Binary \([18,11]_2\) codes do not exist. Nor do \([64,53]_2\) codes.

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Abstract —
Brualdi et. al. [1] define the function \(l(m, r)\) to be the smallest \(n\) such that an \([n, n - m]_r\) code (a binary block code with block length \(n\), dimension \(n - m\) and covering radius \(r\)) exists. For \(r = 2\), the smallest unknown value of \(l(m, r)\) occurs for \(m = 7\). In this correspondence, \(l(7, 2) = 19\) is established by showing that \([18, 11]_2\) codes do not exist. Also, it is shown that \([64, 53]_2\) codes do not exist, implying that \(l(11, 2) \geq 65\).

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Keywords

Linear block codes, covering radius.
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Abstract — Brualdi et. al. [1] define the function \( l(m, r) \) to be the smallest \( n \) such that an \([n, n - m]r\) code (a binary block code with block length \( n \), dimension \( n - m \) and covering radius \( r \)) exists. For \( r = 2 \), the smallest unknown value of \( l(m, r) \) occurs for \( m = 7 \). In this correspondence, \( l(7, 2) = 19 \) is established by showing that \([18, 11]2\) codes do not exist. Also, it is shown that \([64, 53]2\) codes do not exist, implying that \( l(11, 2) \geq 65 \).

Index terms—Linear block codes, covering radius.
Two of the other questions raised in [1] are whether or not there exist $[23, 15]_2$ or $[64, 53]_2$ codes. The first of these questions was answered (negatively) in [2], while the second one will be dealt with in section III.

But first, some preliminaries.

II. PRELIMINARIES

The following well-known facts turn out to be helpful.

Let $C$ be an $[n, n - m]_2$ code and let $H$ be a parity check matrix for $C$. For every vector $\pi = (\pi_1, \ldots, \pi_t) \in F_2^t, 1 \leq t \leq m$, let $\#_t(H, \pi)$ be the number of columns of $H$ starting out with $\pi$ in the first $t$ positions. Then

$$N_{2,t}(H, \pi) \triangleq \begin{cases} 1 + \#_t(H, 0) + \sum_{\alpha \in F_2^t} \left( \#_t(H, \alpha) \right) & \text{if } \pi = 0 \\ 1 + \#_t(H, \pi) + \frac{1}{2} \sum_{\alpha \in F_2^t} \#_t(H, \alpha) \cdot \#_t(H, \alpha \oplus \pi) & \text{if } \pi \neq 0 \end{cases}$$

(where the $\oplus$ denotes binary vector addition) represents the number of sums of two or fewer columns of $H$, such that the sums correspond to $\pi$ in the first $t$ positions. Since there are $2^{m-t}$ binary vectors of length $m$ corresponding to $\pi$ in the first $t$ positions, Lemma 1 follows.

**Lemma 1.** If $H$ is a parity check matrix for an $[n, n - m]_2$ code, then for all $t, 1 \leq t \leq m$, it holds that

$$N_{2,t}(H, \pi) \geq 2^{m-t}, \forall \pi \in F_2^t.$$
Assume that $C$ is an $[n, k, d]$ code containing a codeword $c$ of Hamming weight $w = w(c)$. Then the residual code $C_1 = \text{Res}(C, c)$ is obtained from $C$ by deleting the coordinates where $c$ is nonzero. If only $w$ is important, $C_1$ is denoted by $\text{Res}(C, w)$. (Also, it will be convenient to refer to multi-level residual codes, such as $\text{Res}(\text{Res}(C, w_1), w_2)$, in the abbreviated form $\text{Res}(C, w_1, w_2)$ (and so on).) It is easy to see that Lemma 2 holds:

**Lemma 2** (see, e. g., [3]). If $w < 2d$, then $\text{Res}(C, w)$ is an $[n - w, k - 1, d_1]$ code where $d_1 \geq d - \lfloor w/2 \rfloor$.

**Definition.** Let $W(C) = \{w(c) | c \in C \setminus \{0\}\}$.

Lemma 2 can be generalized to

**Lemma 3.** For every $w \in W(C)$ such that $d \leq w < 2d$,

(a)

$W(\text{Res}(C, w)) \subseteq \{v_0 \mid \exists v_1 : 0 \leq v_1 \leq w, \text{ and } \{v_0 + v_1, v_0 + w - v_1\} \subseteq W(C) \}.$

(b)

$W(C) \subseteq \{w\} \cup \{v_0 + v_1 \mid 0 \leq v_1 \leq w \text{ and } v_0 \in W(\text{Res}(C, w)) \text{ and } \{v_0 + v_1, v_0 + w - v_1\} \subseteq W(C)\}.$

