Characterization of
\{2(q + 1) + 2, 2; t, q\}\textsuperscript{-}minihypers
in \(\text{PG}(t, q)\) (\(t \geq 3, q \in \{3, 4\}\))

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Abstract

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A set \(F\) of \(f\) points in a finite projective geometry \(\text{PG}(t, q)\) is an \((f, m; t, q)\)-minihyper if \(m \geq 0\) is the
largest integer such that all hyperplanes in \(\text{PG}(t, q)\) contain at least \(m\) points in \(F\). Hamada and Deza
(1988) characterized all \{2(q + 1) + 2, 2; t, q\}\textsuperscript{-}minihypers for \(t \geq 3, q \geq 5\). Hamada (1987, 1989) also
determined the cases of \(t = 2, q \geq 3\). In this paper we characterize \{2(q + 1) + 2, 2; t, q\}\textsuperscript{-}minihypers for
\(t \geq 3, q \in \{3, 4\}\). In addition to the previously known constructions, we describe a new \{10, 2; 3, 3\}\textsuperscript{-}minihyper.

1. Introduction

Let \(V(n; q)\) be an \(n\)-dimensional vector space over \(\text{GF}(q)\). If \(\mathcal{C}\) is a \(k\)-dimensional
subspace in \(V(n; q)\) such that every nonzero vector in \(\mathcal{C}\) has a Hamming weight (i.e.,
the number of nonzero coordinates) of at least \(d\), then \(\mathcal{C}\) is denoted an \([n, k, d; q]\)-code. The well-known Griesmer bound [1] states that

\[
n \geq \sum_{i=0}^{k-1} \left\lfloor \frac{d}{q^i} \right\rfloor
\]

(1.1)

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where \( \lceil x \rceil \) denotes the smallest integer \( \geq x \).

A coding theory problem that has been the subject of considerable research is the following.

**Main Problem.** Characterize all \([n, k, d; q]\)-codes meeting bound (1.1) with equality.

It is easy to show that if an \([n, k, d < qk^{-1}; q]\)-code meets bound (1.1) with equality, then a generator matrix of the code must have \( n \) pairwise linearly independent columns. Thus, in this case\(^1\), it is often convenient to think of a generator matrix as a set of points in the finite projective geometry \( \text{PG}(k - 1, q) \).

A set \( F \) of \( f \) points in a finite projective geometry \( \text{PG}(t, q) \) is an \( \{f, m; t, q\} \)-minihyper (also known as a min-hyper) if \( m \geq 0 \) is the largest integer such that all hyperplanes in \( \text{PG}(t, q) \) contain at least \( m \) points in \( F \).

Let \( v_t = (q^t - 1)/(q - 1) \) be the number of points in a finite projective geometry \( \text{PG}(t, q) \). The matrix whose column set is \( F \) generates an anticode of length \( f \) and maximum distance \( \delta = f - m \). Hence, the complement of an \( \{f, m; k - 1, q\} \)-minihyper is the column set of a generator matrix of a \([v_k - f, k, q^{-1} - f + m; q]\)-code, if its points span \( \text{PG}(k - 1, q) \).

Let \( \mathcal{F}(\mu_1, \ldots, \mu_h; t, q) \) be the family of sets of points in \( \text{PG}(t, q) \) that can be obtained by taking a union of \( h \) disjoint flats of dimensions \( \mu_1, \ldots, \mu_h \), respectively, in \( \text{PG}(t, q) \). (For example, in this notation \( \mathcal{F}(0, 1; t, q) \) is the family of sets consisting of one line, and one point not on the line, in \( \text{PG}(t, q) \).) Also, let \( \mathcal{W}(\theta; t, q) \) be the family of sets \( V \setminus S \) of points that can be obtained by removing a \((q + \theta - \varepsilon)\)-arc \( S \) from a \( \theta \)-flat \( V \) in \( \text{PG}(t, q) \).

It is convenient to apply the theory of minihypers to try to solve the Main Problem. In a series of papers, this approach has been followed; an overview of this work can be found in [4]. Here, we shall focus on the characterization of \( \{2(q + 1) + 2, 2; t, q\} \)-minihypers.

