The Hopf Extension Theorem of Measure

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Summary. The authors have presented some articles about Lebesgue type integration theory. In our previous articles [12, 13, 26], we assumed that some \(\sigma\)-additive measure existed and that a function was measurable on that measure. However the existence of such a measure is not trivial. In general, because the construction of a finite additive measure is comparatively easy, to induce a \(\sigma\)-additive measure a finite additive measure is used. This is known as an E. Hopf’s expansion theorem of measure [15].

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The articles [11], [23], [1], [24], [22], [8], [25], [10], [9], [2], [20], [26], [6], [5], [7], [13], [4], [12], [3], [16], [19], [18], [27], [21], [17], and [14] provide the notation and terminology for this paper.

1. The Outer Measure Induced by the Finite Additive Measure

For simplicity, we adopt the following convention: \(X\) denotes a set, \(F\) denotes a field of subsets of \(X\), \(M\) denotes a measure on \(F\), \(A, B\) denote subsets of \(X\), \(S_1\) denotes a sequence of subsets of \(X\), \(s_1, s_2, s_3\) denote sequences of extended reals, and \(n, k\) denote natural numbers.

One can prove the following three propositions:

1. \(\text{Ser } s_1 = (\sum_{\alpha=0}^{k}(s_1)(\alpha))_{\alpha \in N}\).
2. If \(s_1\) is non-negative, then \(s_1\) is summable and \(\sum s_1 = \sum s_1\).
(3) Suppose \( s_2 \) is non-negative and \( s_3 \) is non-negative and for every natural number \( n \) holds \( s_1(n) = s_2(n) + s_3(n) \). Then \( s_1(n) \) is non-negative and \( \sum s_1 = \sum s_2 + \sum s_3 \) and \( \sum s_1 = (\sum s_2) + \sum s_3 \).

Let us consider \( X, F \). Note that there exists a function from \( \mathbb{N} \) into \( F \) which is disjoint valued.

Let us consider \( X, F \). A finite sequence of elements of \( 2^X \) is said to be a finite sequence of elements of \( F \) if:

(Def. 1) For every natural number \( k \) such that \( k \in \text{dom} \) it holds \( \text{it}(k) \in F \).

Let us consider \( X, F \). Observe that there exists a finite sequence of elements of \( F \) which is disjoint valued.

Let us consider \( X, F \). A sep fin sequence of \( F \) is a disjoint valued finite sequence of elements of \( F \).

Let us consider \( X, F \). A sequence of separated subsets of \( F \) is a disjoint valued function from \( \mathbb{N} \) into \( F \).

Let us consider \( X, F \). A sequence of subsets of \( X \) is said to be a set sequence of \( F \) if:

(Def. 2) For every natural number \( n \) holds \( \text{it}(n) \) is a covering of \( S_1(n) \) in \( F \).

Let us consider \( X, A, F \). A set sequence of \( F \) is said to be a covering of \( A \) in \( F \) if:

(Def. 3) \( A \subseteq \bigcup \text{rng} \text{it} \).

In the sequel \( F_1 \) is a set sequence of \( F \) and \( C_1 \) is a covering of \( A \) in \( F \).

Let us consider \( X, F, F_1, n \). Then \( F_1(n) \) is an element of \( F \).

Let us consider \( X, F, S_1 \). A function from \( \mathbb{N} \) into \( (2^X)^{\mathbb{N}} \) is said to be a covering of \( S_1 \) in \( F \) if:

(Def. 4) For every element \( n \) of \( \mathbb{N} \) holds \( \text{it}(n) \) is a covering of \( S_1(n) \) in \( F \).

In the sequel \( C_2 \) denotes a covering of \( S_1 \) in \( F \).

Let us consider \( X, F, M, F_1 \). The functor \( \text{vol}(M,F_1) \) yielding a sequence of extended reals is defined by:

(Def. 5) For every \( n \) holds \( \text{vol}(M,F_1)(n) = M(F_1(n)) \).

The following proposition is true

(4) \( \text{vol}(M,F_1) \) is non-negative.

Let us consider \( X, F, S_1, C_2 \) and let \( n \) be an element of \( \mathbb{N} \). Then \( C_2(n) \) is a covering of \( S_1(n) \) in \( F \).

Let us consider \( X, F, S_1, M, C_2 \). The functor \( \text{Volume}(M,C_2) \) yielding a sequence of extended reals is defined as follows:

(Def. 6) For every element \( n \) of \( \mathbb{N} \) holds \( \text{Volume}(M,C_2)(n) = \sum \text{vol}(M,C_2(n)) \).

