Price taking equilibrium in club economies with multiple memberships and unbounded club sizes\(^1\)

by

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Abstract: This paper develops a model of an economy with clubs where individuals may belong to multiple clubs and where there may be ever increasing returns to club size. Clubs may be large, as large as the total agent set. The main condition required is that sufficient wealth can compensate for memberships in larger and larger clubs. Notions of price-taking equilibrium and the core, both with communication costs, are introduced. These notions require that there is a small cost, called a communication cost, of deviating from a given outcome. With some additional standard sorts of assumptions

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\(^2\)Some changes in definitions have been made but incorporating these into the proofs is in process.
on preferences, we demonstrate that, given communication costs parameterized by \( \varepsilon > 0 \), for all sufficiently large economies, the core is non-empty and contains states of the economy that are in the core of the replicated economy for all replications (Edgeworth states of the economy). Moreover, for any given economy, every state of the economy that is in the core for all replications of that economy can be supported as a price-taking equilibrium with communication costs. Together these two results imply that, given the communication costs, for all sufficiently large economies there exists Edgeworth states of the economy and every Edgeworth state can be supported as a price-taking equilibrium.
1 Motivation

It seems compelling that gains to cooperation by large groups of individuals may be substantial. For example, in economies with public goods, coordination of activities and decreasing costs of providing public goods may provide increasing benefits to ever larger club membership. Consider questions of global pollution, global harmonization of productive activities and memberships in networks. If we wish a model to describe clubs such as the World Trade Organization, the United Nations, the World Environmental Organization, or religions that wish to embrace all people, then a model with bounded club sizes, where clubs become infinitesimal in large economies is not appropriate.\(^3\) Of course much economic activity is carried out within small clubs – marriages, small firms, and swimming pool clubs for example. It is also clear that a general model should also allow overlapping clubs so that a participant may belong, for example, to a two-person partnership, a dance club, and a world-wide social movement.

In this paper we explore the boundaries of price-taking equilibrium in club economies where clubs may overlap and also may be large. Providing most agents have many close substitutes, if an economy is sufficiently large then an equilibrium with communication costs and possibly some frictions, captured by the presence of an exceptional set of agents, exists and is in the core. Communication costs are parameterized by \(\varepsilon\) and \(\varepsilon\) can be allowed to zero as the economy becomes large. Moreover, we demonstrate that in large economies the core with communication costs is nonempty and that an Edgeworth equilibrium exists. The set of Edgeworth equilibria is contained in the set of equilibria with communication costs.

Allowing clubs to be as large as the entire agent set leads to a situation that appears, in essence, to be fundamentally different from a private goods economy, or an economy where small groups of agents can exhaust all gains to coalition formation or a pure public goods economy. Even in large economies, discrimination between otherwise identical individuals can persist.

Recent literature suggests that whenever almost all gains to collective activities can be realized by relatively small groups of participants then, when there are many participants, diverse economies resemble markets. This includes economies with indivisibilities, nonconvexities, local public goods, and club economies with multiple memberships. In particular, under apparently

\(^{3}\)In fact, if the economy is essentially superadditive – that is, if an option open to a large club is to divide into smaller clubs – then economies with possibly large clubs can do no worse than those with clubs restricted in size.
mild conditions – essentially just mild conditions of essential superadditivity (that an option open to groups is to subdivide into smaller groups), boundedness of average or per capita payoffs, and thickness of the total agent set (there are many close substitutes for most agents) – approximate cores are nonempty, approximate cores treat similar people similarly and economies, modeled as games with side payments, generate market games. In addition, analogous of the Laws of Demand and Supply hold.4 Models of games with many agents, however, cannot treat the properties of price-taking economic equilibrium, except for situations where the ‘commodities’ to be priced are types of agents. To obtain richer results on price-taking equilibrium, more detailed economic models are required. Our primary focus is the extent to which increasing returns to club formation in larger and larger economies is consistent with existence of price-taking equilibrium and equivalence of the outcomes of price-taking equilibrium with cooperative outcomes.

Our research grows out of the seminal works of Tiebout [1956] and Buchanan [1965]. Tiebout conjectured that, in large economies with sufficient diversity of communities in terms of their local public good offerings, competitive forces would lead to a ‘market-like outcome.’ Buchanan stressed that there may be congestion so that optimal club sizes may exist; that is, there may exist some finite population at which all gains to membership size would be exhausted. There are now many models showing that large economies with small optimal groups (communities, firms, clubs, jurisdictions, and so on) generate markets; club membership is simply another commodity. For example, think of movie theatres. Movies can be provided by clubs or by profit maximizing entrepreneurs. They tend to be provided by non-market organizations when the demand is small – foreign film clubs, for example – and price discrimination of some sort may be required to cover costs. Most models of such situations rule out large clubs that are few in number, for example, the individual States in the United States. Requiring that optimal clubs be small rules out much interesting economic activity, for example, the formation or break up of nations.

Our paper is one of a few allowing the possibility of large clubs, perhaps as large as the entire population, and the first to study price-taking equilibrium in contexts permitting both overlapping clubs and large clubs. Moreover, we allow a compact metric space of agent types so it does not necessarily hold that there are many exact substitutes for any agent. Other than some standard conditions such as desirability of private goods, the

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4We refer the reader to Wooders (1999) for a survey and to Kovalenkov and Wooders (2003a) for more recent developments.
main assumption of our research is that sufficient wealth, measured in terms of private goods, can compensate for ever larger club sizes. This permits ever-increasing returns to club size while ruling out unbounded increasing returns. A simple example is provided.

An interesting feature of our model is that, depending on the affordability of coalition formation costs for potentially improving coalitions, an equilibrium with communication costs may or may not have similar individuals paying similar costs to belong to a club. If communication costs are affordable, in a sense made precise in the paper, then similar agents belonging to the same club must be treated approximately equally by the equilibrium. If however, communication costs are too large, then similar agents may be treated quite unequally.

In the following, Section 2 develops the model, Section 3 introduces games induced by the economy, and states nonemptiness of the core with communication costs. Section 4 introduces the equilibrium concept. Our main Theorems are stated in this section. With one exception, all Theorems are proven in an Appendix. Section 5 relates our results to the literature and Section 6 concludes the main body of the paper and the Appendix follows.

2 A club economy allowing large clubs

2.1 Agents

Let \( \Omega \) be a compact metric space of attributes with metric \( d \). For later use, without loss of generality, we shall normalize the metric so that \( \max_{\omega, \omega' \in \Omega} d(\omega, \omega') = 1 \). An element of \( \Omega \), typically denoted by \( \omega \), is interpreted as a description of an agent. Let \( F(\Omega) \) denote the set of all pairs \((S, \alpha)\) where \( S \) is a finite nonempty set and \( \alpha : S \rightarrow \Omega \) is an attribute function. In interpretation, \( S \) will be a set of agents and \( \alpha(i), (i \in S) \) describes all relevant characteristics of agent \( i \), including an endowment, preferences, productive abilities, crowding attributes, and so on. Given \( \omega \in \Omega \), the set of agents in \( S \) with attributes \( \omega \) is \( S \cap \alpha^{-1}(\omega) \) and \( |S \cap \alpha^{-1}(\omega)| \) is their number. An economy is a pair \((N, \alpha) \in F(\Omega)\) where \( N = \{1, ..., n\} \) is the set of agents and \( \alpha : N \rightarrow \Omega \) is an attribute function.

2.2 Clubs and club structures

Let \((N, \alpha)\) be an economy. With each nonempty subset of \( N \) there is an associated activity. We call a nonempty subset and its associated activity a club. The activity alone is called the club activity. This activity could be
consumption of a local public good or some shared activity, such as listening to music or swimming in a pool belonging to the club. We note that a club and a coalition will have distinct interpretations. A coalition is simply a nonempty subset of \( N \) while a club is a nonempty subset associated with an activity. We will typically denote a club by \( S_k \) and a coalition by simply \( S \). Let \( S \) be a coalition and let \( \{ S_k \} \) denote a covering of \( S \) (with no repetitions) by clubs.\(^5\) Such a covering is called a club structure of \( S \). Let \( C(S) \) denote the set of all possible club structures of \( S \). We denote a generic element of \( C(S) \) by \( C(S) \).

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Note that an agent may belong to a number of clubs and thus participate in a number of different club activities. For example, an agent may be a member of a marriage, a firm, and a dance club. Observe also that there are no a priori restrictions on club size; for any economy \((N, \alpha)\) the total agent set \( N \) may constitute a club.

Given \((N, \alpha)\), \( S \subset N \), a club structure \( C(S) = \{ S_1, \ldots, S_k, \ldots, S_K \} \in C(S) \), and \( i \in S \), let

\[
C[i; S] = \{ S_k \mid S_k \in C(S) \text{ and } i \in S_k \}
\]

(1)

denote the set of all profiles of clubs in \( C(S) \) that contain consumer \( i \). The set \( C[i; S] \) describes the club memberships of agent \( i \) with respect to \( C(S) \). The set \( \mathcal{C}[i; S] = \cup_{C(S) \in C(S)} C[i; S] \), where the union is taken over all club structures \( C(S) \) of \( S \), is called the club consumption set relative to \( S \) for an agent \( i \in N \).

### 2.3 Attributes

For each attribute \( \omega \in \Omega \) the description of an agent provided by \( \omega \) includes specification of a positive endowment of each of a finite number \( L \) of private goods (there are no endowments of club activities), and a utility function.

Let \((N, \alpha)\) be an economy. Let \( e^i = (e^i_1, \ldots, e^i_k, \ldots, e^i_L) \in \mathbb{R}^L_{++} \) be the endowment of an agent \( i \in N \). The utility function of agent \( i \in N \) is denoted by \( u^i(\cdot, \cdot) \) and maps \( \mathbb{R}^L_x \times \mathcal{C}[i; N] \). Thus, \( \mathbb{R}^L_x \times \mathcal{C}[i; N] \) is the private good consumption set for agent \( i \) and \( \mathcal{C}[i; N] \) is his club consumption set.

