ be
the final graph. By Lemma 2.1,
\[
\frac{m^4}{48} \geq |G_m| \geq \left(\frac{1}{2} - \varepsilon\right)\left(\frac{n}{4}\right) - \left(\frac{1}{2} - \gamma\right)\sum_{i=m+1}^{n} \binom{i}{3}
\]
\[
= (\gamma - \varepsilon)\frac{n^4}{24} + (1 - 2\gamma)\frac{m^4}{48} + O(n^3).
\]

It follows that
\[
m/n \geq (1 - \varepsilon/\gamma)^{1/4} + o(1) > 1 - \varepsilon/4\gamma = 1 - \delta/9
\]
and \(m > n_1\). Applying Theorem 2.2 to the 4-graph \(G_m\) of minimum degree at least \((1/2 - \gamma)\binom{n}{3}\), we obtain a partition \(X \cup Y\) of \(V(G_1)\) with all but \((\delta/2)\binom{m}{4}\) edges having even intersection with the parts. We removed at most \(\delta n/9\) vertices (and thus at most \((\delta/2)\binom{n}{4}\) edges) from \(G\) to form \(G_m\). Therefore, we can remove at most \(\delta\binom{n}{4}\) edges from \(G\) to make it odd. \(\square\)

3 A Magnification Lemma

Given a vertex partition of \(V(G)\), call an edge odd if it intersects either part in an odd number of vertices, and even otherwise. Let \(\mathcal{M}\) denote the set of quadruples intersecting either part in an odd number of points that are not in \(G\). Let \(\mathcal{B}\) denote the set of even edges in \(G\). Call a partition \(V(G) = X \cup Y\) a maximum cut of \(G\) if it minimizes \(|\mathcal{B}|\). Sometimes we denote a typical edge \(\{w, x, y, z\}\) simply by \(wxyz\). Let \(a \pm b\) denote the interval \((a - b, a + b)\) of reals.

Lemma 3.1 Let \(n\) be sufficiently large and let \(G\) be an \(n\)-vertex 4-graph with independent neighborhoods and \(d_{\text{min}}(G) \geq (1/2 - 10^{-40})\binom{n}{3}\). Let \(X, Y\) be a maximum cut of \(G\), and suppose that \(|X|\) and \(|Y|\) are both in \((1/2 \pm 10^{-15})n\). If \(|\mathcal{M}| \leq n^4/10^{40}\), then every vertex \(w\) of \(G\) satisfies \(\text{deg}_{\mathcal{B}}(w) \leq n^3/10^9\).

Proof. Suppose, for a contradiction, that there is a vertex \(w \in X\) with \(\text{deg}_{\mathcal{B}}(w) > n^3/10^9\). Say that an edge is of the form \(X'Y'\) if it has \(i\) points in \(X\) and \(j\) points in \(Y\) (for \(i + j = 4\)). We partition the argument into two cases.

Case 1. At least \(n^3/(2 \cdot 10^9)\) edges of \(\mathcal{B}\) containing \(w\) are of the form \(XXXX\).

Now \(w\) is in at least as many odd edges as even edges, else we could move \(w\) from \(X\) to \(Y\). So in particular, since \(\text{deg}_{\mathcal{G}}(w) \geq d_{\text{min}}(G) > 2\binom{n}{3}/5\), we conclude that \(w\) is in at least \(\binom{n}{3}/5\) odd edges. At least \(\binom{n}{3}/10\) of these are \(XYYY\) edges or at least \(\binom{n}{3}/10\) of these are \(XXXXY\) edges. Depending on which choice occurs, call the resulting set of edges \(\mathcal{H}\).

