Business-Cycle Models: Closing the Gap Between the Different Approaches*

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Summary: This paper is concerned with the subject of how the three main approaches to model aggregate economic fluctuations are related. That is, we are interested in the relation between real business cycle models, endogenous business cycle models and sunspot equilibria models. It is shown that there exists an open set of parameters where the three approaches are very much related: the time series have much resemblance. Outside this set of parameters the approaches seem mutually exclusive. In particular, we show that if the interior steady state is unique and has a saddle structure then no sophisticated deterministic fluctuations can occur. On the other hand if the nonlinear model is close to having multiple eigenvalues close to 1 in absolute value then so-called Bogdanov-Takens bifurcations can occur very close to the interior steady state: sophisticated deterministic fluctuations near unit roots occur. We will argue that if these types of bifurcations occur then linearizing requires checking for the size of the basin of attraction. In addition, for parameters close to these bifurcations, the linearized model may have the same structure as the real business-cycle model and generate similar time series. However, in this case cycles are driven by animal spirits and, therefore, do not require persistent shocks.

1 Introduction

In the dominant macro-economic paradigm agents are rational beings that optimally respond to fundamental changes in a market clearing economy with Pareto optimal allocations of contingent commodities. Translating this view into a system of dynamic equations leads to a neoclassical model where the solutions for (aggregate) capital, and all other variables are, near the unique steady state, linearly approximated, while the (forcing) stochastic process is described by an autoregressive (AR) model with roots close to (but smaller than) one. Since the unique steady state of the, on empirical data calibrated, neoclassical model has a saddle structure attention is restricted to the dynamics on the stable manifold(s), see the real business cycle (**RBC**) approach promoted, following Lucas [30], by, for example, Kydland and Prescott [28] and King, Plosser and Rebelo [25]. To find economic rationale and evidence for the presence (and properties) of the stochastic process has turned out to be a cumbersome task[1] and in fact has reinforced the necessity to find alternative methods to model

[1] See, for instance, the critiques by Sims [37].

aggregate business cycles. This has resulted in an expansion of a line of research that is referred to as the endogenous business cycle (**EBC**) approach. Similar to the RBC models markets clear, agents are rational beings and allocations of contingent commodities are Pareto optimal. In this framework, however, fundamental changes of the economy are disregarded: technology, preferences and endowments are assumed to be stationary. The occurrence of fluctuations is originated by self-fulfilling changes in (deterministic) agents' expectations, endogenously accounted for by nonlinear laws that underly the economy, see Benhabib and Day [4] and Grandmont [15] for early examples. Although this avoids introducing in the model unexplained shocks to the fundamentals, it in turn creates new problems. In fact, it can be shown, at least in one-dimensional endogenous models, that the cycles can only arise under quite unrealistic economic assumptions (such as the violation of the gross substitutability axiom, unrealistic parameter values, long time scale) and that the statistical properties of the time paths have almost no resemblance with those observed empirically. However, some of this criticism has been countered in the past decade or so, see for example Reichlin [36], Woodford [42], de Vilder [40] or Grandmont, Pintus and de Vilder [16] where the time scale and the gross substitutability issues are settled.

An alternative, and promising, way to tackle the occurrence of shocks to the economy is by assuming that business cycles are in fact induced by "animal" spirits of agents. The approach is based on the Keynesian thought of the self-fulfilling prophecy, see for example Azariadis [1], Cass and Shell [8], Azariadis and Guesnerie [2], Guesnerie [17], Guesnerie and Woodford [42]. The time series generated by these types of economies, also referred to as sunspot equilibrium (**SE**) models, may overcome all of the previously mentioned criticism, see Farmer and Guo [14]. However, some of the modern macro-economic assumptions and Pareto optimality are violated which makes this approach, according to some economists, a theoretical curiosity. Moreover, an important caveat of the approach is that the results reported by Farmer and Guo [14] based on linearization may only be valid, as will be shown in this paper, in an extremely small neighbourhood of the interior steady state.

Although the EBC and SE approaches to model business cycles have, until now, mainly received theoretical recognition it remains an open, interesting and possibly enlightening topic to examine how the three approaches are actually connected. Answering this question, at least partially, is the aim of this paper.

To fix ideas, we shall use a standard two-dimensional (discrete time) benchmark economic model, with capital accumulation and input substitution, that is a variant of representative models both used in the exogenous, the endogenous and the sunspot literature (see for example Kydland and Prescott [28], King, Plosser and Rebelo [25], Farmer and Guo [14], Grandmont, Pintus and de Vilder [16]). For certain parameter configurations this particular model displays all principal local and global bifurcations that have been reported in the economic literature in the past two decades. While for other parameter configurations the interior steady state is unique and possesses a saddle structure as prescribed by the real business cycle approach. The main theme throughout the paper is to locate in parameter space those economies that are able, using the different methodologies, to generate time series that have empirically realistic properties such as AR structures with roots close to one. *The conclusion we make is that for certain sets of parameters there is a fundamental connection between the different approaches.*

We have organized the paper as follows. In the next section we present the benchmark model and locate the different economies in parameter space. In section 3 we discuss the dangers of linearization and introduce the Bogdanov-Takens bifurcation. In section 4 we present a sufficient condition for the absence of chaos. In section 5 we apply the results presented in the previous sections to a general specification of the model. In section 6 we give some concluding remarks. Finally, in the appendix we give proofs and provide an extensive discussion of the Bogdanov-Takens bifurcation.

2 The benchmark model

In this section we introduce the model that we use as a benchmark. The proposed model is a variant of settings that have been used in the endogenous literature (see for example Grandmont, Pintus and de Vilder [16] where a modification of Woodford's infinite horizon model [42] is chosen) and in the exogenous literature (Kydland and Prescott [28]). We shall restrict our attention to the endogenous interpretation and introduce randomness at the level of expectations. We end the section by presenting a local stability analysis and indicate in parameter space the regions that fit the different approaches.

2.1 The deterministic equilibrium model

We restrict attention to the region in phase space where the infinite horizon model proposed by Woodford [42] can be interpreted as an overlapping generations model. In that region workers behave as if they came in two generations. They supply, in period t, $l_t \geq 0$ units of labour, the income of which is saved in the form of $m_t \geq 0$ units of fiat money which in turn is spent on $c_{t+1} \geq 0$ units of consumption good in the next period. Workers may be identified by one typical agent whose preferences are represented by a separable utility function $V_2(c_{t+1}) - V_1(l_t)$, where the V_i's are (dis)utility functions that satisfy the standard properties:

Assumption 2.1 *The utility functions V_i, $i = 1, 2$ are C^r, $r > 3$, for $l, c \geq 0$. With $V_1'(l) > 0$, $V_1''(l) > 0$, $\lim_{l\to\infty} = +\infty$, $V_2'(c) > 0$, $V_2''(0) < 0$ and $-cV_2''(c) < V_2'(c)$.*

We are focusing on intertemporal equilibria with self-fullfilling expectations. In this setting workers maximise their utility under the current and future constraints $w_t l_t = m_t = p_{t+1}c_{t+1}$, where $w_t \geq 0$ is the wage rate and $p_{t+1} \geq 0$ is either the random or perfectly foreseen price of the consumption good. In this subsection we shall treat the case where there is no uncertainty about the future price of the consumption good. In the next subsection the alternative case is introduced. When interior the budget constraints and the first order condition leads to

$$v_1(l_t) = v_2(c_{t+1}), \tag{1}$$

where $v(l) = lV_1'(l)$ and $v_2(c) = cV_2'(c)$.

The production sector of the economy is operated by profit maximising, infinitely long living, entrepreneurs. In the production process labour and capital ($k \geq 0$) are combined in variable proportions to produce $y = lf(k/l)$ units of output, where $f(.)$ is the reduced production function that satisfies:

Assumption 2.2 *The marginal productivity of capital $\rho(a) = f'(a)$ is a decreasing function and the marginal productivity of labour $\omega(a) = f(a) - af'(a)$ is an increasing function of a, where a is defined as the capital-labour ratio k/l.*

In the RBC literature the randomness is introduced at the level of the production function by means of a stochastic (scaling) technology parameter. In the next subsection we shall get back to this matter in more detail. As usual in one-sector models the output is to be interpreted as a perishable good that can be either consumed or reinvested. Moreover, the production process also yields in each period $(1-\delta)k$ units of depreciated capital, with $0 < \delta \leq 1$. Under perfect competition, the real gross return on capital $R = \rho + 1 - \delta$ is in equilibrium equal to $R(a) = \rho(a) + 1 - \delta$ while the real wage w/p is in equilibrium equal to $\omega(a)$. We focus on a limit version of the model where capitalists do not consume and in fact reinvest all their capital income. In equilibrium the behaviour of the capitalists is then described by

$$k_t = R(a_t)k_{t-1} \tag{2}$$

Finally, we assume that the supply of outside money $M > 0$ is constant over time. Since money is held by the household sector the real money balances M/p_t are equal to the consumption of "the old" (c_t) which in turn is equal to real wage income $\omega(a_t)l_t$.

$$c_t = \omega(a_t)l_t = M/p_t \tag{3}$$

By substituting (3) into (1) and using (2) we get an expression for the perfect foresight equilibrium dynamics which is summarized in the following definition:

Definition 2.1 *An inter-temporal equilibrium with perfect foresight is a deterministic sequence $k_t > 0$ and $a_t > 0$ that satisfies*

$$\begin{aligned} k_t &= R(a_t)k_{t-1} \\ \omega(a_{t+1})/a_{t+1} &= \gamma(k_{t-1}/a_t)/k_t \end{aligned} \tag{4}$$

for $t \geq 1$, where $\gamma = v_2^{-1} \circ v_1$ is the workers (invertible) offer curve.

Grandmont et al. [16] show that under appropriate boundary conditions on the marginal utility functions and on the marginal productivity of capital, a unique interior steady state exists. Moreover, if $\omega(a)/a$ is invertible at this steady state then the equations (4) define, at least locally, an invertible determistic dynamical system.