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\(^5\)In principle, our techniques allow there to be two (or more) clubs with identical membership offering different activities – repetitions could be allowed. To introduce this formally would significantly increase notational complexity. Also, in principle, some particular clubs may be inadmissible – for example, three-person marriages may be ruled out, at least legally. Inadmissible clubs can be accommodated within our current framework by simply assigning to them negative utilities so that being a member of such a club would not be individually rational.
Agents with the similar attributes have similar endowments. That is, given \( \varepsilon \geq 0 \), if \( i, i' \in N \) and \( d(\alpha^{-1}(i), \alpha^{-1}(i')) \leq \varepsilon \) then \( \|e^i - e^{i'}\|_1 \leq \varepsilon \). Also, the utility functions of \( i \) and \( i' \) are equivalent when the differences in the domains of their club structures are taken into account. To formalize this, let \( C[i; S] \in C[i; S] \) be an arbitrary club structure for player \( i \). Let \( C[i'; S'] \) be the club structure obtained from \( C[i; S] \) when the names of \( i \) and \( i' \) are permuted.\(^6\) Then for any \( x \in \mathbb{R}_{++}^k \) it holds that \( u^i(x, C[i; S]) = u^{i'}(x, C[i'; S']) \).

To discuss the crowding effects of similar but not necessarily identical agents, we also need to define a metric enabling us to say when any two club structures are close to each other. The Prohorov metric, which we will denote by \( \text{dist} \), serves our purpose well. We shall only informally describe the metric here and present a complete description in the Appendix. First, if two coalitions contain different numbers of agents or if their club structures contain differing numbers of clubs, then they are not close to each other; the distance between them is greater than the maximum distance between any two points in \( \Omega \). Formally, let \( S \) and \( S' \) be two subsets of \( S \) and let \( C(S) \) and \( C(S') \) be club structures of \( S \) and \( S' \) respectively. Let \( a \in \mathbb{Z}^{C(S)} \) be a list of the sizes of clubs in \( C(S) \) (with repetitions), ordered from the smallest club to the largest, and similarly let \( a' \in \mathbb{Z}^{C(S')} \) be a list of the club sizes in \( C(S') \). Suppose \( a \neq a' \). Then \( \text{dist}(C(S), C(S')) = 2 \).\(^7\) Next, suppose \( a = a' \). In this case, the two club structures are within \( \varepsilon \geq 0 \) of each other if there is a one-to-one mapping \( \theta \) from the clubs in \( C(S) \) to the clubs in \( C(S') \) so that if \( \theta(S_k) = S'_{k'} \) then \( |S_k| = |S'_{k'}| \), and, for each \( k \) there is a mapping \( \theta^k \) of the members of \( S_k \) onto the set of members of \( S'_{k'} \). It holds that \( \max_{i \in S_k} d(\alpha^{-1}(i), \alpha^{-1}(\theta^k(i))) \leq \varepsilon \). Then \( \text{dist}(C(S), C(S')) \) is defined as the minimum of \( \varepsilon \) over all mappings \( \theta, \theta^k \) satisfying these conditions.

It is assumed that, given any agent \( i \in N \) and any club structure \( C[i; S] \in C[i; S] \), the utility function \( u^i \) satisfies the usual properties of monotonicity, continuity and convexity. Specifically, for any given \( i \in N \) and club consumption \( C[i; S] \), the utility function \( u^i \) satisfies:

(a) **Monotonicity:** \( u^i(\cdot, C[i; S]) \) is an increasing function, that is, if \( x < x' \) then \( u^i(x, C[i; S]) < u^i(x', C[i; S]) \).

(b) **Continuity:** \( u^i(\cdot, C[i; S]) \) is a continuous function.

(c) **Convexity:** \( u^i(\cdot, C[i; S]) \) is a quasi-concave function.

\(^6\)That is, if \( i \in S_k \) then rename \( i \) by \( i' \) and conversely.

\(^7\)What is important here is that this distance is sufficiently large so that the triangle inequality will be satisfied.
(d) **Desirability of endowment**: There exists a real number \( \tau > 0 \) with the property that if \( u^i(e^i - \tau \Gamma, \{ i \}) < u^i(x, C[i; S]) \), then \( x_i >> 0 \).[8]

In addition, we require that agents who are similar in attribute space are similar.

(e) **Continuity with respect to attributes**: Given \( \varepsilon > 0 \), \( i, i' \in N \) with \( d(\alpha^{-1}(i), \alpha^{-1}(i')) \leq \varepsilon \) and club structures \( C[i; S] \) and \( C[i'; S'] \) satisfying \( \text{dist}(C[i; S], C[i'; S']) \leq \varepsilon \) it holds that

\[
u^i(x, C[i; S]) < u^{i'}(x + \varepsilon \Gamma, C[i'; S']).[9]\]

With the exception of (d), the conditions above are all standard. Condition (d) incorporates the Hammond-Kaneko-Wooders (1989) and Kaneko-Wooders (1989) condition that the endowment is preferred to any outcome which assigns an agent zero of any of the indivisible (club) goods.\(^{10}\) Condition (e) ensures that agents who are similar in terms of their attribute have similar utility functions and also that agents who are ‘close’ in attribute space are ‘crowding substitutes’ for each other. To clarify this, suppose \( i = i' \). Then condition (e) states that, for small as long as the club structures \( C[i; S] \) and \( C[i'; S'] \) are ‘similar’, then agent \( i \) is close to indiffereent between whether one prevails or the other.

### 2.4 Production of club activities

Let \( (N, \alpha) \) be an economy and let \( S_k \subset N \) be a club. The production of the club activity for \( S_k \) requires \( z_{S_k} \in \mathbb{R}_+^L \) inputs of private goods. Note that in fact it is possible that zero inputs of private goods are required. For a purely ‘hedonic’ club – a club where the membership of the club itself is the benefit of the club – this may be especially natural.

### 2.5 States of the economy and communication costs

Let \( (N, \alpha) \) be an economy, let \( S \) be a nonempty subset of \( N \), and let \( C(S) \) be a club structure of \( S \).

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[8] This assumption could be weakened but at the cost of more notation and without significant gain in economic understanding.

[9] In the literature of private goods exchange economies, related, more restrictive conditions go back to Broome (1973). For economies with local public goods/clubs an analogous condition was introduced in Wooders (1978,1980). The Hammond-Kaneko-Wooders (1989) condition is less restrictive.
Definition A state of the economy for $S$ relative to $C(S)$ is an ordered pair $(x^S, C(S))$, where $x^S = (x^i : i \in S)$ is an allocation for $S$ satisfying the property that, for each $i \in N$, $x^i \in \mathbb{R}_{++}^L$. The state $(x^S, C(S))$ is feasible if
\[
\sum_{i \in S} (x^i - e^i) \leq \sum_{S_k \in C(S)} z_{S_k}.
\]
Assume that if a group of agents is to form an alliance – a coalition – then the agents must communicate with each other and possibly reallocate goods among themselves. This motivates the introduction of communication costs required to form a coalition. Denote the communication cost for coalition $S$ by $c(\varepsilon, S)$; we assume that
\[
c(\varepsilon, S) \overset{\text{def}}{=} \varepsilon |S| \bar{z},
\]
where $\bar{z} \in \mathbb{R}_{++}^L$ is given and $\varepsilon$ is a non-negative real number.

Definition A state of the economy $(x^S, C(S))$ is $c(\varepsilon, S)$-feasible if
\[
\sum_{i \in S} (x^i - e^i) \leq \sum_{S_k \in C(S)} z_{S_k} - \varepsilon |S| \bar{z}.
\]
Note that implicitly all coalitions face the same per member communication costs. This can be justified by the assumption that there is a common communication technology. We note however, that the commonality of communication costs could be relaxed, but at the cost of more notation and complexity. In particular, it could be more costly to communicate with some types of agents than with others; what is important is that there is some lower bound on the per agent communication costs.

2.6 The core with communication costs

The following concept of the core can be interpreted as either a notion of an approximate core arising from market frictions or as an exact core relative to communication costs, denoted by $c(\varepsilon)$ and parameterized by $\varepsilon$. Let $(x^N, C(N))$ be a state of the economy relative to the club structure $C(N)$. A coalition $S$ can $c(\varepsilon)$-improve upon the state $(x^N, C(N))$ if there is a club structure $C(S)$ of $S$ and a $c(\varepsilon, S)$-feasible state of the economy $(y^S, C(S))$ for $S$ such that for all consumers $i \in \{1, ..., N\}$ it holds that:
\[
u^i(y^i, C[i; S]) > u^i(x^i, C[i, N]).
\]
A feasible state of the economy \((x^N, C(N))\) is in the \(c(\epsilon)\)-core (of the economy) if it cannot be \(c(\epsilon)\)-improved upon by any coalition \(S\). In spirit, one might think of a state of the economy in the \(c(\epsilon)\)-core as ‘secession proof,’ as in the line of recent research by Haimanko, Le Breton, and Weber (forthcoming) and Le Breton and Weber (forthcoming); some state of the economy is taken as given and the question asked is whether that state is vulnerable to secession.

It is clear that when \(\epsilon = 0\) the notion of the \(c(\epsilon)\)-core coincides with the standard notion of the core.

### 2.7 The communication core with remainders

Given the composition of a population \(N\) it may be that some subset of agents cannot be accommodated in their preferred clubs. If this set relatively small, then a solution concept ignoring an exceptional set of agents may provide reasonable approximations to outcomes of an exact solution. Thus, we weaken our notion of the \(c(\epsilon)\)-core to take account of these observations.

An \(\epsilon_1\)-remainder \(c(\epsilon_0)\)-core state of the economy \((N, \alpha)\) is a feasible state of the economy \((x^N, C(N))\) satisfying the property that for some subset \(N^0 \subset N\) with \(\frac{|N \setminus N^0|}{|N|} < \epsilon_1\), \((x^{N^0}, C(N^0))\) is a \(c(\epsilon_0)\)-core state of the economy \((N^0, \alpha|_{N^0})\), where \(\alpha|_{N^0}\) is the restriction of \(\alpha\) to \(N^0\). An \(\epsilon_1\)-remainder \(c(\epsilon_0)\)-core state of the economy simply ignores an exceptional set of agents.

### 3 Edgeworth equilibrium and replica games

#### 3.1 Motivation for Edgeworth equilibrium

Edgeworth (1881) conjectured that if the agent set of an economy were sufficiently large, then the contract curve (the core in allocation space) would shrink to the competitive equilibrium. Debreu and Scarf (1963) gave a rigorous formulation of Edgeworth’s conjecture and demonstrated that if the set of agents in an economy is replicated, the only allocations in the core of the replicated economy for all replications are competitive equilibrium allocations; that is, if an allocation remains in the core for all replications then there exists a price system that, together with the allocation, constitutes a competitive equilibrium. The first step in Debreu and Scarf’s approach to proving Edgeworth’s conjecture, their Theorem 2, was to demonstrate that (with strictly quasi-concave preferences) when an economy is replicated, all replications of an agent must receive the same consumption bundle in any state of the economy in the core. The result enables Debreu and Scarf to
consider the core in a space of fixed dimension, independent of the number of replications of the economy.