**Lemma 4([1]).** An $[l(m, r), l(m, r) - m]r$ code has minimum distance at least 3.

**Lemma 5 (The MacWilliams identities, [4], ch. 5, eq. (13)).** Let $C$ be an $[n, k]$ code, and let $C^\perp$ be the dual code of $C$. Also, in the rest of this correspondence,
let \( \{A_i\} \) and \( \{B_i\} \) be the weight distributions of \( C \) and \( C_\perp \), respectively. Then

\[
B_j = \frac{1}{2^k} \sum_{i=0}^{n} A_i \cdot P_j(n, i), \quad \forall j : 0 \leq j \leq n,
\]

where

\[
P_j(n, i) = \sum_{l=0}^{j} (-1)^l \binom{i}{l} \binom{n - i}{j - l}
\]

is a Krawtchouk polynomial.

**Definition.** Let \( d[n, k] \) denote the largest \( d \) such that an \( [n, k, d] \) exists. Verhoeff [5] provides a comprehensive table of bounds on \( d[n, k] \) for \( 0 \leq k \leq n \leq 127 \). We shall also use a couple of bounds from a more recent update of these tables [6].

Parts of the proofs in section IV rely on computer search to check (partial) parity-check matrices against Lemma 1.

**III. DO \([64,53]2\) CODES EXIST?**

In this section, assume that \( C \) is a \([64, 11]\) code and that \( C_\perp \) is a \([64, 53]2\) code. From Lemma 1, with \( t = 1 \), \( W(C) \subseteq \{27, \ldots, 38\} \).

First consider the case where the minimum distance \( d(C) = 27 \). Then, from Lemma 2, \( C_1 = \text{Res}(C, 27) \) is a \([37, 10, d_1 \geq 14]\) code. Since \( d[37, 10] = 14 \) ([5]), \( C_1 \) contains at least one codeword of weight 14. Hence, performing column permutations and row operations if necessary, the two first rows of the generator
matrix for $C$ can be assumed to be on the form:

$$
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But by Lemma 3 a), [5], and Lemma 1 with \( t = 2 \), this means that \( \text{Res}(C, 28) \) is a \([36, 10, 14]\) code with \( W(\text{Res}(C, 28)) \subseteq \{14, 18, 20, 22\} \). This contradicts Lemma 5, so we have

**Theorem 1.** There is no \([64, 53, 2]\) code.

### IV. DO \([18, 11]_2\) CODES EXIST?

Assume that \( C \) is an \([18, 7]\) code and that \( C^\perp \) is an \([18, 11]_2\) code. From Lemma 1, \( t = 1 \), it follows that \( W(C) \subseteq \{5, \ldots, 14\} \). Since, by [5], \( d[18, 7] = 7 \), three cases must be considered: \( d(C) = 5 \), \( d(C) = 6 \), and \( d(C) = 7 \). In each case, we attempt, but fail, to construct a generator matrix \( G \) (consisting of rows \( r_1, r_2, \ldots, r_7 \)) for \( C \). During this procedure we shall take the view that two \( t \times n \) matrices are *equivalent* if one can be obtained from the other by linear combinations and/or column permutations.

Of course, from Lemma 4, \( B_1 = B_2 = 0 \).

*The case* \( d(C) = 5 \). Without loss of generality, the first row of \( G \) can be taken to be

\[
r_1 : \quad 11111000000000000000
\]

By Lemma 2, \( C_1 = \text{Res}(C, 5) \) is a \([13, 6, d_1 \geq 3]\) code. However, \( d_1 = 3 \) violates Lemma 1 for \( t = 2 \), and by [5] \( d[13, 6] = 4 \), so \( C_1 \) is a \([13, 6, d_1 = 4]\) code. Then it can be assumed that \( r_2 \) has weight 4 when restricted to \( C_1 \). The only vector
that satisfies this condition as well as Lemma 1 \((t = 2)\) is (up to equivalence)

\[
    r_2 : 11100111100000000.
\]

\(Res(C, 5, 4)\) is a \([9, 5, d_2 \geq 2]\) code. From [5], \(d_2\) is either 2 or 3. In each case, up to equivalence there is only one possible third row of \(G\):

\[
    d_2 = 2 : r_3 : 11010110011000000,
\]

\[
    d_2 = 3 : r_3 : 11010110011000000.
\]

If \(d_2 = 2\), then \(C_3 = Res(C, 5, 4, 2)\) is a \([7, 4, d_2 \geq 1]\) code. Using Lemma 1 with \(t = 4\) for all possible nonequivalent fourth rows reveals that \(W(C_3) \subseteq \{2, 4, 6\}\), and that, if \(r_4\) has weight 2 when restricted to \(C_3\), it is the unique (up to equivalence) vector

\[
    r_4 : 10111111111100000.
\]

A similar reasoning with \(t = 5\), together with [5], shows that \(C_4 = Res(C, 5, 4, 2, 2)\) is a (unique (since the dual code has minimum distance 3)) \([5, 3, 2]\) code, and that also the fifth row is essentially unique. Hence, the fifth row and the lower right corner of \(G\) can be assumed to be on the form:

\[
    r_5 : 0000100111111110000
\]

\[
    r_6 : 10110
\]
However, it is a manageable task to check that there is no way to complete $G$ so as to satisfy Lemma 1.