Hamada and Deza [5] characterized all \( \{2(q + 1) + 2, 2; t, q\} \)-minihypers for \( t \geq 3, q \geq 5 \). Their main result was that \( F \) is a \( \{2(q + 1) + 2, 2; t \geq 3, q \geq 5\} \)-minihyper if and only if \( F \in \mathcal{F}(0, 1; t, q) \).

Hamada [2, 3] also determined the cases of \( t = 2, q \geq 3 \). In particular, he found that \( F \) is a \( \{10, 2; 2, 3\} \)-minihyper if and only if \( F \in \mathcal{W}(2, 2; 2, 3) \). He also gave three classes of \( \{12, 2; 2, 4\} \)-minihypers.

In this paper we characterize \( \{2(q + 1) + 2, 2; t, q\} \)-minihypers for \( t \geq 3, q \in \{3, 4\} \). In addition to the previously known constructions, we describe a new \( \{10, 2; 3, 3\} \)-minihyper.

2. **Main results**

In this context, we shall use the term **irreducible** about a \( \{2q + 4, 2; t, q\} \)-minihyper that cannot be described as a union of disjoint minihypers. Thus, the reducible

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\(^1\) Generalizations to the case \( d > qk^{-1} \) can obviously be made.
Characterization of \((2(q + 1) + 2, 2, t, q)\)-minihypers

Fig. 1. An irreducible \((10, 2, t, 3)\)-minihyper.

\((2q + 4, 2, t, q)\)-minihypers include:
- Those in \(F(0, 0, 1, 1; t, q)\), i.e., those that consist of two lines and two points.
- Those that consist of one line and a \(\{q + 3, 1; t, 3\}\)-minihyper of the type described in Fig. 2.
- For \(q = 3\): Those that consist of a \(\{9, 2; t, 3\}\)-minihyper in \(\mathcal{U}(2, 1; t, 3)\) together with a single point.

We now present our two main theorems.

**Theorem 2.1.** There are no irreducible \((12, 2; t, 4)\)-minihypers for \(t \geq 3\).

**Theorem 2.2.** There is exactly one class of irreducible \((10, 2; t, 3)\)-minihypers for \(t \geq 3\). Such a minihyper can be described as a set of 10 points, \((v_0), (v_1), (v_2), (v_3), (v_0 + c_1 v_1), (v_0 + c_2 v_2), (v_0 + c_3 v_3), (c_2 v_2 + 2c_1 v_1), (c_3 v_3 + 2c_1 v_1), (c_3 v_3 + 2c_2 v_2)\), where \((v_0), (v_1), (v_2), (v_3)\) are any linearly independent points in \(PG(t, q)\) and \(c_1, c_2, c_3 \in \{1, 2\}\). Fig. 1 describes this class of minihypers.

Before we proceed to prove these theorems, we need some basic lemmas.