A new approach to showing convergence of cores of economies to competitive equilibrium outcomes is treated in Aliprantis, Brown and Burkinshaw (1987), who introduced the concept of the ‘Edgeworth equilibrium’ in the context of economies with infinite dimensional commodity spaces. An Edgeworth equilibrium is defined as a state of the economy with the property that each replica of that state is in the core of the corresponding replicated economy. Aliprantis, Burkinshaw and Brown show that every Edgeworth equilibrium is an equilibrium.11 Treating Edgeworth equilibria avoids the necessity of a symmetry result, such as Debreu and Scarf’s Theorem 2. Nevertheless, since these papers assume quasi-concavity, it is apparent that every Edgeworth equilibrium state of the economy satisfies the equal treatment property; that is, agents with identical endowments and preferences achieve the same utilities from an Edgeworth equilibrium state.12 We highlight that, of course, if a state of the economy is to be an (exact) equilibrium, then it is necessary that identical agents are assigned bundles yielding the same utilities. Also, note that using the concept of Edgeworth equilibrium, the authors can separate the question of existence of states of the economy in the core for all replications from the question of decentralizing prices for such states of the economy.

In this paper, we use the notion of an approximate Edgeworth equilibrium, defined as a state of the economy with the property that all replications of that state are in approximate cores of the corresponding replica economies. We encounter new problems, however, that require new approaches. First, we cannot embed the core in a fixed dimensional finite space. In our framework, new commodities – in particular, new possible clubs – emerge when the agent set grows large. Thus, the dimensionality of the space of possible club memberships goes to infinity as the population grows large. Second, we cannot embed the core in a fixed dimensional finite space. In our framework, new commodities – in particular, new possible clubs – emerge when the agent set grows large. Thus, the dimensionality of the space of possible club memberships goes to infinity as the population grows large. Second, we cannot embed the core in a fixed dimensional finite space. In our framework, new commodities – in particular, new possible clubs – emerge when the agent set grows large. Thus, the dimensionality of the space of possible club memberships goes to infinity as the population grows large. Second, we cannot embed the core in a fixed dimensional finite space. In our framework, new commodities – in particular, new possible clubs – emerge when the agent set grows large. Thus, the dimensionality of the space of possible club memberships goes to infinity as the population grows large. Second, we cannot embed the core in a fixed dimensional finite space. In our framework, new commodities – in particular, new possible clubs – emerge when the agent set grows large. Thus, the dimensionality of the space of possible club memberships goes to infinity as the population grows large. Second, we cannot embed the core in a fixed dimensional finite space. In our framework, new commodities – in particular, new possible clubs – emerge when the agent set grows large. Thus, the dimensionality of the space of possible club memberships goes to infinity as the population grows large. Second, we cannot embed the core in a fixed dimensional finite space. In our framework, new commodities – in particular, new possible clubs – emerge when the agent set grows large. Thus, the dimensionality of the space of possible club memberships goes to infinity as the population grows large. Second, we cannot embed the core in a fixed dimensional finite space. In our framework, new commodities – in particular, new possible clubs – emerge when the agent set grows large. Thus, the dimensionality of the space of possible club memberships goes to infinity as the population grows large. Second, we cannot embed the core in a fixed dimensional finite space. In our framework, new commodities – in particular, new possible clubs – emerge when the agent set grows large. Thus, the dimensionality of the space of possible club memberships goes to infinity as the population grows large. Second,
although we require quasi-concavity of utility functions over private commodities, we do not make such an assumption for club memberships as we believe this would be too restrictive (even if we took account of the indivisibility of agents). In addition, the usual problems of existence of equilibrium in an economy with clubs or local public goods are present; except under special assumptions, the core (without communication costs) may be empty.

Our assumptions, most notably desirability of wealth, and our approach allows us to both to circumvent the dimensionality issue and the equal treatment issue and separate the question of states of the economy in the \( c(\varepsilon_0) \)-core for all replications from the question of existence of equilibrium prices. Also, we are able to demonstrate an approximate equal treatment property under the same assumptions and an additional assumption on the size of coalition formation costs. This assumption, which dictates that a potentially improving coalition can afford the communication costs, illustrates also that large costs of coalition formation hinder fairness or equal treatment of similar individuals. Desirability of wealth also allows us to bound the rate the possible increase in utilities due to the possibility of larger and larger clubs and thus enables us to demonstrate nonemptiness of cores with communication costs in sufficiently large economies.

### 3.2 Replication economies

Our central results depend on extending the replication case (with a fixed distribution of agents on attribute space) to the case of a compact metric space of attributes. Since one of our main assumptions, ensuring ‘per capita boundedness’ of payoff sets of derived games, is required only for replication sequences, we now turn to this case.

Given \((N, \alpha) \in F(\Omega)\), for each positive integer \(r\) we define the \(r^{th}\) replica economy, denoted by \((N_r, \alpha_r) \in F(\Omega)\) as the economy with agent set

\[
N_r = \{(i, q) : i = 1, \ldots, N \text{ and } q = 1, \ldots, r\},
\]

and attribute function \(\alpha_r : N_r \rightarrow \Omega\) where \(\alpha_r(i, q) = \alpha(i), \ q = 1, \ldots, r\) (i.e. all agents \((i, q), (i, q')\), are identical in terms of attributes). The agent \((i, q)\) is called the \(q^{th}\) agent of attribute \(i\). To replicate a state of the economy, in addition to replicating the agent set we also replicate the club structure and consumptions so that all replicas of an agent are in clubs with identical profiles, and are allocated identical consumptions.

Let \(C(N) = \{J_1, \ldots, J_g, \ldots, J_G\}\) be a club structure of \(N\) and let \(r\) be a positive integer. Let \(C(N_r)\) be a club structure of \(N_r\) containing \(rG\) clubs
and denoted by:

\[ C(N_r) = \{ J_{gj} : j = 1, ..., r \text{ and } g = 1, ..., G \}, \]

where for each \( j = 1, ..., r \) and each \( g = 1, ..., G \) the profile of \( J_{gj} \) equals the profile of \( J_g \). Then \( C(N_r) \) is the \( r^{th} \) replication of \( C(N) \).

Let \((x^N, C(N))\) be a state of the economy \((N, \alpha)\). A state of the replicated economy \((N_r, \alpha_r)\), denoted by \((x^{N_r}, C(N_r))\), is an \( r^{th} \) replication of \((x^N, C(N))\) if

(a) for each \( g = 1, ..., G \) and each \( j = 1, ..., r \),

\[ z_{J_{gj}} = z_{J_g}; \]

(b) for each consumer \( i \in N \) there are \( r \) agents \((i, q)\) where \( q = 1, ..., r \), in the replicated agent set \( N_r \) who are allocated the same private goods bundle as \( i \).

A state of the economy \((x^N, C(N))\) is in the \( c(\varepsilon)\)-core for all replications if, for each positive integer \( r \), it holds that an \( r^{th} \) replication of \((x, C(N))\) is in the \( c(\varepsilon)\)-core of the \( r^{th} \) replication of the economy.

We will also require some minimal assumption on the economy to ensure that equal-treatment utilities derived from the economy do not become infinite as club sizes become large. To this purpose we introduce the following assumption:

**Desirability of wealth:** Assume that there is a bundle of private goods, \( x^* \) and a replication number \( r^* \), such that for some club structure \( C(N_{r^*}) \) of the \( r^{th} \) economy, it holds that, for any \( r \), for each \( i \in N \),

\[ u^i(x^i + x^*, C[i, N_{r^*}]) \geq u^i(x^i, C[i, N_r]) \]

for any \( x^i \) and any club structure \( C(N_r) \) of the \( r^{th} \) economy.

Informally, this assumption ensures that wealth, in terms of private goods, can substitute for ‘large’ clubs, no matter how large the economy. Because of the possibility of ever-increasing returns to club size, due to public goods for example, in our model agents may derive more and more utility from larger and larger clubs. Informally, desirability of wealth dictates that if an agent were sufficiently wealthy, however, he could provide club goods for himself and just a few others (no more than \( r^* \) of each agent that appears in the economy) and achieve a preferred outcome. Note that \( x^* \) is independent
of the attribute of the agent; this is for simplicity of statement. Also, note that $x^*$ may not be feasible for the $r^{th}$ economy. Desirability of wealth is considerably weaker than bounding club sizes.

**Example 1.** As a simple example, consider an economy with two identical agents and suppose agents derive utility only from money and from sharing some common activity with other agents. Suppose in particular that the utility function of a representative agent $i$ can be described by

$$u_i(\xi, C[i, N]) = \xi - \frac{1}{|C[i, N]|}$$

where $|C[i, N]|$ is the number of clubs to which the agent belongs (including the club consisting of himself alone) and $\xi$ is money. Suppose that his endowment of money is $\omega_i = 1$. Possible values for $r^*$ and $x^*$, in the definition of desirability of wealth, are $r^* = 5$ and $x^* = 1$. Also, let $C(N_5)$ be the club structure containing all possible clubs. For any club structure $C(N_r)$ of the $r^{th}$ economy it holds that

$$u(\xi + x^*, C[i, N_5]) = \xi + 1 - \frac{1}{25} \geq u(\xi, C[i, N_r]) = \xi - \frac{1}{|C[i, N_r]|}.$$

We highlight that desirability of wealth is satisfied and the feasible per capita utility level as a function of the economy size does not achieve a maximum – desirability of wealth does not imply the existence of an optimal club size.

### 3.3 Edgeworth states of the economy

First, we define the notion of $(\varepsilon_1, \varepsilon_0)$-Edgeworth state of an economy $(N, \alpha)$.

**Definition** Let $(N, \alpha)$ be an economy and let $\varepsilon_1, \varepsilon_0 \geq 0$ be given. An allocation $(x^N, C(N))$ is an $(\varepsilon_1, \varepsilon_0)$-Edgeworth state of the economy $(N, \alpha)$ if there exists a set $N^0 \subset N$ satisfying

1. $|N \setminus N^0| < \varepsilon_1 |N|$ and
2. each replica $(x^{N^0}, C(N^0_r))$ of $(x^{N_0}, C(N_0))$ is in the $c(\varepsilon_0)$-core of the corresponding replication of the economy $(N^0, \alpha^0)$, where $\alpha^0$ denotes the restriction of $\alpha$ to $N^0$ and $\alpha^0$ denotes its $r^{th}$ replica.
Less formally, for any economy \((N, \alpha)\) with sufficiently many agents, an \((\varepsilon_1, \varepsilon_0)\)-Edgeworth state of the economy \((N, \alpha)\) is in an approximate core (the \(\varepsilon_1\)-remainder \(c(\varepsilon_0)\)-core) for all replications of the economy.