2.2 The stochastic equilibrium model

Let us now show how randomness can be introduced into the benchmark model (4). We introduce the uncertainty at the level of expectations. In this setting the young maximize at date t the mathematical expectation of their utility conditional upon the available information which we denote by H_t and subject to the current and future budget constraints $w_t l_t = m_t = p_{t+1}c_{t+1}$. Instead of the case where agents have perfect foresight about future prices of the consumption good it is now assumed that p_{t+1} is random. This together with the first order condition then leads to

$$v_1(l_t) = E_t(v_2(c_{t+1})) \tag{5}$$

where the v_i's are as in (1). Now take a sequence of random variables s_t, $t = 0, 1, \dots, \infty$ that lies in a compact set S. So at time t a typical household has as an information set $H_t = (s_t, s_{t-1}, \dots)$. Since the uncertainty is about future prices we can still substitute (3) into (5) and use (2) to get an expression for the intertemporal equilibrium dynamics which is summarized by:

Definition 2.2 *An intertemporal equilibrium with self-fullfilling expectations is a sequence of random variables $k_{t-1} > 0$ and $a_t > 0$ that satisfies*

$$\begin{aligned} k_t &= R(a_t)k_{t-1} \\ E[v_2(k_t\omega(a_{t+1})/a_{t+1})|H_t] &= v_1(k_{t-1}/a_t) \end{aligned} \tag{6}$$

An equivalent way to write this is by introducing a sequence of innovations ϵ_t with $E[\epsilon_{t+1}|H_t] = 0$. This then results in:

$$\begin{aligned} k_t &= R(a_t)k_{t-1} \\ v_2(k_t\omega(a_{t+1})/a_{t+1}) &= v_1(k_{t-1}/a_t) + \epsilon_{t+1}, \text{ where } E[\epsilon_{t+1}|H_t] = 0 \end{aligned} \tag{7}$$

Observe that in this two-dimensional stochastic difference equation the random innovations only influence the capital-labour ratio a and not the predetermined variable k.

2.3 Linearized dynamics

We now show that the steady state of (4) can undergo various bifurcations. We state the next lemma in terms of σ (the elasticity of input substitution), ϵ_γ (the elasticity of the offer curve), s (the share of capital in output), δ (the rate of capital depreciation), β_c (the rate of discounting).

Lemma 2.1 *Assume $\theta(1-s) < s$, where $\theta = 1-\beta_c(1-\delta)$. Then the following holds:*

1. *If $\sigma \in [0, \sigma_F = \theta(1-s)/2]$, a stable supercritical Hopf bifurcation occurs at $\epsilon_{\gamma_H} = (s-\sigma)/(\theta(1-s)-\sigma)$; for ϵ_γ slightly larger than $\epsilon_{\gamma H}$ an attracting invariant closed curve surrounds the unstable steady state, on which the dynamics is either periodic or quasi-periodic. Moreover, the steady state is stable when $1 < \epsilon_\gamma < \epsilon_{\gamma H}$.*
2. *If $\sigma \in (\sigma_F, \sigma_H)$, a stable supercritical Hopf bifurcation occurs at $\epsilon_{\gamma H}$, followed by an unstable subcritical flip bifurcation at $\epsilon_{\gamma F} = (2s+\theta(1-s)-2\sigma)/(2\sigma-\theta(1-0s)) > \epsilon_{\gamma H}$, where $\sigma_H = s[1+\theta(1-s)/s-\sqrt{1-\theta(1-s)/s}]/2$; for ϵ_γ slightly larger than $\epsilon_{\gamma H}$ an attracting invariant closed curve surrounds the unstable steady state, while for ϵ_γ slightly smaller than $\epsilon_{\gamma F}$ there exists a period two saddle orbit near the unstable steady state.*
3. *If $\sigma = \sigma_H$ there exists a value ϵ_γ such that a Hopf and a flip bifurcation coincide.*
4. *If $\sigma \in (\sigma_H, \sigma_I)$ an unstable subcritical flip bifurcation occurs at $\epsilon_\gamma = \epsilon_{\gamma F}$ where $\sigma_I = [\theta + s(1-\theta)]/2$; for ϵ_γ slightly smaller than $\epsilon_{\gamma F}$ there exists a period two saddle near the stable steady state.*

The proof can be found in Grandmont, Pintus and de Vilder [16]. In what follows we fix the propensity to save s at $1/3$, the depreciation rate of capital δ at $1/10$ and the discounting parameter β_c at 1. This then leads to the bifurcation diagram in figure 1: along the segment (B, C) the supercritical Hopf bifurcation occurs and along the segment (A, B) the subcritical flip bifurcation. In this paper we have special interest in the case associated with point B where both curves meet. Here we just refer to figure 1 where we have drawn the different bifurcation curves. We also indicate in this figure the stability of the steady state in the regions.

RBC domain:
RBC proponents concentrate on parameters in the region where the unique interior fixed point is a saddle, see figure 1. According to that approach all the economic action takes place on the stable manifold. However, if we would add i.i.d. random noise to the second equation of (4) orbits will not be feasible with probability 1: the fixed point is determinate. Similarly, if we would perturb the technology with an i.i.d. noise term then again all orbits will escape with probability 1. The RBC literature solves this problem by explicitly taking into account in the optimization procedure that the technology is subjected to random perturbations, and then determine the optimal paths of the *expectations* of these random variables. From this, together with their standing assumptions, they conclude that the only economically feasible paths are on the stable manifold of the unique fixed point p (that we denote $W^s(p)$; see subsection 3.2 for the definition). In terms of equations this approach may be expressed in the following simple form:

$$\begin{aligned} k_t &= f_{A_t}(k_{t-1}) \\ A_{t+1} &= \beta A_t + \epsilon_t, \end{aligned} \tag{8}$$

where A denotes total factor productivity, $f_A : W^s(p) \to W^s(p)$ (f is linear), $k_{t+1} = f_{A_t}(k_t)$, $k_t \in W^s(p)$, β smaller than but close to 1 and ϵ_t i.i.d. random noise. This particular configuration is obtained when labour supply is sufficiently inelastic w.r.t. real wage, i.e. if ε_γ is large enough, or if σ (substitutability of inputs) is large enough.

EBC domain:
EBC proponents are interested in the regions where the interior steady state is either stable or when it bifurcates to another attractor, see figure 1. That is, in terms of figure 1, when the left line, while increasing the parameter ε_γ, lies inside the triangle (A, B, C) or if it is close to the lines (B, C) or (A, B). In the first case the fixed point is attracting, in the second case (on the line (B, C)) generically a Hopf bifurcation occurs and in the latter case (on the line (A, B)) generically a period-doubling bifurcation occurs. If the Hopf bifurcation is stable an attracting circle bifurcates out of the fixed point and all asymptotic economic dynamics takes place on this circle. In fact, this circle might transform into a very complicated set which also bears the interest of the EBC proponents, see Pintus, Sands and de Vilder [35]. Similar for the flip bifurcation, the asymptotic

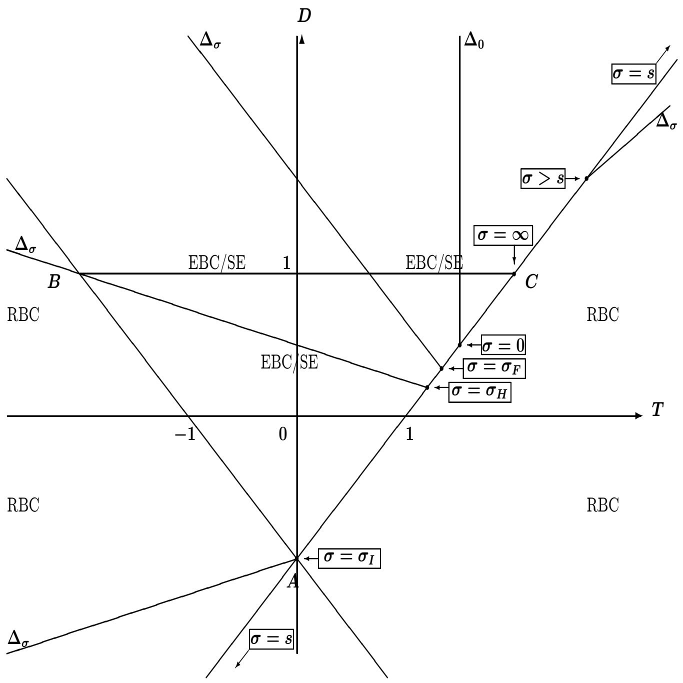

Figure 1: The bifurcation curves in the trace-determinant plane. Are also depicted the left-line Δ_σ for different values of σ and regions in which the different approaches are located, in the trace-determinant plane.

dynamics may take place on a stable period two cycle that might transform into a very complicated set through a whole series of flip bifurcations, see for example Grandmont [15]. In both cases the stable manifolds of the dynamic objects are two-dimensional. This configuration is obtained when labour supply is less inelastic (ε_γ close to 1) than is the case in the RBC approach and only if σ is small enough.

SE domain:
Proponents of the SE approach are in a sense close to the EBC proponents. That is, their economies lie in regions of the trace determinant space that are a subset of the EBC economies, see figure 1b. As mentioned we restrict attention to the case where the SE have a continuous support: i.i.d. random shocks are applied. We start in the triangular region (A, B, C) in figure 1 where the unique interior steady state is stable: the fixed point is indeterminate. In the case the fixed point undergoes an aforementioned stable Hopf bifurcation (along the line (B, C)) the fixed point becomes determinate. However, since a small attracting circle surrounds the fixed point one can still have SE in a annular region containing the circle, see Grandmont et al. [16]. A similar story can be told if the fixed point would loose stability through a stable period-doubling bifurcation: an indeterminate period two orbit surrounds the determinate steady state. Although numerical simulations indicate that one can still construct SE if the period two orbit and the invariant circle are transformed into strange (chaotic) sets there is no formal proof of this assertion yet. In terms of elasticities this configuration only arises under similair assumptions as in the EBC approach.

3 Linearization, normal forms, bifurcations and random perturbations

3.1 The dangers of linearizing

In the RBC literature it is a standard procedure to use the linear terms of the Taylor expansion, associated with the stable direction, as an approximation of the original nonlinear dynamic economic model. The in this fashion obtained linear dynamic economic model is then coupled with an auto-regressive process of the first order with roots close to unity: the AR(1) process is introduced to describe the shocks to the fundamentals mechanism, see system (8). Since in the RBC setting all feasible orbits lie on the stable manifold of the saddle one constructs in this way a stable system. In the EBC and SE literature one generally is not interested in restricting attention to the linear terms of the Taylor expansion. That is, in linear dynamic economic models the dynamics without shocks is trivial: either all orbits converge to a stable fixed point or all orbits escape to infinity. More specifically, all interesting dynamic objects, created through the various local or global bifurcations, are ignored. Finally, one has to keep in mind that in nonlinear dynamics *the size of the basin of attraction of a fixed point can be small if the eigenvalues are close to one*: if both eigenvalues tend to 1 or -1, then the area of the basin in general shrinks very fast to zero (often exponentialy fast). In linear models the basin of attraction is either a point or the whole space whereas in nonlinear models the basin can be determined by stable manifolds of other dynamic objects and can be very small. This implies that if one adds i.i.d. normally distributed random noise to the level of the expectations in an indeterminate SE model, orbits will escape the basin even if it is the positive quadrant, with probability 1. Hence, in order to have that the linearized model and the original model display identical fluctuations one has to compute the basin of attraction of the latter model and one should not add normally distributed noise but only, for example, noise with small amplitude. Summarizing, if eigenvalues are close to unity, then it makes sense to consider normal forms which also contain certain higher order terms, so that one does not ignore crucial local behaviour and bifurcations.