For every choice of \(x, y, z \in X\), with \(E = \{w, x, y, z\} \in \mathcal{B} \subset G\), and for every choice of \(E' = \{v_1, v_2, v_3, w\} \in \mathcal{H} \subset G\) with \(E \cap E' = \{w\}\), consider the five quadruples

\(v_1v_2v_3w, v_1v_2v_3x, v_1v_2v_3y, v_1v_2v_3z, wxyz\).
Regardless of whether $E'$ is of the form $XYYY$ or $XXXY$, the first four quadruples are odd. The first and fifth quadruple are both in $G$, so one of the middle three must be in $M$. On the other hand, each such quadruple $D$ is counted at most $3n^2$ times (note that $w$ is fixed, so in the case of $XXXY$ edges we only have to choose the remaining two points in $E$; in the case of $XXYY$ edges, we also may choose the unique point of $E \cap D$ thereby giving the additional factor of 3).

Putting this together we have

$$|M| \geq \frac{n^3}{2 \cdot 10^9} \times \left(\frac{n}{3}\right)^2/10 - 2n^2 > \frac{n^4}{10^{40}}$$

which is a contradiction.

**Case 2.** At least $n^3/(2 \cdot 10^9)$ edges of $B$ containing $w$ are of the form $XXYY$.

First suppose that at least $(n^3)/10^{20}$ odd edges containing $w$ are of the form $XYYY$. For every choice of $x \in X$, $y, z \in Y$, with $E = \{w, x, y, z\} \in B \subset G$, and for every choice of an odd edge $E' = \{v_1, v_2, v_3, w\} \in G$ with $E \cap E' = \{w\}$, consider the five quadruples

$$xyzw, xyzv_1, xyzv_2, xyzv_3, wv_1v_2v_3.$$ 

One of the three middle quadruples must be in $M$ and each such quadruple is counted at most $3n^2$ times (note that $w$ is fixed, so we only have to choose the remaining two points in $E'$ and the two points of $E \cap \{y, z, v_i\}$). Putting this together we have

$$|M| \geq \frac{n^3}{2 \cdot 10^9} \times \left(\frac{n}{3}\right)^2/10^{20} - 2n^2 > \frac{n^4}{10^{40}}$$

which is a contradiction. Consequently, we may assume that

(i) the number of $XYYY$ edges containing $w$ is at most $(n^3)/10^{20}$, and

(ii) the number of $XXXX$ edges containing $w$ is at most $n^3/(2 \cdot 10^9)$ (otherwise we use Case 1).

Statements (i) and (ii) imply that the edges of $G$ containing $w$ are essentially of two types: $XXXY$, and $XXYY$. Define the 3-graph $L(w) = \{\{a, b, c\} : \{w, a, b, c\} \in G\}$. By hypothesis

$$|L(w)| = \text{deg}_G(w) \geq \left(\frac{1}{2} - \frac{1}{10^{40}}\right) \binom{n}{3}.$$ 

Partition $L(w)$ as

$$L_{XXX} \cup L_{XXY} \cup L_{XYY} \cup L_{YYY},$$

where $L_{X_iY_j}$ is the set of edges of $L$ with $i$ points in $X$ and $j$ points in $Y$ ($i + j = 3$). Again, (i) and (ii) imply that $|L_{XXX}| + |L_{YYY}| < \binom{n}{3}/10^5$, so

$$|L_{XXY}| + |L_{XYY}| > \left(\frac{1}{2} - \frac{1}{10^{4}}\right) \binom{n}{3}.$$ 

For every pair $a \in X, b \in Y$, let $d(a, b)$ denote the number of triples $\{a, b, c\} \in L(w)$. Then

$$\sum_{a \in X, b \in Y} d(a, b) = 2(|L_{XXY}| + |L_{XYY}|) > \left(1 - \frac{2}{10^{4}}\right) \binom{n}{3}.$$

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Consequently, recalling that $|X|$ and $|Y|$ are both in $(1/2 \pm 10^{-15})n$, there exist $a_0 \in X$ and $b_0 \in Y$, for which

$$d(a_0, b_0) > \frac{1 - 2 \cdot 10^{-4}}{|X||Y|} \left( \frac{n}{3} \right) > \frac{1 - 2 \cdot 10^{-4}}{(1/4 + 2 \cdot 10^{-15})n^2} \left( \frac{n}{3} \right) > \left( \frac{2}{3} - \frac{1}{10^3} \right)n.$$ 

We conclude that there exist $S \subset X$ and $T \subset Y$, each of size at least $(2/3 - 1/2 - 10^{-2})n = (1/6 - 10^{-2})n$ such that $\{w, a_0, b_0, s\}, \{w, a_0, b_0, t\} \in G$ for every $s \in S$ and $t \in T$.