Our first Theorem, showing existence of \((\varepsilon_1, \varepsilon_0)\)-Edgeworth state of the economy, is central to demonstrate existence of equilibrium.

**Theorem 1.** Assume desirability of wealth. Then, given any \(\varepsilon_1, \varepsilon_0 > 0\) there is an \(n(\varepsilon_1, \varepsilon_0)\) such that: for any set of agents \(N\), if \(|N| > n(\varepsilon_1, \varepsilon_0)\) then for any attribute function \(\alpha : N \to \Omega\) there exists an \((\varepsilon_1, \varepsilon_0)\)-Edgeworth state of the economy \((N, \alpha)\).

Note that the above Theorem does not depend on replication.

### 3.4 Equal treatment property of Edgeworth states of the economy

In this section, we demonstrate that if communication costs are affordable, then near equal treatment of similar individuals must prevail in any state of the economy in the \(c(\varepsilon)\)-core for all replications.

Let \((N, \alpha)\) be an economy. A state of the economy \((x^N, C(N))\) has the **equal treatment property** if, for all \(i, i' \in \{1, \ldots, N\}\), whenever \(\alpha(i) = \alpha(i')\),

\[
u^{i'}(x^{i'}, C[i', N]) = u^i(x^i, C[i, N]).\]

To show that every state of the economy in the core (the core with zero communication costs) has the equal treatment property, it would be necessary to make some assumptions ensuring that core utility payoffs can be achieved by strict subsets of the population. Consider Example 1 of this paper; it is easy to see that the core need not assign identical agents the same utility levels. Nevertheless, it is well known that, under arguably mild conditions, if relatively small groups of agents are nearly effective for the realization of utility levels of allocations in the core, then cores and approximate cores must assign most (or all) similar individuals similar utility levels. Nevertheless it is desirable to extend such results to the class of economies considered in this paper. Our results below demonstrate that

---

13 Or, if there are no ‘scarce player types’ and per capita payoffs are bounded, which is an apparently extremely mild condition.

14 See, for example, Wooders (1994b) or Kovalenkov and Wooders (2001) and references therein.
states of the economy in the \( c(\varepsilon_0) \)-core for all replications cannot differ substantially from equal treatment states. We first present the results and then a discussion.

Given an economy \((N, \alpha) \in F(\Omega)\) and \(\varepsilon_0 \geq 0\), let \((x^N, C(N))\) be a state of the economy in the \(c(\varepsilon_0)\)-core for all replications of the economy. For each attribute \(\omega \in \Omega\) that appears in the economy (i.e., each \(\omega \in \Omega\) for which \(N \cap \alpha^{-1}(\omega) \neq \emptyset\)) select two agents \(i_\omega\) and \(i_\omega\) as follows
\[
  u_{i_\omega}(x_{i_\omega}, C[N]) = \min_{i \in N \cap \alpha^{-1}(\omega)} u_i(x_i, C[i, N])
\]
and
\[
  u_{i_\omega}(x_{i_\omega}, C[N]) = \max_{i \in N \cap \alpha^{-1}(\omega)} u_i(x_i, C[i, N]).
\]
That is, according to the state of the economy \((x^N, C(N))\), agent \(i_\omega\) is the worst off agent with attribute \(\omega\) and \(i_\omega\) is the best of agent with attribute \(\omega\). Suppose that \(u_{i_\omega}(x_{i_\omega}, C[N]) < u_{i_\omega}(x_{i_\omega}, C[N])\). In this case, let us call \(i_\omega\) the representative of the poor with attribute \(\omega\) and similarly let us call \(i_\omega\) the representative of the rich with attribute \(\omega\). In such cases, we say that communication is affordable for \(i_\omega\) if
\[
  x_{i_\omega} - \varepsilon_0 N \bar{z} \geq 0.
\]

**Proposition 1.** Given an economy \((N, \alpha) \in F(\Omega)\) and \(\varepsilon_0 \geq 0\), let \((x^N, C(N))\) be a state of the economy in the \(c(\varepsilon_0)\)-core for all replications of the economy. For each attribute \(\omega\) for which there exists a representative of the poor \(i_\omega\), if communication is affordable for \(i_\omega\) then
\[
  0 \leq u_{i_\omega}(x_{i_\omega}, C[N]) - u_{i_\omega}(x_{i_\omega}, C[N]) \leq u_{i_\omega}(x_{i_\omega}, C[N]) - u_{i_\omega}(x_{i_\omega} - \varepsilon_0 N \bar{z}, C[N]).
\]
That is, the difference in utilities between the representatives of the poor and of the rich is bounded by the utility loss of the representative of the rich if he had to pay communication costs.

Proof. Suppose not. Then there exists \(\omega \in \Omega\), such that \(N \cap \alpha^{-1}(\omega) \neq \emptyset\) and
\[
  0 \leq u_{i_\omega}(x_{i_\omega}, C[N]) - u_{i_\omega}(x_{i_\omega} - \varepsilon_0 N \bar{z}, C[N]) < u_{i_\omega}(x_{i_\omega}, C[N]) - u_{i_\omega}(x_{i_\omega}, C[N]).
\]
Since \((x^N, C(N))\) is in the \(c(\varepsilon_0)\)-core for all replications of the economy, it follows that the twice replication of \((x^N, C(N))\) is in the \(c(\varepsilon_0)\)-core of the corresponding replica economy. Let \(S\) be a coalition consisting of \(N \setminus \{i, \omega\} \cup \{i'\}\) where \(i'\) is the replica of \(i, \omega\). From the fact that communication is affordable for \(i, \omega\), the allocation \((x^{i, \omega}, y^{i'}, C(S))\) is \(c(\varepsilon_0, S)\)-feasible state for \(S\), where \(y^{i'} = x^{i, \omega} - \varepsilon_0|N| \tilde{z}\) and \(C(S)\) is equal to \(C(N)\) when \(i, \omega\) is replaced by \(i'\). Note that, from (inequality), this state of the economy makes agent \(i'\) strictly better off. Therefore, from continuity and monotonicity of preferences with respect to private commodities, coalition \(S\) can \(c(\varepsilon_0)\)-improve upon the second replica of \((x^N, C(N))\). This is a contradiction.

From the above Proposition, when \(\varepsilon_0 = 0\), a state of the economy in \(c(\varepsilon_0)\)-core for all replications of the economy treats all agents with the same attribute equally in terms of their utilities. The following Proposition demonstrates that, whether or not communication is affordable, if there exists states of the economy in the \(c(\varepsilon_0)\)-core for all replications, then there exists such states with the equal-treatment property.

**Proposition 2**. Given an economy \((N, \alpha) \in F(\Omega)\) and \(\varepsilon_0 \geq 0\), let \((x^N, C(N))\) be a state of the economy in the \(c(\varepsilon_0)\)-core for all replications of the economy. Then there exists an equal treatment state of the economy \((y^N, C(N))\) in the \(c(\varepsilon_0)\)-core for all replications; that is, for any two agents \(i, j \in N\) with \(\alpha(i) = \alpha(j)\) it holds that

\[
u_i(x^i, C[i, N]) = u_j(y^i, C[j, N]).\]

Moreover, if \(u_i(x^i, C(N)) > u_j(x_j, C(N))\) the allocation \(y_i\) can be chosen as a fraction of the allocation \(x_i\).

**Proof**. Suppose for some \(\omega \in \Omega\) and some agent \(i \in N \cap \alpha^{-1}(\omega)\) it holds that \(u^i(x^i, C[i, N]) > u^i(x^{i, \omega}, C[i, \omega], N)\). Since \((x^N, C(N))\) is individually rational it follows that

\[
u^i(x^i, C[i, N]) > u^i(x^{i, \omega}, C[i, \omega], N) \geq u^i(e^{i, \omega} - \varepsilon_0\Gamma, \{i, \omega\}) = u^i(e^i - \varepsilon_0\Gamma, \{i\}),\]

where the final equality results from the fact that \(i, \omega\) and \(i\) have the same attribute. From desirability of endowments it follows that \(x_i > 0\). From continuity of utility functions and the mean value theorem there exists \(0 < \lambda_i \leq 1\) such that \(u^i(\lambda_ix^i, C[i, N]) = u^i(x^{i, \omega}, C[i, \omega], N)\). We consider an allocation \((y^N, C(N))\) satisfying \(y^i = \lambda_ix^i\) if \(u^i(x^i, C[i, N]) > u^i(x^{i, \omega}, C[i, \omega], N)\) and \(y^i = x^i\) otherwise. It is obvious that the allocation \((y^N, C(N))\) is feasible
and satisfies the equal treatment property. Moreover, one can easily show that \((y^N, C(N))\) is in the \(c(\varepsilon_0)\)-core of the economy for all replications of the economy. [If not, eventually a coalition consisting of the worst-off agents with each attribute and their replicas could improve.]

**Remark.** We note that the aggregate deviation of the allocations received by the better off agents cannot exceed the communication costs. More formally, continuing from the above proof, for each agent \(i \in N\) with attribute \(\omega\) and for whom \(u^i(x^i, C[i, N]) > u^i_{\omega}(x^i_{\omega}, C[i_{\omega}, N])\) define \(\Delta_i = (1 - \lambda_i)x_i\); otherwise define \(\Delta_i = 0\). Define \(\Delta = \sum_i \Delta_i\). It cannot hold that \(\Delta \geq c(\varepsilon_0, N)\). If it were the case \(\Delta \geq c(\varepsilon_0, N)\) then, along similar lines as the proof of the first Proposition, we could demonstrate a contradiction. In the case of one-private-good, it must hold that \(\Delta < c(\varepsilon_0, N)\).

Our first Proposition demonstrates that, within the context of our model, given \(\varepsilon_0 > 0\), if a state of the economy in the \(c(\varepsilon_0)\)-core for all replications and the best off agent with a given attribute can feasibly cover communication costs, then all agents with the same attribute must be treated nearly equally. This allows us to place a bound on the differences between the utilities of the best off and worst off agents with that attribute. Since \(\varepsilon_0\) can be made arbitrarily small, the bound can be made ‘small’ and approximate equal treatment holds for all agents with the given attribute.