3.2 Bifurcations near unit roots

Let us return to what happens along the curve $[B, C]$ in figure 1 where both eigenvalues are on the unit circle. Let these eigenvalues be λ_1, λ_2. If $\lambda_1 = \lambda_1^k \lambda_2^m$ (i.e. if we write λ_i as $e^{i\phi_i}$ then this means that $(1-k)\phi_1 = m\phi_2 \bmod 2\pi$) then we say that one has a k:m resonance. If $k, m \notin \{0, 1, 2, 3, 4\}$ then a standard Hopf bifurcation occurs (provided some further conditions are satisfied). However, along the curve BC the ratio k:m is not constant and in fact there is a parameter for which one has a 1:2 resonance. In fact, at the point B, in figure 1 one expects such a 1:2 resonance (both eigenvalues are -1): a flip and a Hopf bifurcation happen at the same time. While at the point C, in figure 1 both eigenvalues are $+1$ and one expects a 1:1 resonance: a saddle-node and a Hopf bifurcation happen simultaneously. If some genericity conditions are satisfied then these are called Bogdanov-Takens bifurcations. In the appendices $A.2$-$A.3$, we discuss in some detail the local structure near the 1:2 resonance. Moreover, the 1:2 and the 1:1 resonance bifurcations are very much related: if one takes the second iterate of the system that displays the 1:2 resonance bifurcations one obtains a system that undergoes the 1:1 resonance bifurcation with the exception that one has two additional saddle fixed points instead of one. In this paper we will be studying parameters near B in figure 1 because there all these domains meet. That is, we shall focus on points in parameter space which combines the best features of the different approaches. Thus we will unify and compare the RBC, EBC, SE approaches. Of course this may seem a very special choice of parameters, but we should emphasize that even not very close to these parameters, the dynamics will be strongly affected by the behaviour at the bifurcation parameter, as we shall see in section 5. We should also emphasize that in typical two-parameters families one does expect 1:2 resonance bifurcations to occur, and indeed in for example, Benhabib, Schmitt-Grohe and Uribe [5], Brock and Hommes [7], Cazzavillan and Pintus [9] and Mortensen [32] an analysis similar to the one given in this paper can be given. Also if one chooses a different production function (with increasing returns to scale) then one can also encounter in system (4) other resonance Hopf bifurcations (see Cazzavillan, Lloyd-Braga and Pintus [9]). Contrary to the standard Hopf bifurcations, one has chaotic dynamics arbitrary near the 1:2 resonance bifurcation. Let us give a short review of some standard results in dynamical systems. Consider a C^1 diffeomorphism $G : \mathbf{R}^2 \to \mathbf{R}^2$. Suppose $P = \{p, G(p)\}$ is a hyperbolic periodic orbit of period 2 of G, so $G^2(p) = p$ and the Jacobian, evaluated at P, has no eigenvalues of norm 1. Then the stable and unstable sets of p are defined by

$$W^s(p) = \{x; G^{2n}(x) \to p, \text{ as } n \to +\infty\}$$

and

$$W^u(p) = \{x; G^{2n}(x) \to p, \text{ as } n \to -\infty\}$$

are smooth manifolds. By replacing p by $G(p)$ in this definition one obtains the stable and unstable manifolds of $G(p)$. In a nonlinear framework stable and unstable manifolds of p and $G(p)$ may intersect. That is, $W^u(p) \cap W^s(G(p))$ may not be equal to the empty set; points x that satisfy this condition are referred to as *heteroclinic points*. In fact, if the stable and unstable manifolds of a periodic orbit have points of heteroclinic intersection then this leads to a so-called *heteroclinic tangle*, see figure 2 for a graphical illustration. Moreover, it can be shown that if a periodic orbit has heteroclinic points then there also

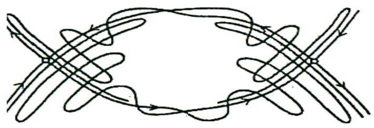

Figure 2: Here we display a heteroclinic tangle, in the plane, generated by the stable and unstable manifolds of a period two saddle orbit.

exists points of *homoclinic* intersection: $W^u(p) \cap W^s(p) \neq \emptyset$. It is well known that if a dynamical system displays a heteroclinic tangle all sorts of complicated dynamic phenomena are present, see for example Kuznetsov [27], Palis and Takens [33] or for an economic application Pintus, Sands and de Vilder [35]. As we shall see in the appendix, provided some number is non-zero, there are two possibilities for the dynamics near such a fixed point (for parameters near the bifurcation value). We shall concentrate on just one case which is summarized in figure 3, were ϵ_1, ϵ_2 are functions of the original parameters. Figure 3 must be read as follows: first the attracting fixed point is surrounded by a period two saddle orbit (i). By altering the parameters (ϵ_1, ϵ_2) such that the line H is crossed and region (ii) is entered an attracting invariant circle emerges out of the fixed point. Along the line C the stable and unstable manifolds of the periodic saddle orbit and the invariant closed curve coincide: a heteroclinic bifurcation occurs in which the circle is destructed. Finally, the period two saddle orbit is destructed through a flip bifurcation when the parameters enter region (iv) leaving region (iii) and crossing F_+. In particular, generically the diffeomorphism will have homoclinic intersections for an open set of parameters. It follows from the existing literature on the subject (see e.g. Palis and Takens [33]) that there must exist generically unfolding quadratic tangencies giving rise to all sorts of (stable) complicated deterministic dynamic phenomena. Although the invariant structures are "thin" they will strongly influence the observable dynamic behaviour in phase space.

4 Absense of chaos: a sufficient condition

In recent literature on dynamic economic models homoclinic bifurcations associated with a fixed (periodic) saddle point have been reported, see for example Brock and Hommes [7], Cazzavillan et al. [9], Tuinstra [39] and de Vilder [40].

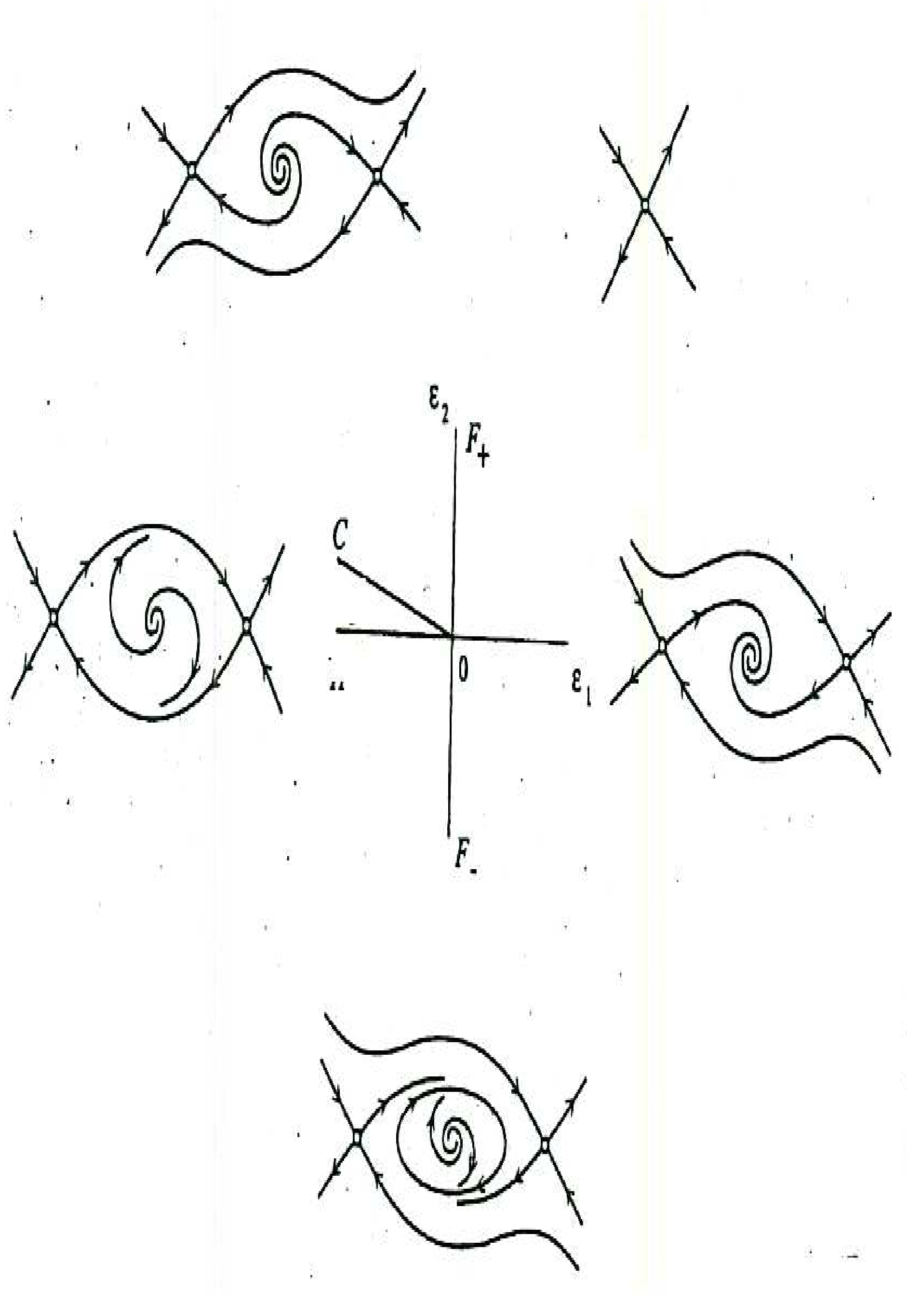

Figure 3: Depending on the sign of $C(0)$ in the normal form for the 1:2 resonance bifurcation one can have two distinct bifurcation scenarios; here we have depicted the situation when $C(0) > 0$.

Without exception in these examples multiple steady states or periodic orbits are also present. The natural question then arises whether homoclinic bifurcations are also a possible outcome if the model has a unique fixed point which has a saddle structure. That is, do economies that fit the RBC regime, display homoclinic bifurcations and associated sophisticated behaviour? The answer to this question is subtle since economic models often have an additional fixed point which is referred to as the autarkic steady state. Although this state is on economic grounds not attainable in a mathematical sense it may influence the dynamics, see de Vilder [40]. Moreover, globally the dynamic model may not be invertible which also has important dynamic implications. Let us start by dealing with these problems by introducing two definitions concerning feasibility. Let U be a subset of $\mathbf{R}_+^2$ and let f be a smooth map defined on U with a unique fixed point p in U, then

Definition 4.1 *The global feasible set* $\hat{\Lambda}$ *is the set of points:* $\{x; f^n(x) \in U, \forall n \geq 0\}$.