We now state the proposition

(5) \( 0 \leq \text{Volume}(M,C_2)(n) \).
Let us consider $X$, $F$, $M$, $A$. The functor $Svc(M, A)$ yielding a subset of $\mathbb{R}$ is defined as follows:

(Def. 7) For every extended real number $x$ holds $x \in Svc(M, A)$ iff there exists a covering $C_1$ of $A$ in $F$ such that $x = \sum \text{vol}(M, C_1)$.

Let us consider $X$, $A$, $F$, $M$. Note that $Svc(M, A)$ is non-empty.

Let us consider $X$, $F$, $M$.

The functor $Svc(M, A)$ is defined as follows:

(Def. 8) For every extended real number $x$ holds $x \in Svc(M, A)$ iff there exists a covering $C_1$ of $A$ in $F$ such that $x = \sum \text{vol}(M, C_1)$.

Let us consider $X$, $A$, $F$, $M$. Note that $Svc(M, A)$ is non empty.

Let us consider $X$, $A$, $F$, $M$. The function $\text{InvPairFunc}$ from $\mathbb{N}$ into $\mathbb{N} \times \mathbb{N}$ is defined by:

(Def. 9) $\text{InvPairFunc} = \text{PairFunc}^{-1}$.

Let us consider $X$, $S_1$, $C_2$. The functor $\text{On}_{C_2}$ yields a covering of $\bigcup \text{rng} S_1$ in $F$ and is defined by:

(Def. 10) For every natural number $n$ holds $(\text{On}_{C_2}(n)) = C_2(\text{pr1}(\text{InvPairFunc})(n))(\text{pr2}(\text{InvPairFunc})(n))$.

We now state several propositions:

(6) Let $k$ be an element of $\mathbb{N}$. Then there exists a natural number $m$ such that for every sequence $S_1$ of subsets of $X$ and for every covering $C_2$ of $S_1$ in $F$ holds $(\sum_{\alpha=0}^{\kappa} \text{vol}(M, C_2)(\alpha))_{\kappa \in \mathbb{N}}(k) \leq (\sum_{\alpha=0}^{\kappa} \text{Volume}(M, C_2)(\alpha))_{\kappa \in \mathbb{N}}(m)$.

(7) $\inf Svc(M, \bigcup \text{rng} S_1) \leq \sum \text{Volume}(M, C_2)$.

(8) If $A \in F$, then $A, \emptyset_X$ followed by $\emptyset_X$ is a covering of $A$ in $F$.

(9) Let $X$ be a set, $F$ be a field of subsets of $X$, $M$ be a measure on $F$, and $A$ be a set. If $A \in F$, then $(\text{the c meas } M)(A) \leq M(A)$.

(10) The c meas $M$ is non-negative.

(11) $(\text{the c meas } M)(\emptyset) = 0$.

(12) If $A \subseteq B$, then $(\text{the c meas } M)(A) \leq (\text{the c meas } M)(B)$.

(13) $(\text{the c meas } M)(\bigcup \text{rng} S_1) \leq \sum((\text{the c meas } M) \cdot S_1)$.

(14) The c meas $M$ is a Caratheodor’s measure on $X$.

Let $X$ be a set, let $F$ be a field of subsets of $X$, and let $M$ be a measure on $F$. Then the c meas $M$ is a Caratheodor’s measure on $X$.

2. The Hopf Extension Theorem

Let $X$ be a set, let $F$ be a field of subsets of $X$, and let $M$ be a measure on $F$. We say that $M$ is completely-additive if and only if:

(Def. 11) For every sequence $F_1$ of separated subsets of $F$ such that $\bigcup \text{rng} F_1 \in F$ holds $\sum (M \cdot F_1) = M(\bigcup \text{rng} F_1)$.

The following four propositions are true:

(15) The partial unions of $F_1$ are a set sequence of $F$. 

(16) The partial diff-unions of $F_1$ are a set sequence of $F$.

(17) Suppose $A \in F$. Then there exists a sequence $F_1$ of separated subsets of $F$ such that $A = \bigcup \text{rng } F_1$ and for every natural number $n$ holds $F_1(n) \subseteq C_1(n)$.

(18) If $M$ is completely-additive, then for every set $A$ such that $A \in F$ holds $M(A) = \text{(the c meas } M)(A)$.

In the sequel $C$ denotes a Caratheodor’s measure on $X$.

We now state three propositions:

(19) If for every subset $B$ of $X$ holds $C(B \cap A) + C(B \cap (X \setminus A)) \leq C(B)$, then $A \in \sigma$-Field($C$).

(20) $F \subseteq \sigma$-Field($\text{the c meas } M$).

(21) Let $X$ be a set, $F$ be a field of subsets of $X$, $F_1$ be a set sequence of $F$, and $M$ be a function from $F$ into $\mathbb{R}$. Then $M \cdot F_1$ is a sequence of extended reals.