We also demonstrate that, if any agents are treated better than the worst-off agents of each attribute by a state of the economy in the \(c(\varepsilon_0)\)-core for all replications, then an allocation where the private goods allocations of the ‘better off’ agent are reduced until equal treatment is satisfied is also in the \(c(\varepsilon_0)\)-core for all replications. This indicates that utilities cannot differ ‘substantially’ from equal treatment.

From our assumptions on preferences, it follows that any state of the economy in the \(c(\varepsilon_0)\)-core must assign each agent a strictly positive amount of each private good so for sufficiently small \(\varepsilon_0\), the communication costs would be affordable for every agent \(i\). Thus, given our assumptions, affordability of the communication costs is not restrictive. The main idea, however, which could result from a number of different assumptions on endowments, preferences, and communication costs, is that the cost of forming coalitions bounds the extent of inequality of states of the economy in the \(c(\varepsilon_0)\)-core for all replications.

Turning to an \((\varepsilon_1, \varepsilon_0)\)-Edgeworth state of the economy for \(\varepsilon_1 > 0, \varepsilon_0 \geq 0\), the Propositions above will apply to a subset of agents \(N^0 \subset N\) for whom the restriction of the \((\varepsilon_1, \varepsilon_0)\)-Edgeworth state to the members of that subset
is in the $c(\varepsilon_0)$-core for all replications of the economy. The remainders, $N \setminus N^0$, may be treated vastly unequally.

4 Equilibrium with communication costs

In this section we first define a communication cost equilibrium, called the $c(\varepsilon_0)$-equilibrium, and then state our Edgeworth equivalence results.

A price system for private goods is a vector $p \in \mathbb{R}_+^L$. A participation price system is a set

$$
\Pi = \{\pi^i(S_k) \in \mathbb{R} : S_k \subset N \text{ and } i \in S_k\},
$$

stating a participation price, positive, negative, or zero, for each agent in each club $S_k$.

A $c(\varepsilon_0)$-equilibrium (for an economy with club goods) is an ordered triple $((x^N, C(N)), p, \Pi)$ consisting of: a state of the economy $(x^N, C(N))$, $C(N) = \{J_1, \ldots, J_g, \ldots, J_G\}$; a price system $p \in \mathbb{R}_+^L \setminus \{0\}$ for private goods and: a participation price system $\Pi$, and satisfying:

(i) $\sum_{i \in N}(x^i - e^i) \leq \sum_{J_g \in C(N)} z_{J_g}$ (feasibility);

(ii) for each possible club $S_k \subset N$,

$$
p \cdot z_{S_k} + \sum_{i \in S_k} \pi^i(S_k) \leq 0
$$

(no possible club profit);

(iii) for any agent $i \in N$, any $S \subset N$ with $i \in S$, and any club structure $C(S)$ of $S$, if

$$
\mu^i(y^i, C[i; S]) > \mu^i(x^i, C[i; N])
$$

then

$$
p \cdot y^i + \sum_{S_k \in C[i; S]} \pi^i(S_k) > p \cdot e^i - \varepsilon_0 p \cdot \bar{z}
$$

(maximization given costs of coalition formation)$^{15}$;

(iv) $\sum_g p \cdot z_{J_g} + \sum_g \sum_{i \in J_g} \pi^i(J_g) \geq - \sum_{i \in N} \varepsilon_0 p \cdot \bar{z}$, and

$^{15}$Since any club structure $C(S)$ of $S$ can be embedded in a club structure of $N$, say $C(N)$, so that $C[i; S] = C[i; N]$, this condition could also be stated in terms of club structures of the total agent set $N$. 

19
\[ \sum_{i \in N} p \cdot x^i + \sum_{i \in N} \sum_{g} \pi^i (J_g) \leq \sum_{i \in N} (p \cdot e^j - \varepsilon_0 p \cdot \bar{z}). \]

(agents cannot be significantly far, in aggregate, inside their budget sets and similarly for clubs).

Our notion of \( c(\varepsilon_0) \)-equilibrium allows at least some agents to spend less than their entire income at the given prices. This is motivated by communication costs, which affect not only the opportunities to change club memberships but also opportunities to purchase difference commodity bundles.

An \( \varepsilon_1 \)-remainder \( c(\varepsilon_0) \)-equilibrium is an ordered triple \((x^N, C(N)), p, \Pi)\) such that for some subset of agents \( N^0 \subset N \) satisfying \( \frac{|N \setminus N^0|}{|N|} < \varepsilon_1 \) there exists a \( c(\varepsilon_0) \)-equilibrium \((x^{N^0}, C(N^0)), p, \Pi)\) as defined above.

Our notion of \( \varepsilon_1 \)-remainder \( c(\varepsilon_0) \)-equilibrium dictates that all agents in the economy are ‘competitive’ or almost competitive except perhaps small proportions of ‘left over’ agents. Concepts of approximate equilibrium or cores involving left over agents are common in the literature of game theory and economics. The left-overs may have unsatisfied demands. Such situations may arise from imperfections in markets.

In Theorem 2 we demonstrate that an \( \varepsilon_1 \)-remainder \( c(\varepsilon_0) \)-equilibrium state of the economy is in the \( \varepsilon_1 \)-remainder \( c(\varepsilon_0) \)-core.

**Theorem 2:** Let \((N, \alpha)\) be an economy. An \( \varepsilon_1 \)-remainder \( c(\varepsilon_0) \)-equilibrium state of the economy is in the \( \varepsilon_1 \)-remainder \( c(\varepsilon_0) \)-core.

**Proof:** Suppose the Theorem is false. Then there exists at least one \( \varepsilon_1 \)-remainder \( c(\varepsilon_0) \)-equilibrium (otherwise the result would be vacuously true). Let \((x^{N^0}, C(N^0)), p, \Pi)\) be the associated \( c(\varepsilon_0) \)-equilibrium with \( \frac{|N \setminus N^0|}{|N|} < \varepsilon_1 \). If the state of the economy \((x^{N^0}, C(N^0))\) is not in the \( c(\varepsilon_0) \)-core, then there is a coalition \( S \subset N^0 \), a club structure \( C(S) \) of \( S \) and an allocation \((y^S, C(S))\) such that

\[ \sum_{i \in S} (y^i - e^i) \leq \sum_{S_k \in C(S)} z_{S_k} - \varepsilon_0 |S| \bar{z} \]

and

\[ u^i(y^i, C[i; S]) > u^i(x^i, C[i, N^0]). \]

From (ii) of the definition of an \( c(\varepsilon_0) \)-equilibrium it holds that
\[ p \cdot z_{S_k} + \sum_{i \in S_k} \pi^i(S_k) \leq 0 \]

and from utility maximization, it holds that

\[ p \cdot y^i + \sum_{S_k \in C\{i:S\}} \pi^i(S_k) > p \cdot e^i - \varepsilon_0 p \cdot \bar{z}. \]

Summing up these above inequalities, one will have

\[ \sum_{i \in S} p \cdot (y^i - e^i) > \sum_{S_k \in \mathcal{C}(S)} p \cdot z_{S_k} - p \cdot \varepsilon_0 |S| \bar{z}, \]

which is a contradiction.

**Theorem 3.** Let \((N, \alpha)\) be an economy and let \(\varepsilon_1, \varepsilon_0 > 0\) be given. If \((x^N, C(N))\) is an \((\varepsilon_1, \varepsilon_0)\) Edgeworth state of the economy then there exists a price system for private goods \(p\) and participation prices \(\Pi\) with the property that \((x^N, C(N), p, \Pi)\) is an \(\varepsilon_1\)-remainder \(c(\varepsilon_0)\)-equilibrium.

We highlight that: Theorem 1 provides an existence result for \((\varepsilon_1, \varepsilon_0)\)-Edgeworth states of the economy. Theorem 2 states that an \(\varepsilon_1\)-remainder \(c(\varepsilon_0)\)-equilibrium is in the \(\varepsilon_1\)-remainder \(c(\varepsilon_0)\)-core; Theorem 3 states that an \((\varepsilon_1, \varepsilon_0)\)-Edgeworth state of the economy is an equilibrium state of the economy.

## 5 Relationships to the literature

The literature on economies with clubs or local public goods, in particular the seminal contributions of Tiebout (1956) and Buchanan (1965), relating to this paper is discussed at some length in survey papers by Kovalenkov and Wooders (2005) and Conley and Smith (2005); thus, we discuss here only the most closely related papers.

### 5.1 Multiple memberships

To the best of our knowledge, the first paper to allow multiple memberships in jurisdictions or clubs is Shubik and Wooders (1982), which demonstrated nonemptiness of approximate cores of economies with many agents. Most recently, Kovalenkov and Wooders (2003) demonstrated conditions under which large finite games and economies with clubs and possibly multiple...
memberships in clubs have nonempty approximate cores. Subsequently, Ellickson et al. (2001) introduced a model of an economy with multiple memberships and obtained approximate versions of existence of equilibrium and equivalence of the core and the set of equilibrium outcomes. Their model is more restrictive than the prior model of Shubik and Wooders (1982) and Kovalenkov and Wooders in the sense that Ellickson et al. (2001) allow only a bounded number of distinct sorts of clubs; thus clubs become negligible as the economy grows large. One interpretation of the Ellickson et al. approach is that the space of clubs becomes analogous to a finite dimensional space of private goods. In the case of one-private-good, the restrictions of Ellickson et al. (2001) transform the economy into an essentially private goods economy with indivisibilities and a consistency condition on club memberships that yield an appropriate feasibility condition.

5.2 Unbounded club sizes

In view of the prior literature on large games and large economies one might hope for approximate equivalence in large finite economies even with multiple memberships in clubs and with potentially ever-increasing returns to club size. The crucial restriction appears to be that almost all gains to collective activities are realized by groups bounded in size; that is, small groups are effective. Our research demonstrates an asymptotic equivalence when arbitrarily large clubs and ever increasing returns to club size are allowed.

Since we allow unbounded club sizes and ever-increasing returns to club sizes the prior approaches in the literature treating price-taking equilibrium in situations with multiple memberships in clubs will not suffice. The fact that the set of possible clubs grows without bound is one of the motivating features of our approach to equilibrium.