Definition 4.2 *The feasible set* Λ *is the path-connected component*[2] *of* $\hat{\Lambda}$ *containing the fixed point and restricted to* Λ *the map* f *is invertible.*

In figure 4 we give a schematic sketch of these two definitions. In this picture the global feasible set $\hat{\Lambda} = \cup_{i=1}^{5} U_i$ whereas the feasible set Λ is equal to U_1. From an economic point of view the definition of Λ can be rationalized as follows: we are only interested in those initial states $x \in \mathbf{R}^2$ that lie in a (possibly large) neighbourhood of the fixed point p and can reach p if time increments would decrease to zero. We are now ready to present a theorem that excludes the occurrence of chaos associated with homoclinic intersections of the stable and unstable manifolds of the saddle fixed point in certain classes of dynamic economic models.

Theorem 4.1 *Assume that* p *is a saddle point with positive eigenvalues and that definition 4.2 is satisfied, then* $\Lambda \subset W^s(p)$.

[2] Λ is the set of all points $x \in \hat{\Lambda}$ such that p and x can be connected by an arc which lies inside $\hat{\Lambda}$.

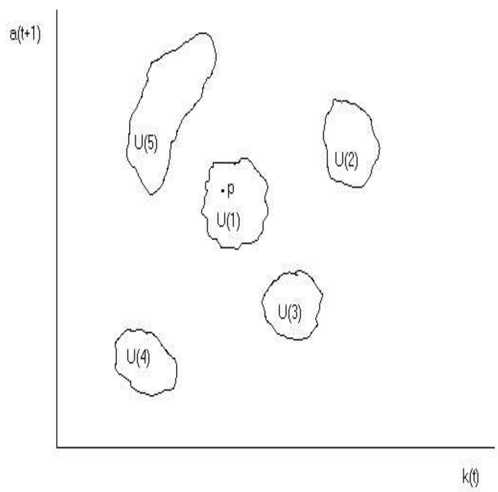

Figure 4: A schematic drawing of the feasible sets Λ and $\hat{\Lambda}$. In this picture $\hat{\Lambda} = \cup_{i=1}^{5} U_i$ and $\Lambda = U_1$ assuming that f is invertible on U_1. The complementary regions consists of those points that are not feasible: they may escape to infinity or converge to different dynamic objects.

Proof: The proof can be found below, in the first appendix. □

So no sophisticated behaviour, such as periodic or chaotic fluctuations, can occur under the outlined circumstances. Moreover, if the saddle has negative eigenvalues and there are no periodic orbits of period two then one can replace the map f by its second iterate and apply the previous theorem. In the next section we will specify the production and utility functions for our model and show that this theorem applies for cetrain parameter configurations to our setting.

5 A worked out example

In this section we shall apply the theory of the previous sections to a general specification of our model.

5.1 Model specification

We specify the following widely-used reduced CES production function:

$$f(a) = A(sa^{-\eta} + 1 - s)^{-1/\eta} \text{ if } \eta \neq 0 \text{ and } f(a) = Aa^s \text{ if } \eta = 0. \tag{9}$$

Here $\eta > 0$ determines the elasticity of input substitution through $\sigma = 1/(1+\eta)$, A is a productivity parameter and $0 < s < 1$ is the share of capital income in total production. As utility functions we use:

$$V_1(l) = l^{1+\alpha_1}/(1+\alpha_1) \text{ and } V_2(c) = c^{1-\alpha_2}/(1-\alpha_2) \tag{10}$$

where $\alpha_1 \geq 0$ and $0 < \alpha_2 < 1$. This specification then leads to an offer curve with constant elasticity $\varepsilon_\gamma = (1+\alpha_1)/(1-\alpha_2) > 1$. As in Grandmont et al. [16] we choose

$$\delta = 0.1, A = \delta/s \text{ and } s = 1/3.$$

Then substituting (9) and (10) into the equations (4), we get a dynamical system:

$$\begin{aligned} k_t &= (sAa_t^{-\eta-1}(sa_t^{-\eta} + 1 - s)^{-1-1/\eta} + 1 - \delta)k_{t-1} \\ A(1-s)a_{t+1}^{-1}(sa_{t+1}^{-\eta} &+ 1 - s)^{-1-1/\eta} = (k_{t-1}/a_t)^{\varepsilon_\gamma}(k_t)^{-1} \end{aligned} \tag{11}$$

It is easy to see that the monetary steady state exists and is actually equal to $(1, \bar{k})$ independently of η, where $\bar{k} = (A(1-s))^{1/(\varepsilon_\gamma - 1)}$. If we choose δ and s as above and $\eta \neq (1-s)/s$ the system (11) has a unique interior fixed point and is invertible in a neighbourhood $\hat{U}$ of $(1, \bar{k})$: observe that the feasible set $\Lambda \subset \hat{U}$. In fact, one can fix the parameters $(\eta, \varepsilon_\gamma)$ such that both eigenvalues go through -1, see Grandmont et al. [16].

5.2 Description of dynamics for various choices of parameters

For this choice of the utility and production functions the only feasible parameters in the trace determinant plane are located to the left of the half line (A, C) in figure 1. As we pointed out in section 2 for parameters inside the triangle (A, B, C) the unique interior steady state is stable. For parameters left to the segment (A, B) we have the following proposition:

Proposition 5.1 *To the left of the half line* (A, B) *in figure 1 the set* $\Lambda \in W^s(p)$.

Proof: In Grandmont et al. [16] it is shown that for all parameters considered there exists a set U containing the interior steady state such that restricted to U the map (11) is invertible. From section 2 we know that to the left of the half line (A, B) the interior steady state is a saddle and from the discussion above that it is unique: the autarkic steady state does not lie inside U. Next, as pointed out, along the half line (A, B) a period-doubling bifurcation of the interior fixed point occurs. Using a mathematica code we have computed the coefficients in the normal form of this bifurcation (see Guckenheimer and Holmes [19, p.158] and it turns out that the bifurcation is always subcritical and gives rise to a period two saddle orbit. This implies that for parameters to the left of the line (A, B) the unique fixed point is a saddle with negative eigenvalues and there are no other periodic orbits of period 2, see figure[3]5a. So for parameters to the left of the segment (A, B) theorem 1 applies and consequently the feasible set $\Lambda \in W^s(p)$ and we are done.

[3]All the phase space figures presented in this paper have been obtained by using the program Dunro [11].

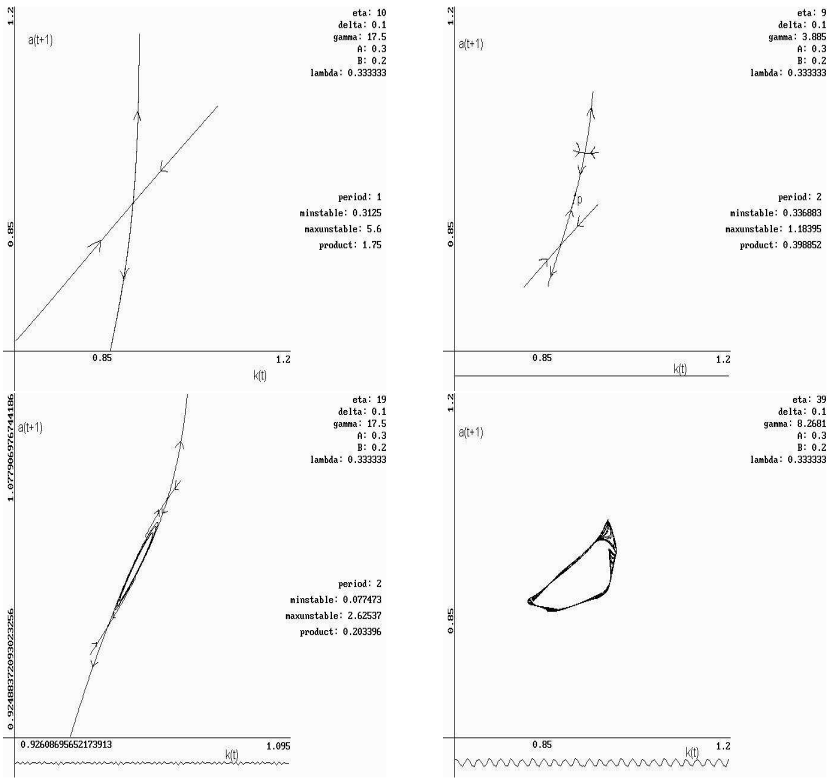

Figure 5: Here we present, in phase space, the different regimes. In (a) we are in the RBC regime where no sophisticated dynamics occurs, in (b) and (c) we are in the EBC and SE regimes, while in (d) we are again in the EBC regime. Observe that there is a one-to-one correspondence with figure 1.

□

For parameters inside the triangle (A, B, C) the attracting fixed point coexists with a period two orbit which actually has a saddle structure, see figure 5b To complete the picture we want to point out that for parameters bounded by the half lines (A, B), (A, C) and above the line segment (B, C) there is a repelling fixed point, a saddle orbit of period two and provided that the parameters are just above (B, C) there is an attracting circle with a rational or irrational rotation, see figure 5c. For parameters further above the circle may break up into a strange attractor (figure 5d), see Pintus, Sands and de Vilder [35].

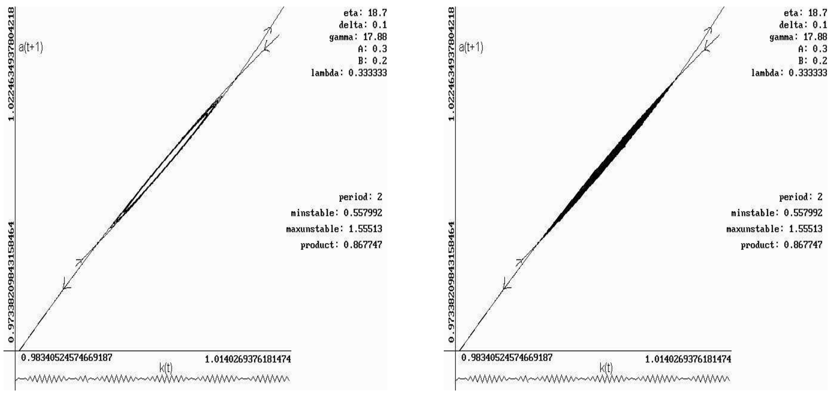

Figure 6: Close to the Bogdanov-Takens bifurcation the basin of attraction is determined by segments of the stable and unstable manifolds that surround the attracting fixed point (a), the basin corresponds to the shaded area in (b).