### 3. Preliminaries

**Lemma 3.1** \([4, \text{Theorems 3.1, 4.1, 5.4, 5.9 and 5.15}]\). For \(q \geq 3\),

1. For \(q + 1 \leq f \leq q + 2\), an \(\{f; 2; t, q\}\)-minihyper consists of a line and \(f - q - 1\) points.
2. A \(\{6, 1; t, 3\}\)-minihyper is either a line and two points, or a set of points of the form \(\{(v_1), (v_0 + v_1), (2v_0 + v_1), (v_2), (v_1 + v_2), (cv_0 + 2v_1 + v_2)\}\) for \(c \in \{1, 2\}\) and noncollinear points \((v_0), (v_1), (v_2) \in PG(t, 3)\). (This construction, denoted ‘type B’, is described in Fig. 2).
3. An \(\{8, 2; t, 3\}\)-minihyper consists of two lines. A \(\{9, 2; t, 3\}\)-minihyper is in \(\mathcal{F}(0, 1, 1; t, 3) \cup \mathcal{U}(2, 1; t, 3)\).
4. A \(\{7, 1; t, 4\}\)-minihyper is either a line and two points, or a set of points of the form \(\{(v_0 + v_1), (xv_0 + v_1), (x^3v_0 + v_1), (v_2), (cv_0 + v_1 + v_2), (cxv_0 + x^2v_1 + v_2)\}\) for \(c \in \{1, \alpha, \alpha^2\}\) and noncollinear points \((v_0), (v_1), (v_2) \in PG(t, 4)\), where \(\alpha = \alpha^2 + 1\) is a primitive element of \(GF(2^2)\). (This construction, denoted ‘type B’, is described in Fig. 2).
5. For \(10 \leq f \leq 11\), an \(\{f, 2; t, 4\}\)-minihyper consists of two lines and \(f - 10\) points.
Lemma 3.2 (Similar to Lemma 3.1 in [5]). Let $F$ be a $\{2(q + 1)+2, 2; t, q\}$-minihyper, where $t \geq 3$ and $q \in \{3, 4\}$. Then the following statements are true:

1. For an arbitrary hyperplane $H \subset \text{PG}(t, q)$, let $F^* = F \cap H$ and $f^* = |F^*|$. If $m(q + 1) + \delta(q, m) < f^* < (m + 1)(q + 1)$, where $m \in \{0, 1, 2\}$ and
   \[
   \delta(q, m) = \begin{cases} 
   1 & \text{if } q = 3, m = 1, \\
   0 & \text{otherwise}, 
   \end{cases}
   \]
   then $F^*$ is a $\{f^*, m; t, q\}$-minihyper.

2. There is no hyperplane $H \subset \text{PG}(t, q)$ such that $mq + 4 < |F \cap H| < (m + 1)(q + 1)$ for any integer $m \in \{0, 1\}$.

3. There is a hyperplane $H \subset \text{PG}(t, q)$ such that $|F \cap H| \geq q + 3$. If $t \geq 4$, then there is a hyperplane $H \subset \text{PG}(t, q)$ such that $|F \cap H| \geq q + 4$.

Proof. For $q \geq 5$ this is shown in [5]. The proof for $q \in \{3, 4\}$ is similar:

1. The '<' part of the condition is due to the fact (e.g. Theorem 2.2 in [4]) that $f^* > (m + 1)(q + 1)$ in a $\{f^*, m + 1; t, q\}$-minihyper.

   If $m = 0$, there is nothing to prove, so assume $m \in \{1, 2\}$. Suppose there is a $(t - 2)$-flat $G \subset H$ such that $|F^* \cap G| \leq m - 1$. Let $H_1, \ldots, H_q$ be $q$ other hyperplanes containing $G$. Then, for each $H_i$, $1 \leq i \leq q$, $|F \cap H_i| \geq 2$ and $|F \cap (H_i \cap G)| \geq 2 - (m - 1)$. Thus, $|F| = |F \cap H| + \sum_{i=1}^{q} |F \cap (H_i \cap G)| \geq f^* + q(3 - m) \geq 3q + m + \delta(m, q)$, a contradiction. (Note: We shall use this method sufficiently often to have a name for it: We call this the '(t-2)-flat argument'.) Figure 3 describes this method for $q = 4, m = 1$.

2. For $q = 3$ there is nothing to prove. So assume that $q = 4$ and that for some hyperplane $H$, $|F \cap H| = 9$. From (1), $F \cap H$ is a $\{9, 1; t, 4\}$-minihyper, hence there is a $(t - 2)$-flat $G \subset H$ such that $|F^* \cap G| - 1$. The $(t - 2)$-flat argument leads to a contradiction, as shown in Fig. 4.