In the literature on approximate cores of games and economies with collective activities and clubs, there are a number of models in the literature permitting ever-increasing gains to coalition and club sizes (cf., Wooders 1983, 1994 and Kovalenkov and Wooders 2001(a,b), 2003(a,b)). These models permit games derived from economies where individuals may belong to overlapping clubs and where there may be ever-increasing gains to club size. In addition, following Shubik and Wooders (1982), Kovalenkov and Wooders (2003a) explicitly allow an individual to belong to multiple clubs. None of these papers, however, treat price-taking equilibrium. Other papers treating price taking equilibrium require significantly stronger assumptions on gains to club size.\(^{16}\) While Wooders (1989) allows club sizes to be unbounded,

\(^{16}\) See, for example, Conley and Wooders (1997,2001), Ellickson et al. (2001), and Wood-
to demonstrate existence of states of the economy in $c(\varepsilon)$-core for all replications she requires that there is a ‘minimum efficient scale’ – utility levels that can be realized in an arbitrarily large economy can be realized with clubs bounded in size. Similar restrictions are made in Wooders (1997). Moreover, the problem of multiple memberships is not treated in either of these papers.

### 5.3 The techniques of our decentralization result

To ensure that the games derived from the economies satisfy per capita boundedness – simply boundedness of the set of equal treatment payoffs – we make an assumption of ‘desirability of wealth’. Informally, this assumption dictates that there is some level of wealth, measured in terms of a bundle of private goods, such that an individual would prefer that level of wealth and membership in some bounded number of clubs, all bounded in size, to any feasible equal-treatment outcome in any economy, no matter how large. Loosely, desirability of wealth implies that private goods can compensate for membership in large clubs and also for membership in more and more clubs. At the nub of our proof are results from Wooders (1983) showing convergence of equal treatment utility vectors for replica games satisfying per capita boundedness.

A crucial innovation in the current paper is our construction of the commodity space. Part of this innovation is in extending and further developing the Foley (1970)-Wooders (1985) proof technique of defining ‘preferred sets of allocations of private goods’ for coalitions.

To place our proof techniques in the literature, our research builds on the research of Debreu and Scarf (1963), Foley (1970) and Wooders (1985). Recall that, given a state of the economy that is in the core for all replications of the total agent set, Debreu and Scarf (1963) define the set of preferred net trades of each agent in the economy and show that the convex hull of union of these sets can be separated from the origin. For an economy with pure public goods, Foley (1970) extends the commodity space to make the public good a separate good for each consumer. Wooders (1985) further extends the commodity space to make local public goods for each consumer in each possible jurisdiction separate commodities. In this paper, we build on these three approaches. Precisely, we extend the public good space so that each club and its membership is a different commodity for each agent in the club. Having done so, extensions of the techniques of Debreu and Scarf (1963) can...
be applied. We also introduce a virtual production set. Even though we have no production in the current paper, our virtual production set plays a similar role to the extended production sets in Foley (1970) and Wooders (1985). In particular, the feasibility requirements ensuring the club choices are consistent are imposed on the virtual production set.

6 Conclusions

The major economic importance of our research is that equilibrium clubs may be unbounded — they do not necessarily become infinitesimal as the economy grows large. In this respect, our model is similar to those with a fixed number of jurisdictions, see, for example, Konishi, Le Breton and Weber (1998) and references therein. This aspect of our modeling is especially relevant for questions of political economy, for example, and to issues of regulation of large firms, such as multinationals. We hope to study these issues, as well as other issues relating to labor markets in economies with large firms/jurisdictions in future research.

7 Appendix

7.1 The metric on the space of club structures

To be included.

7.2 Proofs

Now, we state and prove our first result.

Theorem 1. Assume desirability of wealth. Then, given any $\varepsilon_1, \varepsilon_0 > 0$ there is an integer $n(\varepsilon_1, \varepsilon_0)$ such that: for any set of agents $N$, if $|N| > n(\varepsilon_1, \varepsilon_0)$ then for any attribute function $\alpha : N \to \Omega$ there exists an $(\varepsilon_1, \varepsilon_0)$-Edgeworth state of the economy $(N, \alpha)$.

Proof of Theorem 1. The proof is divided into two steps.

STEP 1.

Suppose the claim of the Theorem is not true. Then there exists $\varepsilon_1, \varepsilon_0 > 0$ and a sequence of economies $(N_{\nu}, \alpha_{\nu})_{\nu=1}^{\infty}$ such that for every $\nu$ we have $|N_{\nu}| > \nu$ and the $\varepsilon_1$-remainder $c(\varepsilon_0)$-core of $(N_{\nu}, \alpha_{\nu})$ is empty.

Observe that:
From assumption (e), boundedness of marginal utilities with respect to at least one commodity, for any agent set \( N \) there is a positive number \( \rho > 0 \) such that for every club structure \( C(N) \), for each agent \( i \in N \) and every consumption \( x^i \in \mathbb{R}^L_+ \) we obtain

\[
u^i(x^i + \varepsilon_0 \bar{z}, C[i; N]) \geq u^i(x^i, C[i; N]) + \rho.
\]

From continuity with respect to attributes, given \( \varepsilon_0 > 0 \) there exists a partition \( \Omega_1, \ldots, \Omega_T \) be a partition of \( \Omega \) such that if \( \omega, \omega' \in \Omega_i \) then \( d(\omega, \omega') < \frac{\varepsilon_0}{2^T} \).

For each \( t = 1, \ldots, T \) select arbitrarily \( \omega_t \in \Omega_t \). For each \( (N_\nu, \alpha^\nu) \) define another pair \( (N_\nu, \gamma^\nu) \) where the attribute function \( \gamma^\nu \) is defined by \( \gamma^\nu(i) = \omega_t \) whenever \( \alpha^\nu(i) \in \Omega_t \). We note that the range of all the \( \gamma^\nu \) is finite and therefore one may represent this sequence of economies as

\[ N_\nu = \{(t, q): t = 1, \ldots, T \text{ and } q = 1, \ldots, n_\nu^t \} \]

where all agents \((t, q)\) and \((t', q')\) with \( t = t' \) are substitutes for each other – that is, they have the same attributes.

The following Lemma is used to approximate the sequence of economies \( (N_\nu, \gamma^\nu) \) by a sequence of replication economies.

**Lemma 1** (Wooders, 1992 Lemma 1). Let \( \{N_\nu\} \) be a sequence of sets of agents where

\[ N_\nu = \{(t, q): t = 1, \ldots, T \text{ and } q = 1, \ldots, n_\nu^t \} \]

for some integers \( n_\nu^t \), \( t = 1, \ldots, T \). Suppose that, for each \( t = 1, \ldots, T \),

\[ \frac{|N_\nu^t|}{|N_\nu|} \]

converges to a limit \( n_t \),

where \( N_1^\nu \) def \( \{(i, q): q = 1, \ldots, n_1^\nu \} \), the subset of agents in \( N_\nu \) of type \( i \). Then, given \( \varepsilon_1 > 0 \) there exists a vector of integers, \( \pi = (\pi_1, \ldots, \pi_T) \), such that for all \( \nu \) sufficiently large, for some \( r^\nu \in \mathbb{Z}_+ \) and \( \ell^\nu \in \mathbb{Z}_+^T \)

\[ n^\nu = (n_1^\nu, \ldots, n_T^\nu) = r^\nu \pi + \ell^\nu \]

and

\[ \frac{\|\ell^\nu\|}{\|n^\nu\|} < \varepsilon_1 \]

where, for any vector \( n \), \( \|n\| = \sum_t n_t \). (Observe that \( |N_\nu| = \|n^\nu\| \).)
Now let us consider an economy $\mathbf{N}$ with profile $\mathbf{\pi}$. We will need the following definition and Lemma for replication sequences $\mathbf{N}_r$ of $\mathbf{N}$.

A state of the economy $(x^N, C(N))$ satisfies the equal treatment property in utility space whenever, for all $i, i' \in \{1, \ldots, N\}$, if $\alpha(i) = \alpha(i')$ it holds that:

$$u^i(x^{i'}, C[i', N]) = u^i(x^i, C[i, N]).$$

We call $(x^N, C(N))$ a utility equal treatment feasible state of the economy.

**Lemma 2** Assume desirability of wealth. Then there is a positive real number $K$ such that for any replication number $r$ and for any utility equal treatment feasible state of the $r$th economy (with profile $r\mathbf{\pi}$),

$$\sup_i u^i(x^i, \mathbf{N}_r[i]) < K.$$ (per capita boundedness).

**Proof.** First, define $(\mathbf{N}_r, V^e_r)$ as the game induced by the $r$th replication of the economy with agent set $\mathbf{N}$ and profile $\mathbf{\pi}$. To show per-capita boundedness of $(\mathbf{N}_r, V^e_r)_{r=1}^{\infty}$ we construct an auxiliary sequence of replica *-economies* and their induced games, denoted by $(\mathbf{N}_r, V^*_r)_{r=1}^{\infty}$. To obtain the conclusion of the Lemma demonstrate that $V^e_r(\mathbf{N}_r) \subset V^*_r(\mathbf{N}_r)$ and that $(\mathbf{N}_r, V^*_r)_{r=1}^{\infty}$ satisfies per-capita boundedness.

For each *-economy, let the utility function of agent $i$ be defined by

$$u^*_i(x^i) = \max_{r \leq r^*} u^i(x^i + x^*, C[i; \mathbf{N}_r]),$$

where $r^*$ satisfies desirability of wealth.

The utility functions $u^*_i$ are well defined and are quasi-concave. Also, it is clear that given any $(x^i, C[i; \mathbf{N}_r])$ for any $r$ we have

$$u^*_i(x^i) \geq u^i(x^i, C[i; \mathbf{N}_r])$$

from desirability of wealth.

For each $r$, the allocation $(x^{\mathbf{N}_r})$, is *-feasible if

$$\sum_{iq \in \mathbf{N}_r} (x^{iq} - e^{iq}) \leq 0.$$

The set of all *-feasible allocations is denoted by $A^*_r$. Let $K$ be a real number such that

$$K > \max_{i \in \mathbf{N}} \max_{x=(x^1, \ldots, x[^N]) \in A^*_i} u^*_i(x^i).$$

From the closedness of $A^*_r$ and quasi concavity there is a such real number. Obviously, since $V^e_r(\mathbf{N}_r) \subset V^*_r(\mathbf{N}_r)$, $K$ is a per-capita bound for the original sequence of games. $\square$
7.3 The game induced by an economy

Given an economy, we associate a parameterized collection of games with the economy, where the parameter depends on the communication costs. We first select a real number \( \varepsilon_0 > 0 \) sufficiently small so that for every \( \varepsilon \in [0, \varepsilon_0] \) it holds that \( \varepsilon \bar{z} \leq \tau_1 \).