5.3 Dangers of linearization and restricting to stable manifolds: chaos near unit roots

As mentioned linearizing becomes less justifiable when the basin of the attracting fixed point is very small. For example, for parameters just north-east of B one obtains a phase picture as in figure 6a: the attracting fixed point, with complex eigenvalues, is surrounded by a period two saddle orbit. Segments of the stable manifolds of the period two saddle determine the boundary of the basin of attraction of the fixed point and as can be seen is very small indeed: the feasible set Λ (which also contains the stable and unstable manifolds of the period two orbit) is very small, see figure 6b. This particular configuration is very similar to the one reported by Farmer and Guo [14]. The reason for this is that the eigenvalues at their fixed point are close to one and so taking the second iteration of our map corresponds to theirs with the exception that there are now two additional fixed points instead of one. Nevertheless, one must conclude that a linearization only makes sense if the dynamics remains inside Λ; outside Λ almost all, in the sense of Lebesgue, orbits leave the positive quadrant. Moreover, in order to stay inside Λ one can not apply for example normally distributed i.i.d. shocks to the expectations. In fact, the linear part of the map near B can be brought in the following (normal) form

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \mapsto \begin{pmatrix} -1 & 1 \\ \beta_1 & -1+\beta_2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \tag{12}$$

where at B one has $\beta_1 = \beta_2 = 0$. For parameters left to the half line (A, B) and close to B, where the unique fixed point has a saddle structure, the stable and unstable generalized eigenspaces of (12) make a very small angle. So in terms of the nonlinear map the stable and unstable manifolds are also very close together and a point $x \neq p$ in the stable manifold can land on the unstable manifold by a small perturbation. Let us now show that our system displays stable chaotic dynamics while the eigenvalues at the fixed point are close to -1.

Theorem 5.1 *There exists values $(\varepsilon_\gamma, \sigma)$ such that the feasible set of system (11) contains stable chaotic dynamics.*

Proof: From Grandmont et al. [16] we know that there exists values $(\varepsilon_\gamma, \sigma)$ such that the eigenvalues are both -1. From the same paper we know that f is invertible in a neighbourhood U of p. Observe that the feasible set $\Lambda \subset U$. From the discussion in the appendix we learn that we have to transform the map at B (where both eigenvalues are -1) into the normal form (16). Crucial in this normal form are the coefficients given by (15). Using a matlab code we have computed these coefficients. From these computations we learn that $C(0) >> 0$ and $D(0) >> 0$. Therefore, $D_1(0) = -2D(0) - 6C(0) \neq 0$ and $C_1(0) > 0$ (see (19)). So we are in the situation in which the dynamics of the vector field approximation is as in figure 3. In terms of our discrete map this means that we have a generic 1:2 bifurcation. This implies that in a very small neighbourhood of the fixed point all sorts of complicated dynamic objects exists for parameters close to the bifurcation value where both eigenvalues pass through -1. Whether some of these dynamic objects are stable depends on the product of the eigenvalues of the period two saddle orbit. Using the program DUNRO [11] we have checked that the product of the eigenvalues for parameters near the 1:2 resonance bifurcation are larger than 0.9 but smaller than 1.0. Let p be the periodic orbit of saddle-type near the fixed point created by the bifurcation. Generically, see for example [27], there exists parameters so that $W^s(p)$ and $W^u(f(p))$ have tangential intersections which unfold generically (and similarly $W^s(f(p))$ and $W^u(p)$. It is well-known, see [33], that this means that there are strange attractors for nearby parameters. We need to show that these attractors lie in the feasible set Λ. To see this, note that the tangency occurs within U (more precisely, the pieces of $W^s(p)$, $W^s(f(p))$ and $W^u(p)$, $W^u(f(p))$ connecting p and $f(p)$ with two tangency points bound a region which lies in U). So we can take a region R and an integer n so that $R, \ldots, f^n(R)$ lie in U and so that $f^n\colon R \to \mathbf{R}^2$ as the parameter changes is a 'horseshoe family' (and so f^n is invertible on R. The strange attractors constructed by Mora and Viana, see [33], are contained in the set $\Omega = \{x; f^i(x) \in R \cup \cdots \cup f^n(R) \text{ for all } i \geq 0\}$ and so lie in U. $\square$

5.4 Stochastic fluctuations near unit roots

It is generally believed that deviations from common trend in business cycles possess unit roots. In terms of time series this particular feature can be modeled by

$$x_{t+1} = Ax_t + \epsilon_t \tag{13}$$

where $x_t \in \mathbf{R}^n$, A a nxn-matrix with eigenvalues inside and close to the unit circle and ϵ_t for example an independent, identically and uniformly distributed process. Then

$$x_t = A^t x_0 + A^{t-1}\epsilon_0 + A^{t-2}\epsilon_1 + \cdots + \epsilon_{t-1} \tag{14}$$

In our case we shall assume that economic agents can only observe the variable every other time[4], and let A be the second iterate of the linear matrix (12). Then when $\beta_1, \beta_2 = 0$ both eigenvalues of this matrix are equal to one, but there is only one eigenvector. It is easy to see that for these parameters the matrix A^n is equal to $\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$ and so certain coefficients grow linearly with n. This implies that when $\beta_i \approx 0$ and both eigenvalues are less than unity the i.i.d. random shocks have longer-term effects than would be the case if we took simply the random walk model $x_{t+1} = x_t + \epsilon_t$. The time series, buffeted with a sufficiently small (depending on the size of the basin of attraction) i.i.d. uniformly distributed shock at each time step to the second equation of (12), look much more like the ones reported by King et al. [25], see figure 7a. In fact, if we apply on average at each 5-th time step a sufficiently small i.i.d. uniformly distributed shock to the second equation of (12) then we obtain similar time series, see figure 7. Moreover, if we would fix the parameters such that the original system is close to having a Bogdanov-Takens bifurcation then some additional irregularities may be observed if we would use the nonlinear model. If

[4] We could also introduce increasing returns to scale into the model and then we could obtain a 1:1 resonance bifurcation; for reasons of exposition we have omitted this exercise.

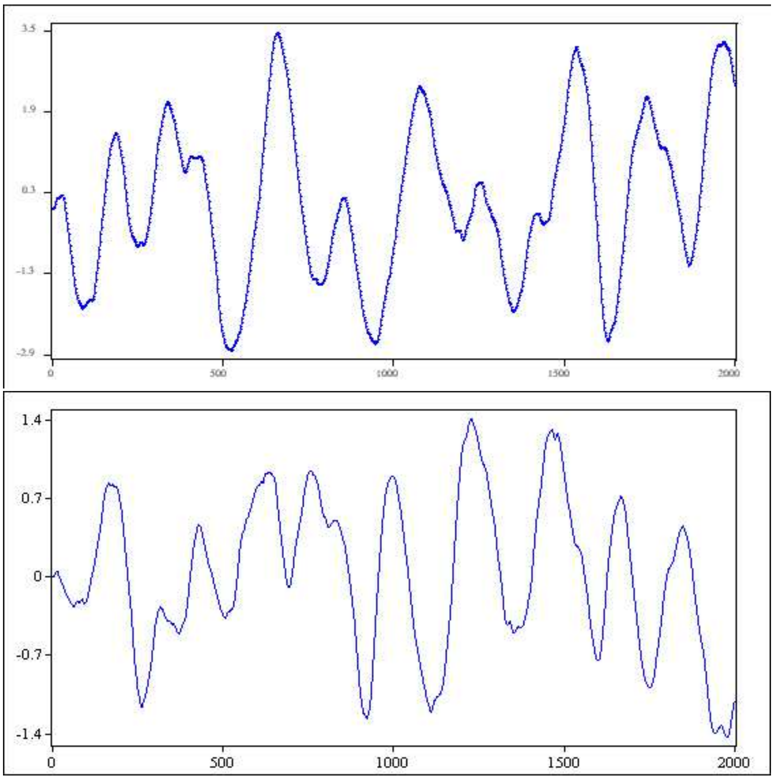

Figure 7: In (a) we present a time-series that we obtained by adding a small i.i.d. uniformly distributed shock to the second equation of (12). In (b) we followed the same procedure but now shocks were applied, on average, each 5-th time step. Compare these time series with the ones reported by Farmer and Guo [14] and King et al. [25].

we apply a unit root test (Phillips-Perron, see Stock [38]) on the series presented in figure 7 (a) it is not rejected on a 10% level, see table 1.

PP test statistic	-2.307281	1% critical value	-4.396
		5% critical value	-2.8648
		10% critical value	-2.5685

Table 1

On the other hand, if we would test for the presence of i.i.d. random increments using $(x_t - x_{t-1})$ the hypothesis will be rejected. Only in the case where we

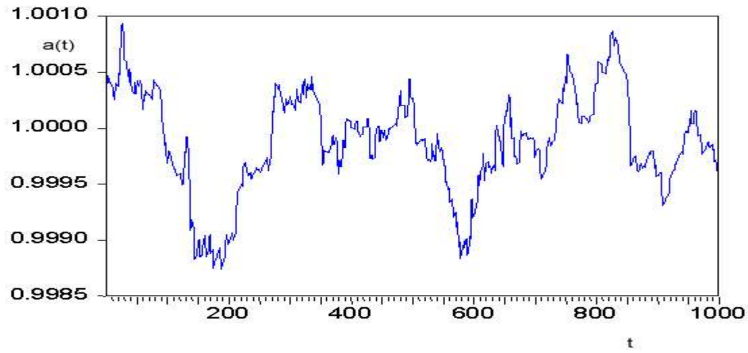

Figure 8: Here we present the time-series obtained by adding at each time step a small amount of i.i.d. uniformly distributed noise to the second equation of (11) shortly after a standard Hopf bifurcation has occurred. We have filtered out the periodic behaviour as explained in the text.

use as increments $(x_t - Ax_{t-1})$ the hypothesis of i.i.d. random variables will not be rejected. It has been argued that if a dynamic economic model displays a standard Hopf bifurcation similar results as above can be obtained. However, it is not hard to see that if the rotation of the circle is not close to 0 then the Phillips-Perron unit root test will be rejected; just because the time series fluctuate too much around the stationary steady state. Moreover, in a standard Hopf bifurcation one will not get an off-diagonal term in the properly chosen normal form as was the case in (12). This implies that if one corrects for the periodic behaviour induced by the Hopf bifurcation a standard random walk with i.i.d. random increments will be obtained. The resulting time series are however far less appealing from an economic point of view as may be seen from figure 8.