3. By definition, there is a hyperplane $H_0 \subset \text{PG}(t, q)$ such that $|F \cap H_0| = 2$. Select a $(t - 2)$-flat $G_0 \subset H_0$ such that $F \cap H_0 \subset G_0$. By the $(t - 2)$-flat argument, it is easy to see that at least one of the (other $q$) hyperplanes containing $G_0$ contains at least

![Fig. 2. Type B minihypers: $\{q + 3, 1, 2, q\}$-minihypers not containing full lines. All points and all lines in $\text{PG}(2, q)$ containing more than one point of the minihyper are shown. The black points are the ones included in the minihyper.](image-url)
Characterization of $\{2(q+1)+2, 2; t, q\}$-minihypers

\[ \left\lfloor \frac{(2q+4-2)}{q} \right\rfloor + 2 \geq 5 \] points of $F$ because the average number of elements of $F$ in $H_i \setminus G_0$, $i = 1, \ldots, q$ is $\left\lfloor \frac{(2q+4-2)}{q} \right\rfloor$. Let $H$ be such a hyperplane. Suppose $|F \cap H| \leq q + 2$. Then by (1), with $m = 1$, and Lemma 3.1, $F \cap H$ contains a line, so it is possible to find a $(t-2)$-flat $G \subset H$ containing the line, i.e. $q + 1$ points. Once again we employ the $(t-2)$-flat argument, to obtain that at least one hyperplane containing $G$ contains at least $q + 3$ points of $F$ because, here, there remain $2q + 4 - q - 2 = q + 2$ elements of $F$ to be distributed among the $q$ sets $H_i \setminus G$, $i = 1, \ldots, q$. In the special case of $t \geq 4$, suppose $|F \cap H'| = q + 3$ for some hyperplane $H'$. Then it follows from (1) and Lemma 3.1 that some $(t-2)$-flat $G' \subset H'$ contains at least $q + 2$ points of $F$. Applying the now familiar $(t-2)$-flat argument, we find that some hyperplane contains at least $q + 4$ points of $F$.

**Lemma 3.3** (Lemma 3.2 in [5]). Let $F^*$ be any $\{q+4, 1; t, q\}$-minihyper such that $F^* \subset H \subset PG(t, q)$, where $t \geq 3$ and $q \geq 3$. Let $R$ be any point such that $R \in H \setminus F^*$. Then there exists a $(t-2)$-flat $G \subset H$ such that $R \in G$ and $|F^* \cap G| = 1$.

**Definition 3.1.** Let $V$ and $W$ be a $\mu$-flat and a $\nu$-flat in $PG(t, q)$, respectively, where $0 \leq \mu, \nu < t - 1$. Let $V \oplus W$ denote the minimum flat in $PG(t, q)$ which contains both $V$ and $W$.

**Lemma 3.4** (Similar to Lemma 3.3 in [5]). Let $F$ be any $\{2(q+1)+2, 2; t, q\}$-minihyper such that there exists a hyperplane $H \subset PG(t, q)$ satisfying $|F^*| = q + 4$, where $t \geq 3$ and $q \geq 3$, and $F^* \subset F \cap H$. Let $Q_1, Q_2$ be any two points in $F \setminus F^*$, and let $R \in H$ be the point such that $(Q_1 \oplus Q_2) \cap H = \{R\}$. Then:

1. $R \in F^*$, and
2. $F^* \setminus \{R\}$ is a $\{q+3, 1; t, q\}$-minihyper in $H$.

**Proof.** Again, the proof is similar to the proof of Lemma 3.3 in [5]. By Lemma 3.2, $F^*$ is a $\{q+4, 1; t, q\}$-minihyper.
(1) Assume that $R \notin F \ast$. Then from Lemma 3.3, there exists a $(t - 2)$-flat $G \subset H$ such that $R \in G$ and $|F \ast \cap G| = \delta$. Noticing that one of the hyperplanes containing $G$ contains the line $Q_1 \oplus Q_2$, we obtain a contradiction by the $(t - 2)$-flat argument.

(2) Assume that some $(t - 2)$-flat $G^\ast \subset H$ is such that $F \ast \cap G^\ast = \{R\}$ (i.e., $F \ast \setminus \{R\}$ is a $(q + 3, 0; t, q)$-minihyper in $H$). Again, one of the hyperplanes containing $G$ contains the line $Q_1 \oplus Q_2$, and the $(t - 2)$-flat argument leads to a contradiction.