Given an economy \( (N, \alpha) \in F(\Omega) \) and \( \varepsilon \in [0, \varepsilon_0] \) we denote the game induced by the economy by \( (N, V^\varepsilon_\alpha) \), where \( V^\varepsilon_\alpha \) is a correspondence mapping subsets \( S \) of \( N \) into \( \mathbb{R}^N_+ \). For each subset \( S \) of \( N \), define \( V^\varepsilon_\alpha \) as the set of vectors \( v \in \mathbb{R}^N_+ \) with the property that for some club structure \( C(S) \) of \( S \) and some \( c(\varepsilon, S) \)-feasible state with associated allocation \( (x^S, C(S)) \) we have \( v_i \leq u^i(x^i, C[i; S]) \) for each \( i \in S \) and \( v_i = 0 \) for \( i \notin S \). When \( \varepsilon = 0 \), we denote \( V^\varepsilon_\alpha \) simply by \( V_\alpha \). Observe that since utility functions are continuous and endowments are bounded, for every \( S, V^\varepsilon_\alpha(S) \) is a compact subset of \( \mathbb{R}^N_+ \).

Now, we state and prove our first result.

7.4 Theorem 1

**Theorem 1.** Assume desirability of wealth. Then, given any \( \varepsilon_1, \varepsilon_0 > 0 \) there is an integer \( n(\varepsilon_1, \varepsilon_0) \) such that: for any set of agents \( N \), if \( |N| > n(\varepsilon_1, \varepsilon_0) \) then for any attribute function \( \alpha : N \to \Omega \) there exists an \( (\varepsilon_1, \varepsilon_0) \)-Edgeworth state of the economy \( (N, \alpha) \).

**Proof of Theorem 1.** The proof is divided into two steps.

**STEP 1.**
Suppose the claim of the Theorem is not true. Then there exists \( \varepsilon_1, \varepsilon_0 > 0 \) and a sequence of economies \( (N_\nu, \alpha^\nu)_{\nu=1}^{\infty} \) such that for every \( \nu \) we have \( |N_\nu| > \nu \) and the \( \varepsilon_1 \)-remainder \( c(\varepsilon_0) \)-core of \( (N_\nu, \alpha^\nu) \) is empty.

Observe that from continuity with respect to attributes, there is a partition \( \Omega_1, \ldots, \Omega_T \) of \( \Omega \) such that if \( \omega, \omega' \in \Omega_t \) then \( d(\omega, \omega') < \frac{\varepsilon_0}{2} \).

For each \( t = 1, \ldots, T \) select arbitrarily \( \omega_t \in \Omega_t \). For each \( (N_\nu, \alpha^\nu) \) define another pair \( (N_\nu, \gamma^\nu) \) where the attribute function \( \gamma^\nu \) is defined by \( \gamma^\nu(i) = \omega_t \) whenever \( \alpha^\nu(i) \in \Omega_t \). We note that the range of all the \( \gamma^\nu \) is finite and therefore one may represent this sequence of economies as

\[
N_\nu = \{(t, q) : t = 1, \ldots, T \text{ and } q = 1, \ldots, n_t^\nu\}
\]

where all agents \((t, q)\) and \((t', q')\) with \( t = t' \) are substitutes for each other – that is, they have the same attributes.
The following Lemma is used to approximate the sequence of economies \((N_\nu, \gamma_\nu)\) by a sequence of replication economies.

**Lemma 3** (Wooders, 1992 Lemma 1). Let \(\{N_\nu\}\) be a sequence of sets of agents where

\[ N_\nu = \{(t, q) : t = 1, \ldots, T \text{ and } q = 1, \ldots, n_\nu^t\} \]

for some integers \(n_\nu^t\), \(t = 1, \ldots, T\). Suppose that, for each \(t = 1, \ldots, T\),

\[ \frac{|N_\nu^t|}{|N_\nu|} \]

converges to a limit \(n_t\), where \(N_\nu^i = \{(i, q) : q = 1, \ldots, n_\nu^i\}\), the subset of agents in \(N_\nu\) of type \(i\).

Then, given \(\epsilon_1 > 0\) there exists a vector of integers, \(n = (n_1, \ldots, n_T)\) such that for all \(\nu\) sufficiently large, for some \(r_\nu \in \mathbb{Z}^+\) and \(\ell_\nu \in \mathbb{Z}^+_T\)

\[ n_\nu = (n_\nu^1, \ldots, n_\nu^T) = r_\nu \pi + \ell_\nu \]

and

\[ \frac{\|\ell_\nu\|}{\|n_\nu\|} < \frac{\epsilon_1}{2} \]

where, for any vector \(n\), \(\|n\| \overset{\text{def}}{=} \sum_t n_t\). (Observe that \(|N_\nu| = \|n_\nu\|\).

Now let us consider an economy \(\overline{N}\) with profile \(\pi\). We call a state of the economy \((x^N, C(N))\) satisfying the equal treatment property an equal treatment feasible state of the economy.

**Lemma 4** Assume desirability of wealth. Then there is a positive real number \(K\) such that for any replication number \(r\) and for any feasible equal treatment state of the \(r\)th economy (with profile \(r\pi\)),

\[ \max_i u^i(x^i, \overline{N}_r[i]) < K. \]

(per capita boundedness).

**Proof.** First, define \((\overline{N}_r, V_r^0)\) as the game induced by the \(r\)th replication of the economy with agent set \(\overline{N}\) and profile \(\pi\). To show per-capita boundedness of \((\overline{N}_r, V_r^0)_{r=1}^\infty\) (and thus of \((\overline{N}_r, V_r^\varepsilon)_{r=1}^\infty\) for any \(\varepsilon > 0\)) we construct an auxiliary sequence of replica ’*’-economies’ and their induced games, denoted by \((\overline{N}_r, V_r^*)_{r=1}^\infty\). To obtain the conclusion of the Lemma demonstrate that \(V_r^0(\overline{N}_r) \subset V_r^*(\overline{N}_r)\) and that \((\overline{N}_r, V_r^*)_{r=1}^\infty\) satisfies per-capita boundedness.
For each *-economy, let the utility function of agent $i$ be defined by

$$u^*_i(x^i) = \max_{r \leq r^*} u^i(x^i + x^*, C[i; N_r]),$$

where $r^*$ satisfies the desirability of wealth condition.

The utility functions $u^*_i$ are well defined and are quasi-concave. Also, it is clear that given any $(x^i, C[i; N_r])$ for any $r$ we have

$$u^*_i(x^i) \geq u^i(x^i, C[i; N_r])$$

from desirability of wealth.

For each $r$, the allocation $(x^{N_r})$ is *-feasible if

$$\sum_{iq \in N_r} (x^{iq} - e^{iq}) \leq 0.$$

The set of all *-feasible allocations is denoted by $A^*_r$. Let $K$ be a real number such that

$$K > \max_{i \in N} \max_{x = (x^i, \ldots, x^{N}) \in A^*_1} u^*_i(x^i).$$

From the closedness of $A^*_1$ and quasi concavity there is a such real number. Obviously, since $V^0_r(N_r) \subseteq V^*_r(N_r)$, $K$ is a per-capita bound for the original sequence of games.

The fundamental paper showing nonemptiness of approximate cores of games with a fixed distribution of player types is Wooders (1983); we rely heavily on results in that paper, especially Lemmas 1-6. We refer the reader to Wooders (1983) or Kovalenkov and Wooders (2001b) for further discussion of the balanced cover of a game and related properties to those used below.

**Lemma 5** There exists an integer $m_0$ such that, for some vector $v \in \mathbb{R}^{m_0 N}$ the $k$th replica of $v$ is in the $c(\varepsilon_0)$ - core of $(N_r, V^0_r)$ for all sufficiently large $r$.

**Proof of Lemma 5.** Let $E^{\varepsilon_0}(r) \subseteq \mathbb{R}^T$ represent the set of equal treatment payoff vectors for the game derived from the economy $(N_r, V^0_r)$ and let $\tilde{E}^{\varepsilon}(r)$ represent the set of equal treatment payoff vectors for the balanced cover game derived from $(N_r, V^0_r)$. Also let $E^0(r)$ denote the set of equal treatment payoff vectors for the game derived from the economy with zero costs of coalition formation. Observe that from monotonicity, $E^{\varepsilon_0}(r)$ is
strictly contained in $E^0(r)$. Moreover, from properties of the balanced cover, there is an equal treatment payoff vector in the core of the balanced cover of a game. (Of course this payoff may not be feasible for the game itself.)

From Wooders (1983), for all replication numbers $r$ it holds that $E^\varepsilon_0(r) \subset E^\varepsilon_0(r + 1)$ and the closed limit of the sequence $\{E^\varepsilon_0(r)\}$ exists; let $L(\varepsilon_0)$ denote this limit. Let $\hat{r}$ be sufficiently large so that $E(\varepsilon_0) \subset E^0(r)$ for all $r > \hat{r}$. This is possible from the fact that, for each $r$, $E^\varepsilon_0(r)$ is strictly contained in $E^0(r)$.

From per capita boundedness, given any $r$, there is an integer $m_r$ such that $E^0(km_r) \subset E^0(km_{r+1})$ for all positive integers $k$ (Wooders 1983, Lemma 517). Let $u^* \in R^T$ denote an equal treatment payoff vector in the core of the balanced cover game derived from the game $(N_r, V_{\varepsilon_0}^r)$ and let $u^*$ denote the limit of any converging subsequence of the sequence $\{u^r\}$. Note that $u^*$ cannot be $c(\varepsilon_0)$-improved upon by any coalition in any game where the players have the same attributes as those of players in $N_r$, no matter how large the total player set; this is because, if some coalition could $c(\varepsilon_0)$ improve upon $u^*$, then ‘on the way to $u^*$’ there would be an economy containing that coalition and the coalition could improve upon the equal treatment core payoff vector $u^*$. Since $L(\varepsilon_0) \subset E^0(r) \subset E^0(km_r)$ it follows that for any positive integer $k$ there is a feasible state of the economy with agent set $N_{km_r}$, say $(x^{N_{km_r}}, C(N_{km_r}))$, that assigns each agent with attribute $\omega_t$ the utility level $u^*_t$ and that cannot be $c(\varepsilon_0)$-improved upon by coalition $S \subset N_{km_r}$. [If such a state could be improved upon, some set of agents with the same profile as $S$ could improve upon the equal treatment payoff $u$ in the game $(N_{km_r}, V_{km_r}^\varepsilon)$ which is impossible from the definition of $u^*$].