If, on the other hand, $d_2 = 3$, then $C_3 = Res(C, 5, 4, 3)$ is a $[6, 4, d_2 = 2]$ code, and the only fourth row of $G$ that survives Lemma 1 ($t = 4$) is, up to equivalence

$$r_4 : \ 10111111110110000,$$

$C_4 = Res(C, 5, 4, 3, 2)$ is a $[4, 3, d_4 \geq 1]$ code. But there is (by checking all the possibilities against Lemma 1 for $t = 5$) no way to add a fifth row with weight restricted to 1 or 4 in $C_4$, hence $C_4$ is a $[4, 3, 2]$ code without the all-one codeword; an obvious contradiction.

The case $d(C) = 6$. Let the first row of $G$ be

$$r_1 : \ 11111100000000000.$$

$C_1 = Res(C, 6)$ is a $[12, 6, d_1]$ code, where $d_1$ is either 3 or 4. If $d_1 = 3$, then $C_2 = Res(C, 6, 3)$ is a $[9, 5, d_2]$ code. If $d_2 = 2$, by arguments similar to those above, parts of $G$ can be specified as follows:

$$r_2 : \ 111000111000000000$$

$$r_3 : \ 110000000$$
Again, it is impossible to complete $G$ to satisfy Lemma 1. (The computer program found two ways to complete $r_3$, a total of 7 ways to complete $r_3$ and $r_4$, a total of 40 ways to complete $r_3, r_4, r_5,$ and $r_6$ but no way to complete all rows).

If $d_2 = 3$, parts of $G$ can be specified as follows:

$r_2 : 1110001110000000000$

$r_3 : 11100000$

$r_4 : 11000$

Now $C_4 = Res(C, 6, 3, 3, 2)$ is a $[4, 3, d_4 \geq 1]$. However, it is not possible to complete $G$ if $r_5$ contains (1000) or (1111) in $C_4$, again a contradiction.

Next in consideration is the subcase $d_1 = 4$. There are two choices for $r_2$, denoted (A) and (B):

$(A) : r_2 : 111001111000000000$
\[(B): \quad r_2 : \quad 1110001110000000\]

In either case, \(C_2 = Res(C, 6, 4)\) is an \([8, 5, 2]\) code. Using Lemma 1 for \(t = 3\), up to equivalence this leaves only the same single choice for \(r_3\) in cases (A) and (B):

\[r_3 : \quad 11001011001100000\]

From Lemma 2, \(C_3 = Res(C, 6, 4, 2)\) is a \([6, 4, d_3 \geq 1]\) code. However, in both cases (A) and (B), it turns out to be impossible to satisfy Lemma 1 (\(t = 4\)) if \(r_4\) has weight 1 when restricted to \(C_3\), so \(d_3 = 2\), and we can assume that \(r_4\) has weight 2 when restricted to \(C_3\). Now, \(C_4 = Res(C, 6, 4, 2, 2)\) is our old friend the \([4, 3]\) code. Again, in both cases (A) and (B), it is impossible to complete \(G\) with \(r_5\) having one of the weights 1 or 4 in \(C_4\).

**The case** \(d(C) = 7\). By Lemmas 1 and 2, and [5], \(W(C) \subseteq \{7, 8, 10, 11, 12, 14\}\). Then \(C_1 = Res(C, 7)\) is an \([11, 6, 4]\) code. From Lemma 2 and Lemma 3 a), \(W(C_1) \subseteq \{4, 5, 6, 8, 9\}\). However, if \(C_1\) has odd-weight codewords, then exactly half of the codewords have odd weight, which contradicts Lemma 5. Lemma 5 shows that in fact \(W(C_1) = \{4, 6, 8\}\). But then from Lemma 3 b), \(W(C) \subseteq \{7, 8, 11, 12\}\). Lemma 5, for \(0 \leq j \leq 3\), together with
the fact that $A_7 + A_{11} = 64$, then imply

$$A_7 + A_8 + A_{11} + A_{12} = 127$$
$$A_8 + 4A_{11} + 5A_{12} = 263$$
$$3A_{11} + 5A_{12} = 194$$
$$5A_{12} = 170 - 24B_3$$
$$0 = 2B_3 - 5.$$  

The last equation is an obvious contradiction. Thus we have shown

**Theorem 2.** There is no $[18, 11]_2$ code.
REFERENCES