4. The proof of Theorem 2.1

We shall assume that $F \subset PG(t, 4)$ is a \{12, 2; t, 4\}-minihyper.

**Lemma 4.1.** If there is a hyperplane $H$ such that $|F \cap H| = 8$, then $F$ is reducible.

**Proof.** In this case, we set $F \setminus H = \{Q_1, Q_2, Q_3, Q_4\}$. Lemma 3.4 implies that $F \cap H = F \ast \cup \{R\}$, where $F \ast$ is a \{7, 1; t, 4\}-minihyper, $R \in Q_1 \oplus Q_2$.

Case I: $F \ast$ of type B.

Assume that $Q_i \notin Q_1 \oplus Q_2$, $i \in \{3, 4\}$. Then denote by $R_i$, the intersection point between $H$ and the line $Q_1 \oplus Q_i$. By Lemma 3.4, $R_i \in F \ast$, and $F || = F \ast \setminus \{R_i\} \cup \{R\}$ is another \{7, 1; t, 4\}-minihyper. Since $F ||$ contains no lines with more than four points, it must be of type B – see Fig. 2. There are exactly three lines through each point of a type B minihyper that contains two other points of the minihyper. However, a close inspection of Fig. 2 reveals that there is only one line through $R$ that contains more than one other point in $F ||$, a contradiction.

Hence, $F$ consists of the type B minihyper $F \ast$, and the line $\{R, Q_1, Q_2, Q_3, Q_4\}$.

Case II: $F \ast$ is a line $L$ and two points $R_2, R_3$.

If $Q_i \oplus Q_j, 1 \leq i < j \leq 4$, intersects $H$ in a point $R_i \in L, i \in \{2, 3\}$, then $F \cap H \setminus \{R_i\}$ is a \{7, 1; t, 4\}-minihyper containing four points on a line, which is impossible.

So all lines $Q_i \oplus Q_j, 1 \leq i < j \leq 4$, intersect $H$ in a point in $\{R, R_2, R_3\}$. Let $M = Q_1 \oplus Q_2$. If $|F \cap M| = 3$, then without loss of generality (w.l.o.g.) $F$ has the structure shown in Fig. 5.

The only way to complete all the lines $Q_i \oplus Q_j$ is to make $\{R, R_2, R_3, Q_1, Q_2, Q_3, Q_4\}$ a type B minihyper.

If $|F \cap M| = 4$, say, $M = \{R, Q_1, Q_2, Q_3\}$, then the three lines $Q_i \oplus Q_4, 1 \leq i \leq 3$, must meet $H$ in distinct points in $\{R_2, R_3\}$, which is impossible.

Finally, if $|F \cap M| = 5$, then $F \in \mathcal{F}(0, 0, 1, 1; t, 4)$. This concludes the proof of Lemma 4.1. □

**Lemma 4.2.** If $t = 3$ and there is no hyperplane $H'$ such that $|F \cap H'| = 8$, but there is at least one hyperplane $H$ such that $|F \cap H| = 7$, then $F$ is reducible.

**Proof.** $F \setminus H = \{Q_1, Q_2, Q_3, Q_4, Q_5\}$. Lemma 3.2 implies that $F \ast = F \cap H$ is a \{7, 1; t, 4\}-minihyper.
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**Fig. 5.** \(q=4, |F \cap H|=8\), and \(F \cap H\) contains a line \(L\).

**Fig. 6.** \(q=4, F^*\) of type B: The \((t-2)\)-flat argument.

**Case 1:** \(F^*\) of type B.

Let \(M=Q_1 \oplus Q_2\) and \(M \cap H = \{R\}\), where by Lemma 3.4 \(R \in F^*\). From Fig. 2, one can always find a line \(G \subset H\) through \(R\) such that \(|F^* \cap G|=1\). Thus, the five hyperplanes, \(H, M \oplus G, H_1, H_2, H_3\) containing \(G\) can be drawn as shown in Fig. 6.