For every choice of distinct $s, s', s'' \in S$, and $t \in T$, consider the five quadruples

$$wa_0b_0s, wa_0b_0s', wa_0b_0s'', wa_0b_0t, ss' s'' t.$$ 

Since the first four are in $G$, we must have $\{s, s', s'', t\} \in \mathcal{M}$. Consequently,

$$|\mathcal{M}| \geq \left( \frac{|S|}{3} \right)|T| > \left( \frac{1}{6} - 10^{-2} \right)n \left( \frac{1}{6} - 10^{-2} \right)n > \frac{n^4}{10^{40}}.$$ 

This contradiction completes the proof of the lemma. \hfill \square

4 The Exact Result

**Proof of Theorem 1.1.** Let $G$ be an $n$-vertex 4-graph with independent neighborhoods and $|G| = b(n)$. Since $B(n)$ is maximal with respect to the property of being $F^4$-free, it suffices to show that $G = B(n)$.

We claim that we may also assume that $d_{\min}(G) \geq b(n) - b(n - 1)$. Indeed, otherwise, assuming we have proved the result under this assumption for $n > n_0$, we can successively remove vertices of small degree to obtain a contradiction. (Note that each removal strictly increases the difference $|G| - b(n)$, where $n$ is the number of vertices in $G$.) We refer the Reader to Keevash and Sudakov [6, Theorem 2.2] for the details. Also in [6] we have the calculations showing that

$$d_{\min}(G) \geq b(n) - b(n - 1) > \frac{1}{12}n^2 - \frac{1}{2}n^2 > \left( \frac{1}{2} - \frac{1}{10^{40}} \right) \left( \frac{n}{3} \right).$$

Choose a maximum cut $X \cup Y$ of $G$. By Theorem 1.2, we may assume that the number of even edges is less than $n^4/10^{40}$ (choose $n$ sufficiently large to guarantee this). It also follows that, for example, $|X|$ and $|Y|$ both lie in $(1/2 \pm 10^{-15})n$ for otherwise a short calculation shows that $|G| < b(n)$. These bounds will be used throughout.

Define $\mathcal{M}$ and $\mathcal{B}$ as in Section 3. Call quadruples in $\mathcal{M}$ *missing* and those in $\mathcal{B}$ *bad*. Since $(G \cup \mathcal{M}) \setminus \mathcal{B}$ is odd and $|G| = |B(n)|$, we conclude that

$$|B(n)| + |\mathcal{M}| - |\mathcal{B}| = |G| + |\mathcal{M}| - |\mathcal{B}| \leq |B(n)|$$

and therefore $|\mathcal{B}| \geq |\mathcal{M}|$. In particular, this implies that $|\mathcal{M}| < n^4/10^{40}$. If $\mathcal{B} = \emptyset$, then $G$ is odd, so $G = B(n)$ and we are done. Hence assume that $\mathcal{B} \neq \emptyset$. In the remainder of the proof, we will obtain a contradiction to $|\mathcal{M}| < n^4/10^{40}$, or to the choice of the partition of $V(G)$. 