6 Conclusion

In this paper we have shown that one can find a region in parameter space of two-dimensional economic models in which the RBC, EBC en SE approaches to model business cycles are very much related. In fact, we have been able to show that for certain parameters in this region chaotic dynamics is a possible outcome, with multiple eigenvalues close to 1, which makes a linearization less justifiable. In addition, for different parameters in this region one is able to find linear approximations that have the same structure as the RBC models. However, in this case cycles were generated by animal spirits. Outside this region the different approaches seem mutually exclusive. That is, if the model has a unique interior steady state which has a saddle structure complicated deterministic phenomena related with the stable and unstable manifolds of the saddle are excluded. Moreover, if the model displays a standard Hopf bifurcation the resulting time series do not have much resemblance with the ones observed in reality. A similar conclusion can be made if the invariant circle, created in the Hopf bifurcation, transforms into a complicated set. It remains to be seen if similar results do occur in alternative and empirically relevant business-cycle models. For instance, one could build upon some results by Pintus [34] which show that resonant bifurcations are likely to be found in generalized versions of the RBC model *la* Farmer and Guo [14].

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A Appendix

A.1 The two-fixed point lemma

Complicated expectations-driven business cycles may occur in various economic models. For example, the general equilibrium models, suggested by Benhabib and Day [4] and Grandmont [15], focused on simple one-dimensional economies and required large income effects in order to get complicated cyclical behaviour. By introducing productive capital into similar types of models, which increases the dimension of the dynamics of the economy to two, one can show that both local regular (e.g. Reichlin [36], Woodford [42], Grandmont, Pintus and de Vilder [16]) and global irregular (de Vilder [40], Pintus, Sands and de Vilder [35]) cycles are compatible with dominant substitution effects. The main mechanism that accounts for the occurrence of complicated deterministic global fluctuations in the two-dimensional framework involves *intersections* of stable and unstable manifolds of a (periodic) saddle equilibrium (see de Vilder [40], Pintus, Sands and de Vilder [35] and Brock and Hommes [7], for example).
By contrast, a broad (widely known) class of two-dimensional dynamic economic models, such as the ones studied by King, Plosser and Rebelo [25] and Kydland and Prescott [28], have a unique fixed point with a saddle structure. It is not clear from the mathematical literature on the subject (see, for example, Guckenheimer and Holmes [19], Katok and Hasselblatt [24] or Palis and Takens [33]), whether complicated deterministic structures associated with intersections of stable and unstable manifolds can also be present in this widely used framework. More specifically, is it possible for the stable and unstable manifolds of a unique fixed point of a two-dimensional C^1 invertible map of the positive orthant of the plane to intersect non-trivially?
In this section it is shown that chaos cannot arise from such intersections in these models. That is, we show that a necessary condition for stable and unstable manifolds of a saddle stationary state to intersect non-trivially is that the map has, at least, one additional steady state with positive index; we refer to

this finding as *the two-fixed point lemma*. We obtain this result by exploiting the Lefschetz index theorem for vector fields [29]. So only in models with multiple steady states (see for example Hornstein [23], Farmer and Guo [14] and Boldrin and Rustichini [6], Cazzavillan et al. [10], Cazzavillan and Pintus [9]) this kind of chaos is possible.
We should emphasize that this appendix appeared as a Warwick preprint on November, 10, 1999, see [26]. Hirsch [21] has reported a similar result in February 2000.
This appendix is organized as follows. In the next section we define the types of dynamic economic models we have in mind. In that section we also introduce the notion of stable and unstable manifolds as well as some related results. In the third section we present the two-fixed point lemma and provide a sketch of the proof. In the section thereafter we give some concluding remarks. Finally, the formal proof of the two-fixed point lemma can be found in the last section.

A.1.1 The Framework

The results of this section apply to any two-dimensional model of the plane satisfying the axioms specified in the next subsection.

The dynamic economic model

In this section we shall assume that the economic model satisfies the following **Standing Assumptions**.

- The phase space is a simply connected open subset U of $\mathbf{R}^2$. For example U can be the positive quadrant $\mathbf{R}^2_+$ without boundary points. Moreover, we assume that time is discrete. We denote the state variables by $(x_n, y_n) \in U, \quad n \in \mathbf{Z}$.
- The dynamics of the economy is described by $(x_{n+1}, y_{n+1}) = f(x_n, y_n)$.
- We assume that fixed points of f are isolated.
- The map $f\colon U \to f(U)$ is C^1 and invertible.

The first three assumptions are extremely general, and are used in a broad class of models. Models that we have in mind are, for example, King et al. [25], Kydland et al. [28] and Weibull [41]. The fourth assumption is more restrictive, because in some models f is not invertible. Whether one believes that the equations of motion also allow for backward motion, is perhaps a matter of taste.

Intersecting stable and unstable manifolds

We introduce the notion of *stable and unstable manifolds* of fixed points. That is, let $p \in U$ be a hyperbolic fixed point of f, so $f(p) = p$ and the Jacobian Df has two real eigenvalues λ_s and λ_u such that $|\lambda_s| < 1 < |\lambda_u|$. Then the stable and unstable manifolds of p are defined as follows.

$$W^s = \{x \in U \ ; \ f^n(x) \in U \text{ for all } n \geq 0 \text{ and } \lim_{n\to\infty} f^n(x) = p\}.$$

Since $f(U)$ can be not equal to U, it is possible that W^s has several connected components. Similarly, let

$$W^u = \{x \in U \ ; \ f^n(x) \in U \text{ for all } n \leq 0 \text{ and } \lim_{n\to-\infty} f^n(x) = p\}.$$

If $f(U) \subset U$, the unstable manifold can only have one connected component, but otherwise it is possible that it has many connected components. These manifold are smooth curves passing through p, tangent to the stable and unstable eigenspaces of $Df(p)$, respectively. Contrary to a linear specified model, in a nonlinear framework, the stable and unstable manifolds of p may intersect outside the saddle; these points of intersection of stable and unstable manifolds are known as homoclinic points. These are those points x for which $f^n(x) \to p$ as $n \to +\infty$ and as $n \to -\infty$, see figure 1. If p is a saddle fixed point and the stable and unstable manifolds of p intersect at a point $q \neq p$ then q is called a *homoclinic point* of p. The orbit of a homoclinic point is called a *homoclinic orbit*; each point in it is homoclinic.

A.1.2 The Main Result

In this section we present the main result.

The two fixed-point lemma

Let $f\colon U \to f(U)$ be as in the standing assumptions, and let $p \in U$ be a fixed hyperbolic saddle point of f with positive eigenvalues. Assume that the stable

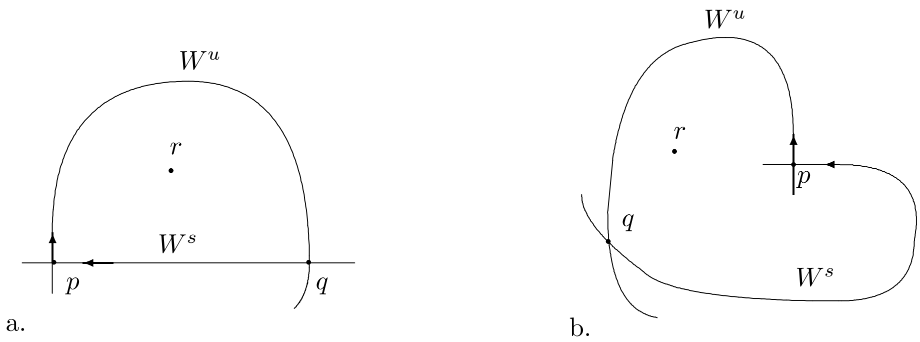

Figure 9: W^u and W^s of the saddle equilibrium p may intersect outside p. We have schematically drawn the situation where W^u intersects W^s in a point q. The point r in the figures is the additional fixed point that exists according to the two-fixed point lemma in section A.1.2.

and unstable manifolds of p have a point $q \neq p$ of intersection and that there are curves $\gamma_s \subset W^s$ and $\gamma_u \subset W^s$ in U connecting q and p. Then f has at least one additional fixed point r of positive index in the interior of the domain bounded by γ_s and γ_u, see figure 9 for a graphical illustration.

The index of a fixed point is defined in the last section.

Remark: If one assumes that $f(U) \subset U$ then the statement of the lemma can be simplified: there is then no need to assume the existence of the connecting curves γ_s and γ_u. Indeed, in this case if the stable and unstable manifold intersect in some point q ($\neq p$), then there exists an integer n such that $f^n(q)$ belongs to the local stable manifolds and there is a piece of this manifold connecting p and $f^n(q)$. Moreover, the piece of the unstable manifold situated between p and $f^n(q)$ is connected because the unstable manifold is connected (here we use again $f(U) \subset U$).

We should also emphasize that we only consider fixed points in the open set U (and not on the boundary). The additional fixed point of positive index the lemma above asserts, is actually in the open set U. So if there are several saddle fixed points (with positive eigenvalues), then since these have index -1, the conclusion of the lemma still applies. If $f(U) = U$ then one can use an extension of a result of Brouwer, see the last lemma in [12]. In that case, f has no recurrent behaviour: for each point $x \neq p$ (where p is a fixed point) there is a neighborhood O so that $f^n(O) \cap f^m(O) = \emptyset$ for all $n \neq m$.

The proof of the lemma can be found in the last section. Here we just give a sketch. The main tool that we use to prove the two-fixed point lemma is the Lefschetz index theorem[5] for vector fields [29]. Roughly speaking, the index of a vector field V on the plane with respect to an oriented Jordan curve Γ in the plane (i.e. a continuous closed curve without self-intersections on which direction is defined) is equal to the number of full turns the vector field produces when Γ is traversed once (for a formal definition see the proof below). However, the index of V cannot be defined if it has a singularity on Γ. The index is always an integer and stays constant if one continuously deforms Γ. Provided this deformation does not create singularities for V on the curve. Assuming that these conditions are satisfied one can apply the Lefschetz index theorem for vector fields, which says the following:

Lefschetz Index Theorem
Let Γ be a Jordan curve and V a continuous vector field defined on Γ and the set bounded by Γ. Suppose that V has no singularities on Γ and that all singularities inside Γ are isolated. Then the sum of the indices of the singularities of V inside Γ is equal to the index of V on Γ.

As a vector field V we shall use $V(x) = f(x) - x$ which implies that the index of V at a singularity s is equal to the index of s as a fixed point of f. The index of a hyperbolic fixed point may be defined in terms of the eigenvalues of the Jacobian matrix evaluated at the fixed point[6], see Table 1.

To apply this theory to our map f, we shall define a curve Γ in the last section by using segments of the stable and unstable manifolds of the fixed point p. Observe that the vector field $V(x) = f(x) - x$ is defined on Γ and its interior D, see figure 10. The final step is the observation that V restricted to Γ makes one full turn when the curve Γ is traversed once (to prove this, we shall use a deformation argument in the proof below). This implies that the index of the

[5] For other applications of this theorem to economic theory, see for example Balasko [3], Guesnerie and Woodford [18] and Mas-Colell [31].

[6] Although not generic we have to point out that one may construct fixed points with, for example, index $+2$.