If \(|(M \setminus H) \cap F| \geq 3\), then \(\sum_{i=1}^{5} |H_i \setminus G| \leq 2\). Hence there is at least one hyperplane containing only one point of \(F\), a contradiction.

So none of the lines \(Q_i \oplus Q_j\) contain more than two points in \(\{Q_1, \ldots, Q_5\}\). By a similar argument a contradiction is obtained if we assume that \((Q_i \oplus Q_j) \cap H = \{R\}\) for \((i, j) \neq (i', j')\), and we have that the ten lines \(Q_i \oplus Q_j\) intersect \(H\) in ten different points. At least three of these ten points are not in \(F^*\), so it is possible to find two points \(R_1 \in (Q_i \oplus Q_j) \cap H\) and \(R_2 \in (Q_i' \oplus Q_j') \cap H\) such that \(|(R_1 \oplus R_2) \cap F^*| < 3\). From Fig. 2, we observe that \(|(R_1 \oplus R_2) \cap F^*| = 1\). There are
two cases, \(|\{i, j\} \cap \{i', j'\}| = 0\) and \(|\{i, j\} \cap \{i', j'\}| = 1\). The first of these cases is shown
in Fig. 7, the second case is similar.

In both cases, we obtain a contradiction by the \((t-2)\)-flat argument.

Case II: \(F^*\) is a line \(V\) and two points \(P_1\) and \(P_2\).

Let \(L = P_1 \oplus P_2\) and \(V \cap L = \{R\}\). In addition to \(H\), there are four hyperplanes that
contain \(L\). These four hyperplanes contain the five points \(Q_1, \ldots, Q_5\), so by the
\((t-2)\)-flat argument, there is a hyperplane \(M\) that contain at least two points \(Q_1, Q_2\).
Then \(5 \leq |F \cap M| \leq 7\).

If \(|F \cap M| \leq 6\), then from Lemma 3.2, \(F \cap M\) is a minihyper containing a full line,
a contradiction.

Assume that \(|F \cap M| = 7\). If \(F \cap M\) is a type B minihyper, then this is equivalent to
Case I. Otherwise, \(F \cap M\) contains a full line \(N\). If \(N\) intersects \(H\) in \(R\), then \(N \oplus V\) is
a hyperplane containing nine points of \(F\). Hence, \(N\) intersects \(H\) in \(P_1\) or \(P_2\), and
\(F \in \mathcal{F}(0, 0, 1, 1; t, q)\).

This concludes the proof of Lemma 4.2.

**Proof of Theorem 2.1.** Let \(H^\text{max}\) be a hyperplane in \(PG(t, 4)\) such that for any
hyperplane \(H\) in \(PG(t, 4)\), \(|F \cap H^\text{max}| \geq |F \cap H|\). Theorem 2.1 follows from considering
the possible values of \(|F \cap H^\text{max}|\):

\[
|F \cap H^\text{max}| = 7: \text{ Lemmas 3.2 and 4.2.}
|F \cap H^\text{max}| = 8: \text{ Lemma 4.1.}
|F \cap H^\text{max}| = 9: \text{ Lemma 3.2(2).}
|F \cap H^\text{max}| \in \{10, 11\}: \text{ Lemma 3.1(5).}
|F \cap H^\text{max}| = 12: \text{ Induction on } t. \quad \Box
\]
5. The proof of Theorem 2.2

The proof is similar to the previous one. We shall assume that $F \subseteq PG(t, 3)$ is a $\{10, 2; t, 3\}$-minihyper.

Lemma 5.1. If there is a hyperplane $H$ such that $|F \cap H| = 7$, then $F$ is reducible.

Proof. We set $F \setminus H = \{Q_1, Q_2, Q_3\}$. Lemma 3.4 implies that $F \cap H = F^* \cup \{R\}$, where $F^*$ is a $\{6, 1; t, 3\}$-minihyper, $R \in Q_1 \oplus Q_2$. By Lemma 3.1, $F^*$ is contained in a 2-flat $T \subset H$.