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Our strategy is to show that each even edge yields many potential copies of $F^4$, and hence many missing quadruples. Define

$$A = \{ z \in V(G) : \deg_M(z) > n^3/10^7 \}.$$

Our first goal is to prove that $A \neq \emptyset$. In fact, we actually will need the following stronger statement:

**Claim.** There exists $B' \subset B$ such that $|B'| > |B|/20$ and

$$\forall E \in B', \quad |E \cap A| \geq 1. \quad (4)$$

**Proof of Claim.** Write $B = B_{XXX} \cup B_{YYY} \cup B_{XYX}$ (with the obvious meaning).

**Case 1.** $|B_{XXX}| + |B_{YYY}| \geq |B|/10$.

Pick $E = \{ w, x, y, z \} \in B_{XXX} \cup B_{YYY}$. Assume without loss of generality that $\{ w, x, y, z \} \in B_{XXX}$. For every choice of $v_1, v_2, v_3 \in Y$ the five quadruples

$$v_1v_2v_3w, v_1v_2v_3x, v_1v_2v_3y, v_1v_2v_3z, wxyz \quad (5)$$

form a potential copy of $F^4$, so one of the first four must be in $M$. This gives $|M| \geq \binom{|Y|}{3}$, and so at least $\binom{|Y|}{3}/4 > n^3/10^7$ of these quadruples of $M$ contain the same vertex of $E$, say $w$. Thus $\deg_M(w) > n^3/10^7$. Now let $B' = B_{XXX} \cup B_{YYY}$. Then $|B'| \geq |B|/10 > |B|/20$ as claimed.

**Case 2.** $|B_{XYX}| > 9|B|/10$.

Let $B' = \{ E \in B : |E \cap A| \geq 1 \}$. If $|B'| \geq |B_{XYX}|/10$, then

$$|B'| \geq \frac{|B_{XYX}|}{10} \geq \frac{1}{10} \times \frac{9}{10} |B| > \frac{|B|}{20}$$

and we are done. Hence we may assume that $|B'| < |B_{XYX}|/10$. Let $B'' = B_{XYX} \setminus B'$. Thus $|B''| > 9|B_{XYX}|/10$. Given a set $S$ of vertices, write $\deg_M(S)$ for the number of edges of $M$ containing $S$.

**Subclaim.** For every $E \in B''$, and for every $S \in \binom{S}{3}$, we have $\deg_M(S) \geq (1/2 - 10^{-2})n$.

**Proof of Subclaim.** Suppose to the contrary that there exists $E \in B''$ and $S \in \binom{E}{3}$ with $\deg_M(S) < (1/2 - 10^{-2})n$. Assume that $E = \{ w, x, y, z \}$ with $w, x \in X$ and $y, z \in Y$ and $S = \{ x, y, z \}$. Let $Y' = \{ v \in Y : \{ x, y, z, v \} \in G \}$. Then

$$|Y'| \geq |Y| - \deg_M(S) - 2 > \left( \frac{1}{2} - \frac{1}{10^4} - \frac{1}{2} + \frac{1}{10^2} \right) n = \left( \frac{1}{10^2} - \frac{1}{10^{14}} \right) n.$$ 

For every choice of $v_1, v_2, v_3 \in Y'$ the five quadruples

$$xyzv_1, xyzv_2, xyzv_3, xyzw, v_1v_2v_3w.$$ 

form a potential copy of $F^4$, so the last one must be in $M$. This gives

$$\deg_M(w) > \binom{|Y'|}{3} > \binom{10^{-2} - 10^{-14})n}{3} > \frac{n^3}{10^7}.$$
Consequently, $E \in B'$ which contradicts the fact that $B' \cap B'' = \emptyset$.