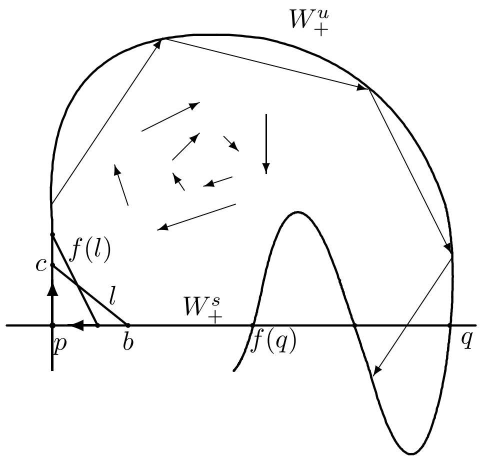

Figure 10: The closed curve Γ consists of l and the pieces $W^s(b,q)$ and $W^u(c,q)$ of stable and unstable manifolds. If the curve Γ is traversed once, V makes one full turn and so the index of V with respect to Γ is equal to $+1$.

vector field V with respect to Γ is $+1$. Hence, from the Lefschetz Index Theorem it follows that there must be at least one singularity of V in the interior of γ of positive index. This singularity corresponds, by the definition of the vector field, with a fixed point of the map f.

Remark: By taking the second iterate of the map f one can also account for the orientation reversing cases as well as for the case where both eigenvalues of the saddle are negative by considering the second iterate of f. In these cases the statements will be different since a priori one cannot exclude the presence of a period two orbit instead of an additional fixed point.

A.1.3 Conclusion

In this section we have shown that stable and unstable manifolds of a saddle equilibrium of an invertible two-dimensional dynamic economic model, cannot intersect non-trivially if the model has no other steady states. This means that the system cannot have homoclinic intersections causing chaotic dynamics unless the map has a fixed point of positive index in the (open) domain of definition.

	Eigenvalues	Index of fixed point	Description
1	$\|\lambda_1\| < 1$, $\|\lambda_2\| < 1$	$Ind_{Df}(p) = 1$	contracting
2	$\|\lambda_1\| > 1$, $\|\lambda_2\| > 1$	$Ind_{Df}(p) = 1$	expanding
3	$\|\lambda_1\| = \|\lambda_2\| = 1 \neq \lambda_i$	$Ind_{Df}(p) = 1$	elliptic
4	$0 < \lambda_1 < 1 < \lambda_2$	$Ind_{Df}(p) = -1$	hyperbolic saddle
5	$\lambda_1 < -1 < \lambda_2 < 0$	$Ind_{Df}(p) = 1$	hyperbolic saddle with rotation

Table 1: The index (called $Ind_{Df}(p)$) of a fixed point p of a map f may be defined in terms of the eigenvalues of the Jacobian matrix Df evaluated at p. For orientation preserving maps, generically, five different cases can be distinguished for the index of a fixed point.

So behaviour as observed by Grandmont, Brock, de Vilder and others for expectation driven business cycle models, cannot occur in that case. In other words, only if the two-dimensional dynamic economic model has multiple steady states or if the model does not satisfy the conditions stated in standing assumptions, global analysis is required. Only then one might have "unexpected" complicated deterministic structures.

A.1.4 Proof of the two-fixed point lemma

We start by providing some useful definitions. First, we define the degree of a circle map $\phi : S^1 \to S^1$, where S^1 is equal to $\mathbf{R}$ modulo 1. We identify S^1 also with the unit circle in $\mathbf{R}^2$ which has the anti-clockwise orientation.

Definition A.1 *Let ϕ be a continuous map from the circle S^1 into itself. Let Φ be any lift of ϕ to $\mathbf{R}$ (so $\phi(x) = \Phi(x) \pmod 1$ and Φ is continuous). The degree of ϕ is $\Phi(x+1) - \Phi(x)$, where $x \in \mathbf{R}$ is any point. The degree is independent of the choice of Φ and of x.*

A Jordan curve is an injective map $\gamma \colon S^1 \to \mathbf{R}^2$. We shall write $\Gamma = \gamma(S^1)$. By Jordan's theorem (see any book on topology or for example page 730 of [24]) we know that such a curve divides the plane into two components: one bounded and one unbounded (if for example U is the positive quadrant, then the 'unbounded component' is $\partial\mathbf{R}^2_+$). We shall only consider piecewise smooth curves, and say that $\gamma \colon S^1 \to \mathbf{R}^2$ is positively oriented if going forward along the curve, the unbounded component is on the right hand side (so positive orientation is the anti-clockwise orientation). In a similar fashion we can define the *negative* orientation of a curve. Next we define the index of a vector field with respect to γ.

Definition A.2 *Let $\gamma\colon S^1 \to \mathbf{R}^2$ be a Jordan curve, $\Gamma = \gamma(S^1)$ and $V : \Gamma \to \mathbf{R}^2$ be a vector field which nowhere takes the value* 0 *(has no singularities). Let Γ be parameterized by some map $\gamma : S^1 \to \Gamma$ which preserves orientation. The* index *of V with respect to Γ is equal to the degree of the circle map ϕ defined by*

$$\phi : x \mapsto \frac{V(\gamma(x))}{|V(\gamma(x))|}.$$

Next we define the index of a singularity of a vector field.

Definition A.3 *Let V be a vector field defined on an open set U and let $p \in U$ be an isolated singularity of V. Let Γ be a Jordan curve surrounding p in U, separating p from any other singularities of V. The* index *of V at p is defined to be the index of V on Γ. This index is an integer, and is independent of the choice of Γ.*

We define the vector field V by $V(x) = f(x) - x$. Then by definition, the index of a fixed point p of f is equal to the index of the vector field V at p. In the case that the Jacobian Df at p has no eigenvalues equal to 1 (the fixed point is hyperbolic), the index can be defined as $(-1)^{card(i|\lambda_i>1,\, \lambda_i\in\mathbf{R})}$ where λ_i $(i = 1, 2)$ are the eigenvalues of Df evaluated at p. In table 1 (above in the text) we have summarized the 5 generically occurring cases.

We are now ready to prove the two-fixed point lemma which we recall here:

The two fixed-point lemma

Let $f\colon U \to f(U)$ be as in the standing assumptions, and let $p \in U$ be a fixed hyperbolic saddle point of f with positive eigenvalues. Assume that the stable and unstable manifolds of p have a point $q \neq p$ of intersection and that there are curves $\gamma_s \subset W^s$ and $\gamma_u \subset W^s$ in U connecting q and p. Then f has at least one additional fixed point r of positive index in the interior of the domain bounded by γ_s and γ_u, see figure 9 for a graphical illustration.

Proof of the two fixed-point lemma: Let us introduce some notation that we will use later on. If x, y are points on W^u then $W^u(x, y)$ denotes a piece of W^u bounded by the points x and y. The same notation is applied to W^s.

Let W_+^s and W_+^u be components of the stable and unstable manifolds of p which intersect. Remember that W_+^u is contained in U. We have the 2 curves γ_s and γ_u connecting p and q. Moreover, we can assume that these curves intersect only in the points p and q. Indeed, if it is not the case, then we can take the first point q' of intersection of γ_u with γ_s so that $W^u(p, q') \cap \gamma_s = \{p, q'\}$. Now, if we denote $\gamma_u' = W^u(p, q')$ and $\gamma_s' = W^s(p, q')$, we obtain two curves intersecting only in p and q'.

Let us now define a closed Jordan curve Γ and a domain D bounded by Γ. Let O be a neighborhood of p on which Hartman-Grobman linearization is possible (see Katok and Hasselblatt [24, ch.6,p.260]) and so that $W^u(p, f(q)) \cap O$ and $W^s(p, q) \cap O$ have only one component. Take a straight line segment $l = [b, c]$ close to p in O connecting $c \in W_+^u$ with $b \in W_+^s$, where the points b and c are very close to the origin and such that $f(l)$ and $W^s(p, f^{-1}(b))$ are also in O. The curve $\Gamma = W^u(c, q) \cup W^s(q, b) \cup l$ forms a closed Jordan curve in the simply connected domain U and by Jordan's theorem Γ bounds a simply connected region (i.e. a disc) D which is contained in U. We choose on Γ a positive orientation. The origin may or may not be in D, see figure 1 for the two possible cases.

Next we define a vector field V on the closure of D by considering $V(x) = f(x) - x$. Any zero of V is a fixed point of f. By construction V has no zeroes on the boundary Γ of D. This implies that the index $ind_\Gamma(V)$ is well-defined, where the index of a vector field is defined as in definition A.2. Our aim is to show that $ind_\Gamma(V)$ is equal to $+1$. Using the Lefschetz index theorem for vector fields (see the core of the text) this implies the following proposition

Proposition A.1 *There is a singularity of V in the interior of D of positive index corresponding to a fixed point $r \neq p$ of f.*

Proof: By the Lefschetz index theorem for vector fields the sum of the indices of singularities of V in the interior of D is equal to $ind_\Gamma(V)$ which is equal to $+1$ as we shall show below. By the definition of V we have that a singularity of

V corresponds to a fixed point of f. Hence, if $p \notin D$ then this gives the result immediately. If $p \in D$ then since the index of the fixed point p is -1 (it is a saddle with positive eigenvalues, see table 1), we must have other fixed points in D in order to have that the sum of the indices equals $+1$. $\square$

To prove that the index of V w.r.t. the boundary of D is equal to $+1$ we will continuously deform the vector field on the boundary without creating new singularities. We first define the notion of a *rotational vector field*.

Definition A.4 *A vector field V defined on a Jordan curve Γ is called* rotational *if it has no singularities on Γ and if for any $x \in \Gamma$ the point $x+V(x)$ also belongs to Γ.*

Proposition A.2 *A rotational vector field has index $+1$.*

Proof: By an isotopy of the curve (and a corresponding one of the vector field) we may assume that Γ is a circle. Define a new vector field N which at a base point x points to the center of the circle. Next consider a deformation of the vector field defined by

$$V_\lambda(x) = (1-\lambda)V(x) + \lambda N(x),$$

where $0 \leq \lambda \leq 1$. Notice that $V_0(x) = V(x)$ and that $V_1(x) = N(x)$ and that $V_\lambda(x)$ is not equal to 0 for $0 < \lambda < 1$ for all $x \in \Gamma$. Indeed, if this would not be true then $(1-\lambda)V(x)+\lambda N(x) = 0$ and so $V(x) = -\lambda/(1-\lambda)N(x)$ which would mean that the vector field points outside the unit disc, a contradiction. Since the index of a vector field V_λ is an integer and depends continuously on the parameter, the index of V_λ is a constant. In particular, the indices of $V_0 = V$ and $V_1 = N$ are equal. Obviously the index of N is equal to $+1$, it follows that the index of V is also $+1$. $\square$

To apply this proposition to the vector field V we have to continuously deform V into a rotational vector field without creating singularities. To see why it fails to be rotational, notice that a point in $\Gamma \cap W_+^s$ close to q, is not mapped into Γ. Therefore we deform V, and subdivide Γ in four segments

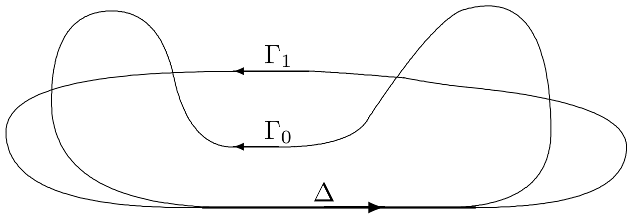

Figure 11: The curves Γ_i from proposition A.3 and the deformation of the vector field V.