First, observe that if $Q_3 \in Q_1 \oplus Q_2$, then $F$ consists of a line and a known $\{6, 1; t, 3\}$-minihyper. Otherwise, assume that $Q_3 \notin Q_1 \oplus Q_2$. Then the three points $Q_1, Q_2, Q_3$ span a 2-flat, which meets $T$ in a line. Thus the three points $R_1(= R), R_2, R_3$ are collinear, where $H \cap (Q_2 \oplus Q_3) = \{R_2\}$ and $H \cap (Q_1 \oplus Q_3) = \{R_3\}$.

Case I. $F^*$ is a line $L$ and two points.

If $R_2 \in L$, then three of the lines (in $T$) through $R_2$ contain all the points of $F^*$. Hence the fourth line in $T$ through $R_2$, denoted $L_0$, satisfies $|L_0 \cap F| = 1$. Then it is also possible to find a $(t-2)$-flat $G \subset H$ that contains only the single point $R_2$ in $F^*$. Let $H, G \oplus (R_2 \oplus Q_3), H_1, H_2$ be the four hyperplanes containing $G$. By the $(t-2)$-flat argument, one of the hyperplanes $H_1$ or $H_2$ contains only one point, $R_2$, in $F$, a contradiction.

If $R_2, R_3 \notin L$, then $F$ consists of $L$ and the set of points $\{R_1, R_2, R_3, Q_1, Q_2, Q_3\}$, which from Fig. 8 is a type B minihyper.

Case II: $F^*$ of type B.

Since $R_2, R_3 \in F^*, R_1$ is on one of the three lines in $T$ containing $R_2$ and at least one other point in $F^*$. Thus, it can be seen from Fig. 2 that there is a line $L_0 \in T$ through $R_2$ such that $|L_0 \cap F| = 1$. Hence there is a $(t-2)$-flat $G \subset H$ that contains only the single point $R_2$ in $F^*$, and by the $(t-2)$-flat argument, it can be shown that one hyperplane contains only one point, $R_2$, in $F$. This is a contradiction.

This concludes the proof of Lemma 5.1. $\square$

Lemma 5.2. If $t = 3$ and there is no hyperplane $H'$ such that $|F \cap H'| = 7$, but there is at least one hyperplane $H$ such that $|F \cap H| = 6$, then either $F$ is reducible, or it is a minihyper of the type described in Fig. 1.

Proof. $F \setminus H = \{Q_1, Q_2, Q_3, Q_4\}$. Lemma 3.2 implies that $F^* = F \cap H$ is a $\{6, 1; t, 3\}$-minihyper.

Case I: $F^*$ is a line $V$ and two points, $P_1, P_2$.

Let $L = P_1 \oplus P_2$ and $\{R\} = V \cap L$. Since the four points $Q_1, Q_2, Q_3, Q_4$ are distributed among three hyperplanes that contain $L$, it follows by the $(t-2)$-flat argument that one such hyperplane, denoted $M$, contains at least two of these four points, say, $Q_1$ and $Q_2$. Thus, $5 \leq |F \cap M| \leq 6$. But $|F \cap M| \neq 5$, for otherwise $F \cap M$ is a $\{5, 1; t, 3\}$-minihyper, which by Lemma 3.1 is a line and a point. This is a contradiction.
If $|F \cap M| = 6$, then $F \cap M$ is a $\{6, 1; t, 3\}$-minihyper; there are two possibilities. If $F \cap M$ is a line and two points, then the line does not contain $R$ (by the assumption that no hyperplanes contain more than six points), so $F$ consists of two lines and two points. If on the other hand $F \cap M$ is a type $B$ minihyper, then we have Case II, which we will investigate next.

Case II: $F^*$ of type $B$.

Let $\{R_{ij}\} = (Q_i \oplus Q_j) \cap H$, $1 \leq i < j \leq 4$, and let $(i_1, \ldots, i_4)$ be some permutation of $(1, \ldots, 4)$.