Counting edges of $\mathcal{M}$ from subsets of edges of $B''$ yields

$$
\left(\frac{3}{2}\right) \cdot \max\{|X|, |Y|\} \cdot |\mathcal{M}| \geq \sum_{E \in B''} \sum_{S \in \binom{E}{3}} \deg_{\mathcal{M}}(S),
$$

since the right hand side counts an edge of $\mathcal{M}$ at most $3 \max\{|X|, |Y|\}$ times. For example, an edge $\{a, b, c, d\} \in \mathcal{M}$ with $a \in X$ and $b, c, d \in Y$ is counted on the right-hand side by choosing $E \in B''$ where $|E \cap \{b, c, d\}| = 2$ and $a \in E$. Using $|B''| \geq (0.9) |\mathcal{M}|$, and the Subclaim, we get

$$
|M| \geq \left(0.9\right)^2 \cdot 4 \left(\frac{1}{2} - 10^{-2}\right)n |\mathcal{M}| = 1.08 \left(\frac{1}{2} - 10^{-2}\right) |\mathcal{M}| > |\mathcal{M}|.
$$

This contradiction concludes the proof of Case 2 and of the Claim. □

Counting missing edges from vertices of $A$, we have

$$
4|\mathcal{M}| \geq \sum_{x \in A} \deg_{\mathcal{M}}(x) > \frac{|A| n^3}{10^7}.
$$

Recalling that $|B'| > |\mathcal{B}|/20$ and $|\mathcal{B}| \geq |\mathcal{M}|$, we obtain

$$
|B'| > \frac{|\mathcal{M}|}{20} > \frac{|A| n^3}{80 \cdot 10^7}.
$$

Now the Claim (see (4)) implies that

$$
\sum_{x \in A} \deg_{\mathcal{B}'}(x) \geq |B'| > \frac{|A| n^3}{80 \cdot 10^7}.
$$

Consequently, there exists $w \in V(G)$ for which $\deg_{\mathcal{B}'}(w) \geq \deg_{\mathcal{B}'}(w) > n^3/(80 \cdot 10^7) > n^3/10^9$. This contradicts Lemma 3.1 and completes the proof of the theorem. □

5 The Sharp Structure

Proof of Theorem 1.3. Let $\delta = 12/10^{40}$, and choose $\varepsilon < \delta/12$ from Theorem 1.2. Now $|G| > (1/2 - \varepsilon)_{\binom{n}{4}}$, so by Theorem 1.2 $G$ has a vertex partition $X \cup Y$ with the number of even edges less than $\delta_{\binom{n}{4}} < n^4/(2 \cdot 10^{40})$. Easy calculations show that $|X|$ and $|Y|$ are both in $(1/2 \pm 10^{-15})n$. We may also assume that $X, Y$ is a maximum cut. We will show that both $X$ and $Y$ are independent sets. As in (3), we have

$$
\left(\frac{1}{2} - \varepsilon\right) \binom{n}{4} + |\mathcal{M}| - |\mathcal{B}| < |G| + |\mathcal{M}| - |\mathcal{B}| \leq b(n)
$$

which implies that

$$
|\mathcal{M}| \leq |\mathcal{B}| + b(n) - \left(\frac{1}{2} - \varepsilon\right) \binom{n}{4} \leq \frac{n^4}{2 \cdot 10^{40}} + \varepsilon \binom{n}{4} + O(n^3) < \frac{n^4}{10^{40}}.
$$
Suppose now that there is an edge $E$ of $G$ in $\binom{X}{4} \cup \binom{Y}{4}$. Assume by symmetry that $E \in \binom{X}{4}$. Then by the same argument as in (5), we obtain $\deg_M(w) > (\frac{|Y|}{3})/4 > n^3/10^5$ for some $w \in E$.

Now

$$\left(\frac{1}{2} - \varepsilon\right)\left(\frac{n}{3}\right) < \deg_G(w) = \deg_B(w) + \left(\frac{|Y|}{3}\right) + \left(\frac{|X| - 1}{2}\right)|Y| - \deg_M(w).$$

As $\binom{|Y|}{3} + \binom{|X| - 1}{2}|Y| < \left(1/2 + \varepsilon\right)n^3/3$ we obtain $\deg_B(w) \geq n^3/10^5 - 2\varepsilon\left(\frac{n}{3}\right) > n^3/10^9$. This contradicts Lemma 3.1 and completes the proof. □

References