$W^u(c, f^{-1}(q))$, $W^u(f^{-1}(q), q)$, $W^s(q, f^{-1}(b))$ and $W^s(f^{-1}(b), b) \cup l$. Obviously V is well defined on these segments and it has the rotational property on the segments $W^u(c, f^{-1}(q))$ and $W^s(q, f^{-1}(b))$. It remains to be shown that V can be deformed such that it also has the rotational property on the remaining two segments. We use the following proposition that is a standard result from topology:

Proposition A.3 *Take two Jordan curves (Γ_0 and Γ_1) which have the same orientation, suppose that these two curves have a common segment Δ and moreover suppose that along this segment Γ_0 and Γ_1 are oriented in the same way (see figure 11). Then there exists a homotopy from the curve $\Gamma_0 \setminus \Delta$ to the curve $\Gamma_1 \setminus \Delta$ without creating intersections with the segment Δ.*

Proof: Since we assume that Γ_i are piecewise smooth, there are curves $\tilde{\Gamma}_i$ near Γ_i homotopic to Γ_i which intersect transversally. Then proceed as in chapter 8 of [20]. $\square$

Notice that Γ_0 and Γ_1 are allowed to intersect in some points which are not in Δ.

We next apply this result to the curves $\Gamma_0 = W^u(p, q) \cup W^s(p, q)$, $\Gamma_1 = W^u(p, f(q)) \cup W^s(f(q), p)$ and $\Delta = W^s(f(q), p) \cup W^u(p, q)$. Γ_0 and Γ_1 are closed curves without self-intersection because of the choice of the point q. Notice also that $\Gamma_1 = f(\Gamma_0)$. If we orient these curves in the direction of the unstable manifold, then they will have the same orientation because f is an orientation preserving. Moreover, these curves have the same orientation along Δ. Thus we can apply the previous proposition and obtain a homotopy from $W^u(q, f(q))$ to $W^s(q, f(q))$ without crossing $W^u(p, q)$. Let us denote this homotopy by $\psi_\lambda : W^u(q, f(q)) \to \mathbf{R}^2$, $\lambda \in [0, 1]$, so $\psi_0 = Id$, $\psi_1(W^u(q, f(q))) = W^s(q, f(q))$.

Now we can define a deformation of the vector field V on the segment $W^u(f^{-1}(q), q)$ by

$$V_\lambda(x) = \psi_\lambda(f(x)) - x,$$

where $x \in W^u(f^{-1}(q), q)$ and $0 \leq \lambda \leq 1$. Obviously, $V_0 = V$ because $\psi_0 = Id$, V_λ is never singular because $\psi_\lambda(x)$ is never in $W^u(p, q)$ and V_1 satisfies the rotational property on the segment $W^u(f^{-1}(q), q)$ for the curve $\Gamma = \partial D$.

Since in the small neighborhood O of p (the segments $l = [b, c]$, $f(l)$ and $W^s(p, f^{-1}(b))$ are contained in O), f is almost a linear map, it is easy to see that one can continuously deform the vector field V on $l \cup W^s(f^{-1}(b), b)$ without creating singularities in such a way that the vector field becomes rotational: simply again use proposition A.3.

So, we have shown that V can be continuously deformed on the curve $\Gamma = \partial D$ to a rotational vector field without creating singularities. Once again, since the index depends continuously on the parameter this implies that the index of V on Γ is equal to the index of the rotational vector field which is $+1$. Together with proposition A.1 this proofs the lemma. $\square$

A.2 The normal form near a Bogdanov-Takens bifurcation

Even if two eigenvalues are equal to -1 this is not sufficient for having a 'generic' 1:2 resonance bifurcation. To check this, one also needs to compute coefficients in the normal form, for parameters at the 1:2 bifurcation value. One algorithm for doing the normal form computation is described in for example Kuznetsov [27, pp.369-373]. The first step is to translate the steady state to $\underline{0} = (0, 0)$ and the 1:2 bifurcation parameter value of α to $(0, 0)$ (and relabel this parameter as β). Next linear changes of coordinates are performed such that the system is transformed into:

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \mapsto \begin{pmatrix} -1 & 1 \\ \beta_1 & -1+\beta_2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} F(u,\beta) \\ H(u,\beta) \end{pmatrix}, \tag{15}$$

where $\beta = (\beta_1(\alpha), \beta_2(\alpha))$ and $F, H = O(||u||^2)$. A smooth invertible nonlinear change of coordinates smoothly depending on the parameters is used to transform (15) into the following map (see lemma 9.8 in [27])

$$\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \mapsto \begin{pmatrix} -1 & 1 \\ \beta_1 & -1+\beta_2 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} + \begin{pmatrix} 0 \\ C(\beta)\xi_1^3 + D(\beta)\xi_1^2\xi_2 \end{pmatrix} + O(||\xi||^4). \tag{16}$$

where $C(\beta)$ and $D(\beta)$ are smooth functions. In fact, $C(\beta)$ and $D(\beta)$ depend on the third order Taylor expansion of F and H w.r.t. the variable u. In particular, on page 377 of [27] it is shown that if $C(0) \neq 0$ and $D(0) + 3C(0) \neq 0$ then a 1:2 resonance bifurcation occurs. Here $C(0)$ and $D(0)$ are given by:

$$\begin{array}{rl} C(0) = & h_{30}(0) + f_{20}(0)h_{20}(0) + h_{20}^2(0)/2 + h_{20}(0)h_{11}(0)/2 \\ D(0) = & h_{21}(0) + 3f_{30}(0) + f_{20}(0)h_{11}(0)/2 + \frac{5}{4}h_{20}(0)h_{11}(0) \\ & h_{20}(0)h_{02}(0) + 3f_{20}^2(0) + \frac{5}{2}f_{20}(0)h_{20}(0) + \frac{5}{2}f_{11}(0)h_{20}(0) \\ & +h_{20}^2(0) + h_{11}^2(0)/2. \end{array} \tag{17}$$

A.3 The bifurcation diagram near a Bogdanov-Takens map

The general approach to the study of such resonant bifurcations is to embed (some iterate of) the map in a flow, and then to use the bifurcation analysis of the corresponding vector field (in this case the Bogdanov-Takens bifurcation). A description of the Bogdanov-Takens can be found in for example the books by Guckenheimer and Holmes [19] and Kuznetsov [27, pp. 373-381]. We shall follow the exposition in the latter book. In what follows we assume that a 1:2 resonance bifurcation occurs: the conditions on the coefficients in the normal form (16) are satisfied. First we aim to approximate the 1:2 resonance normal form (16) by a flow. Since at the 1:2 bifurcation value the eigenvalues both are -1 this cannot be done (near a singularity of a flow eigenvalues must be positive). However, if we denote for simplicity the normal form map by $\xi \mapsto \Gamma_\beta(\xi)$, then the second iterate of the map $\xi \mapsto \Gamma_\beta(\xi)$ can be approximated by a flow. In fact, see for example [27, theorem 9.3, p.376], $\Gamma_\beta^2(\xi)$ can be represented for all sufficiently small $||\beta||$ by

$$\xi \mapsto \phi_\beta^1(\xi) + O(||\xi||^4) \tag{18}$$

where ϕ^t_β is the flow of a planar system that is smoothly equivalent to the system

$$\begin{pmatrix} \dot{\zeta}_1 \\ \dot{\zeta}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \tau_1(\beta) & \tau_2(\beta) \end{pmatrix} \begin{pmatrix} \zeta_1 \\ \zeta_2 \end{pmatrix} + \begin{pmatrix} 0 \\ C_1(\beta)\zeta_1^3 + D_1(\beta)\zeta_1^2\zeta_2 \end{pmatrix}. \tag{19}$$

where $C_1(0) = 4C(0)$ and $D_1(0) = -2D(0) - 6C(0)$ (the $C(0)$ and $D(0)$ are the coefficients in the normal form (16)). The system (19) is the normal form **flow** for 1:2 resonances. Under our assumptions on the coefficients $C(0)$ and $D(0)$ one can scale the variables, parameters and time in the normal form for the flow (19) is such a way that the following system is obtained

$$\dot{\xi}_1 = \xi_2 \tag{20}$$

$$\dot{\xi}_2 = \epsilon_1\xi_1 + \epsilon_2\xi_2 + d\xi_1^3 - \xi_1^2\xi_2. \tag{21}$$

Here $d =$ sign $C(0) = 1$ and ϵ_i are parameters. Depending on the sign of d one can have two distinct bifurcation scenarios for the vector fields. In figure 3 we display the case that corresponds with $d = 1$. The alternative case, which essentially leads to the same conclusions, can be found in Kuznetsov [27, p. 379]. For the vector fields, the stable and unstable manifolds of the period two saddle orbits coincide along the curve C. By construction the second iterate of the original diffeomorphism is ϵ-close to the time-one map of the vector field (19). This means that it is close to a map for which the stable and unstable manifolds of the two fixed points of G^2 (which correspond to period two points of the map G) coincide. However, for diffeomorphisms it is ungeneric for stable and unstable manifolds to coincide, and so one has some parameter at which one expects to observe homoclinic intersections. Let us be more precise here. For the family of vector fields one has locally the situation as in figure 3. This means that for the nearby family of diffeomorphisms (compact) segments of the stable and unstable manifolds will change (with a change of the parameter). So at some parameter, the stable and unstable manifolds will intersect, see figure 2. In principle, it is possible that the stable and unstable manifolds will actually coincide, but by a (generic) perturbation one can make sure that these manifolds will intersect transversally. Since transversal homoclinic intersections are structurally stable it follows that generically the diffeomorphism will have homoclinic intersections for an open set of parameters. Moreover, it follows from the existing literature on the subject (see e.g. Palis and Takens [33]) that there must exist generically unfolding quadratic tangencies giving rise to all sorts of (stable) complicated deterministic dynamic phenomena. The corresponding invariant structures will strongly influence the observable dynamic behaviour in phase space.