Assume that $Q_{i_3} \in Q_i \oplus Q_j$. Then a line $G \subset H$ through $R_{i_1i_3}$ can be found such that $|G \cap F^*| = 1$, and by the $(t-2)$-flat argument, at least one hyperplane containing $G$ contains at most one point in $F$, a contradiction. Thus,

$$Q_k \not\in Q_i \oplus Q_j, \quad 1 \leq i < j < k \leq 4. \quad (5.1)$$

The hyperplanes thus spanned by three points $Q_i$, $Q_j$, and $Q_k$ must meet $H$ in a line, i.e.,

$$R_{ij} \in R_{ik} \oplus R_{jk}, \quad 1 \leq i < j < k \leq 4. \quad (5.2)$$

A similar $(t-2)$-flat argument shows

$$R_{ij} \neq R_{i'j'}, \quad \{i, j\} \neq \{i', j'\}. \quad (5.3)$$

Let $I = Q_{i_1} \oplus Q_{i_2} \oplus Q_{i_3}$ be the hyperplane spanned by $Q_{i_1}$, $Q_{i_2}$, and $Q_{i_3}$, and let $L_0 = H \cap I$. If $Q_{i_4} \in I$, then by (5.3) $L_0$ contains six different points $R_{ij}$, $1 \leq i < j \leq 4$. This is a contradiction, so $Q_{i_4} \not\in I$. If $|F \cap L_0| \leq 1$, then by the $(t-2)$-flat argument, at least one hyperplane containing $L_0$ contains at most one point in $F$, a contradiction. Hence, $5 \leq |F \cap I| \leq 6$.

If $|F \cap I| = 5$, then $F \cap I$ is a $\{5, 1; t, 3\}$-minihyper, which by Lemma 3.2 is a line and a point. But by (5.1) this is a contradiction.

If $|F \cap I| = 6$, then by (5.1) and Lemma 3.2, $F \cap I$ is a $\{6, 1; t, 3\}$-minihyper without a full line, i.e., of type $B$.

Since there are four lines $L_1, \ldots, L_4 \in H$ each containing three points of $F^*$, each of these lines can be associated with exactly one subset $\{Q_i, Q_j, Q_k\}$. For instance, the line containing the points $R_1, R_2, R_3$ (see Fig. 10) can w.l.o.g. be assumed to form a type B minihyper with points $Q_1, Q_2, Q_3$.
The only way to complete the construction so as to satisfy (5.2) and (5.3) is shown in Fig. 1.

Next, we show that this construction really is a \{10, 2; 3, 3\}-minihyper. Assume the converse is true, i.e. that there is a hyperplane \( H_0 \) such that \( |F \cap H_0| = 1 \). Let \( G \) be the line defined by \( H \cap H_0 \), obviously \( |F^* \cap G| = 1 \). It follows then that the four points \( Q_1, \ldots, Q_4 \) must be contained in the two other hyperplanes, \( H_1 \) and \( H_2 \), that contains \( G \). Then it is possible to find at least two points \( R_{ij}, R_{i'}, j' \) both in \( |F^* \cap G| \), a contradiction.

This concludes the proof of Lemma 5.2. \( \square \)

**Proof of Theorem 2.2.** Let \( H^{\text{max}} \) be a hyperplane in \( PG(t, 3) \) such that for any hyperplane \( H \) in \( PG(t, 3) \), \( |F \cap H^{\text{max}}| \geq |F \cap H| \).

Theorem 2.2 follows from considering the possible values of \( |F \cap H^{\text{max}}| \):

- \( |F \cap H^{\text{max}}| = 6 \): Lemmas 3.2 and 5.2.
- \( |F \cap H^{\text{max}}| = 7 \): Lemma 5.1.
- \( |F \cap H^{\text{max}}| \in \{8, 9\} \): Lemma 3.1(3).
- \( |F \cap H^{\text{max}}| = 10 \): Induction on \( t \). \( \square \)

**References**

[3] N. Hamada, Characterization of \{(q+1)+1; 1; t, q\}-min. hypers and \((2q+1)+2; 2, q\)-min. hypers in a finite projective geometry, Graphs Combin. 5 (1989) 63–81.