Let $X$ be a set, let $F$ be a field of subsets of $X$, let $F_1$ be a set sequence of $F$, and let $g$ be a function from $F$ into $\mathbb{R}$. Then $g \cdot F_1$ is a sequence of extended reals.

Next we state the proposition

(22) Let $X$ be a set, $S$ be a $\sigma$-field of subsets of $X$, $S_2$ be a sequence of subsets of $S$, and $M$ be a function from $S$ into $\mathbb{R}$. Then $M \cdot S_2$ is a sequence of extended reals.

Let $X$ be a set, let $S$ be a $\sigma$-field of subsets of $X$, let $S_2$ be a sequence of subsets of $S$, and let $g$ be a function from $S$ into $\mathbb{R}$. Then $g \cdot S_2$ is a sequence of extended reals.

The following propositions are true:

(23) Let $F$, $G$ be functions from $\mathbb{N}$ into $\mathbb{R}$ and $n$ be a natural number. Suppose that for every natural number $m$ such that $m \leq n$ holds $F(m) \leq G(m)$. Then $(\text{Ser } F)(n) \leq (\text{Ser } G)(n)$.

(24) For all $X$, $C$ and for every sequence $s_1$ of separated subsets of $\sigma$-Field($C$) holds $\bigcup \text{rng } s_1 \in \sigma$-Field($C$) and $C(\bigcup \text{rng } s_1) = \sum C \cdot s_1$.

(25) For all $X$, $C$ and for every sequence $s_1$ of subsets of $\sigma$-Field($C$) holds $\bigcup s_1 \in \sigma$-Field($C$).

(26) Let $X$ be a non empty set, $S$ be a $\sigma$-field of subsets of $X$, $M$ be a $\sigma$-measure on $S$, and $S_2$ be a sequence of subsets of $S$. If $S_2$ is non-decreasing, then $\lim (M \cdot S_2) = M(\lim S_2)$.

(27) If $F_1$ is non-decreasing, then $M \cdot F_1$ is non-decreasing.

(28) If $F_1$ is descending, then $M \cdot F_1$ is non-increasing.

(29) Let $X$ be a set, $S$ be a $\sigma$-field of subsets of $X$, $M$ be a $\sigma$-measure on $S$, and $S_2$ be a sequence of subsets of $S$. If $S_2$ is non-decreasing, then $M \cdot S_2$
is non-decreasing.

(30) Let $X$ be a set, $S$ be a $\sigma$-field of subsets of $X$, $M$ be a $\sigma$-measure on $S$, and $S_2$ be a sequence of subsets of $S$. If $S_2$ is descending, then $M \cdot S_2$ is non-increasing.

(31) Let $X$ be a non empty set, $S$ be a $\sigma$-field of subsets of $X$, $M$ be a $\sigma$-measure on $S$, and $S_2$ be a sequence of subsets of $S$. If $S_2$ is descending and $M(S_2(0)) < +\infty$, then $\lim(M \cdot S_2) = M(\lim S_2)$.

Let $X$ be a set, let $F$ be a field of subsets of $X$, let $S$ be a $\sigma$-field of subsets of $X$, let $m$ be a measure on $F$, and let $M$ be a $\sigma$-measure on $S$. We say that $M$ is extension of $m$ if and only if:

(Def. 12) For every set $A$ such that $A \in F$ holds $M(A) = m(A)$.

Next we state four propositions:

(32) Let $X$ be a non empty set, $F$ be a field of subsets of $X$, and $m$ be a measure on $F$. If there exists a $\sigma$-measure on $\sigma(F)$ which is extension of $m$, then $m$ is completely-additive.

(33) Let $X$ be a non empty set, $F$ be a field of subsets of $X$, and $m$ be a measure on $F$. Suppose $m$ is completely-additive. Then there exists a $\sigma$-measure $M$ on $\sigma(F)$ such that $M$ is extension of $m$ and $M = \sigma\text{-Meas}(\text{the c meas } m)|\sigma(F)$.

(34) If for every $n$ holds $M(F_1(n)) < +\infty$, then $M((\text{the partial unions of } F_1)(k)) < +\infty$.

(35) Let $X$ be a non empty set, $F$ be a field of subsets of $X$, and $m$ be a measure on $F$. Suppose that

(i) $m$ is completely-additive, and

(ii) there exists a set sequence $A_1$ of $F$ such that for every natural number $n$ holds $m(A_1(n)) < +\infty$ and $X = \bigcup \text{rng } A_1$.

Let $M$ be a $\sigma$-measure on $\sigma(F)$. If $M$ is extension of $m$, then $M = \sigma\text{-Meas}(\text{the c meas } m)|\sigma(F)$.

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