Now select some integer $\tilde{r} > \hat{r}$ sufficiently large so that $\frac{m_{\tilde{r}}}{m_r} < \frac{\varepsilon_1}{2}$ and consider any economy $(N_r, V_{\varepsilon_0}^r)$ where $r \geq \tilde{r}$. The profile of the set of agents in the economy can be written as $n^\nu = k_1 m_{\tilde{r}} - \tilde{\pi} + k_2 \tilde{\pi} + \ell^\nu$ where $k_2 < m_{\tilde{r}}$. From the above argument, there is a state of the economy in the $c(\varepsilon_0)$-core of the subeconomy where the profile of the player set of the sub-economy is $k_1 m_{\tilde{r}} - \tilde{\pi}$. Since

$$\frac{\|k_2\tilde{\pi} + \ell^\nu\|}{\|n^\nu\|} \leq \frac{\|m_{\tilde{r}}\|}{\|n^\nu\|}\frac{\|\ell^\nu\|}{\|n^\nu\|} < \frac{\varepsilon_1}{2} + \frac{\varepsilon_1}{2} = \varepsilon_1$$

we have a contradiction.

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\textsuperscript{17}The integer $m_r$ clears all the denominators of the balancing weights for any minimal balanced collection of subsets of $N_r$. See Wooders (1983) for details.
Now for each of the economies \((N_\nu, \alpha^\nu)_{\nu=1}^\infty\) with \(r \geq \hat{r}\), we ignore a fraction of agents and consider only a subset of agents \(N_0^\nu\) with \(\|N_0^\nu\| = ||k_1 m_\nu|\) and where \(N_0^\nu\) is ‘approximated’ by \(\overline{N}_{km,r}\). Consider a state of the economy with the equal treatment property in the \(c(\varepsilon_0)\)-core of the economy, and with utility payoff vector \(u^\ast\). By construction, there is a feasible state of the economy \((N_\nu^0, \alpha^\nu)\) that cannot be \(c(\varepsilon_0)\)-improved upon by any coalition contained in the economy \(\overline{N}_{k_1 m_\nu}\). We claim that this means we can find a state of the economy \((N_\nu^0, \alpha^{\nu'})\) that cannot be \(c(\varepsilon_0)\)-improved upon by any coalition in \(N_\nu^0\). In particular, since endowments of agents in \(N_\nu^0\) are within \(\varepsilon/2\) of their representatives in \(\overline{N}_{km,r}\) and in construction of the sets \(E^{\varepsilon_0}(r)\) we have effectively subtracted \(\varepsilon/2\) from each agent in each possible club, we can construct a state where each agent is receiving the same allocation as his representative in \(\overline{N}_{k_1 m_\nu}\) plus \(\varepsilon/2\). Therefore each agent is strictly better off than his representative in \(\overline{N}_{k_1 m_\nu}\) and therefore no coalition in \(N_0^\nu\) can \(c(\varepsilon_0)\)-improve upon the final allocation constructed. This contradicts our initial hypothesis.\(\blacksquare\)

7.5 Theorem 3

**Theorem 3.** Let \((N, \alpha)\) be an economy and let \(\varepsilon_1, \varepsilon_0 > 0\) be given. If \((x^N, C(N))\) is an \((\varepsilon_1, \varepsilon_0)\) Edgeworth state of the economy then there exists a price system for private goods \(p\) and participation prices \(\Pi\) with the property that \((x^N, C(N)), p, \Pi)\) is an \(\varepsilon_1\)-remainder \(c(\varepsilon_0)\)-equilibrium.

**Proof of Theorem 3**

The proof of the Theorem is an extension of proofs of convergence of the core to equilibrium states due to Debreu and Scarf (1963) and existence proof of Foley (1970) and Wooders (1989). Without any loss of generality we can assume that there exists \(N_0\) such that \(\|N_0\| < \varepsilon_1\) and \((x^{N_0}, C(N^0))\) in the \(c(\varepsilon_0)\)-core of the economy for all replications of the economy. Let \(\{S_1, ..., S_k, ..., S_K\}\) denote the set of all clubs in \(N^0\) and let \(C(N^0) = \{J_1, ..., J_g, ..., J_G\}\).

**Preliminaries:** We first consider the following space \(A = \mathbb{R}^{N_0 K}\) where \(N_0\) is the number of agents and \(K\) is the number of all possible clubs in \(N^0\). Let \(a = (a^1, ..., a^K)\) be a vector where, for each \(i\), \(a^i = (a^i_1, ..., a^i_k, ..., a^i_K)\) and for each \(k\), \(a^i_k \in \mathbb{R}\). Let \(A_i\) be the set of elements in \(\mathbb{R}^K\) defined by

\[
A_i = \{ a \in \mathbb{R}^{N_0 K} : a^i_k = 0 \text{ if } i' \neq i \text{ or if } i \notin S_k \}.
\]

For a given \(C[i; S] \subset C[i; N]\), we represent \(C[i; S]\) in \(A_i\) by \(a \in A_i\) where \(a^i_k\) equals one if \(S_k\) belongs to \(C[i, S]\) and equals zero otherwise. Observe
that we can represent the total consumption \((x^i, C[i; N^0])\) of each agent \(i\) by \((x^i, \tilde{a}^i) \in \mathbb{R}^{L+N^0K}\).

We next define a ‘virtual’ production set in the extended commodity space. For each \(k\) define \(b[k] \in \mathbb{R}^{N^0K}\) as a vector having the properties that:

(i) \(b[k]^i_{k'} = 0\) if \(k \neq k'\) or if \(i \notin S_k\)

(ii) for (any) \(i\) in \(S_k\), \(b[k]^i_k = 1\).

Define the virtual production set \(Y\) as the convex cone generated by the \(((z_{S_k}, b[k])) : k = 1, \ldots, K\), where \(z_{S_k}\) is the input required to form the club \(S_k\). The set \(Y\) is precisely the set of all positive linear combinations of \(((z_{S_k}, b[k])) : k = 1, \ldots, K\).

**Step 1: The sets of preferred allocations \(\Gamma_i\).** Let \(\Gamma_i = \{(y^i - e^i + \varepsilon_0 \bar{z}, a^i) \in \mathbb{R}^{L_1} \times A_i : \text{for every club structure } C(S) \text{ with the property that} \ C[i, S] = \{S_k | a^i_k = 1\}, \text{we have } u'(y^i, C[i; S]) > u'(x^i, C[i; N^0])\}\).

The set \(\Gamma_i \subseteq \mathbb{R}^{L-N^0K}\) describes the set of net trades of private goods and club memberships for agent \(i\) that are strictly preferred to his allocation in the given state of the economy \((x^{N^0}, C(N^0))\). It is clear that \(\Gamma_i\) is not convex.

**Step 2: The preferred set \(\Gamma\).** Let \(\Gamma\) denote the convex hull of the union of the sets \(\Gamma_i, i = 1, \ldots, N^0\). We now show, in the remainder of Step 2, that

\[\Gamma \cap Y = \emptyset.\]

Suppose, on the contrary, that \((y, a) \in \Gamma \cap Y\). Then, by the definition of \(\Gamma\), there exist an integer \(J\) and \(\lambda \in \mathbb{R}^J\) such that \((y, a) = \sum_{j=1}^J \lambda_j (y^j, a^j)\) with \(\lambda_j > 0\), \(\sum \lambda_j = 1\).

From the definition of \(Y\) there exist a \(K' \in \{1, \ldots, K\}\) and \(\mu \in \mathbb{R}^{K'}\) such that

\[(y, a) = \sum_{k \in K'} \mu_k (z_{S_k}, b[k]).\]

Let us consider \(J[i] = \{j \mid (y^j, a^j) \in \Gamma_i\}\). Then, it follows from

\[\sum_{j=1}^J \lambda_j (y^j, a^j) = \sum_{k \in K'} \mu_k (z_{S_k}, b[k])\]

that for each \(k \in K'\) and each \(i \in S_k\) we have

\[\sum_{j \in J[i]} \lambda_j a^j_{k} = \mu_k\]
For a given \((y^j, a^j)\) in \(\Gamma_i\) and a given sequence \(\{(\beta^n)\}_n\) of real numbers, such that \(\beta^n \geq 1\) for each \(n\) and \(\{(\beta^n)\}_n\) converges to one as \(n\) goes to infinity. Then, because of the continuity of preferences, for all \(n\) sufficiently large, \((\beta^n y^j, a^j) \in \Gamma_i\).

We now show that, since we have supposed that \(\Gamma \cap Y \neq \emptyset\), we can form a blocking coalition for some sufficiently large replication. We will use the following lemma.

**Lemma.** There exists a sequence of rational numbers \((\lambda^n_1, \ldots, \lambda^n_j, \ldots, \lambda^n_J)\) converging to \((\lambda_1, \ldots, \lambda_j, \ldots, \lambda_J)\) and having the properties that:

(i) \(\lambda^n_j \leq \lambda_j\)

(ii) for (any) \(k\), and for any \(i, i' \in S_k\) we have:

\[
\sum_{j \in J[i]} \lambda^n_j a^j_k = \sum_{j \in J[i']} \lambda^n_j a^{j,i'}_k.
\]

**Proof.** Let us consider the closed line segment \([0, \lambda] \times \mathbb{R}^J\). From convexity it follows that, for any \(\alpha \in [0, \lambda] \times \mathbb{R}^J\), for (any) \(k\) and for any \(i, i' \in S_k\) we have

\[
\sum_{j \in J[i]} \alpha_j a^j_k = \sum_{j \in J[i']} \alpha_j a^{j,i'}_k.
\]

But we know that \(Q^J\), where \(Q\) is the set of rational number, is dense in \(\mathbb{R}^J\). Hence, \(Q^J \cap [0, \lambda] \times \mathbb{R}^J\) is dense in \([0, \lambda] \times \mathbb{R}^J\) and therefore we can choose a sequence satisfying (i) and (ii). \(\square\)

Let us consider the sequence \((\lambda^n_1, \ldots, \lambda^n_j, \ldots, \lambda^n_J)\) defined above, and let us select a positive integer \(n\), which will eventually tend to infinity. For each \(j\) define \(y^{jn} = \frac{\lambda^n_j}{\lambda^n_j} y^j\). From the concluding paragraph of the last Step, for all \(n\) sufficiently large it holds that \((y^{jn}, a^j) \in \Gamma_i\). Let \(n\) satisfy the property that \((y^{jn}, a^j) \in \Gamma_i\) for each \(i\). Recall that \(\lambda^n_j\) is a rational number.

Now, let us define \(\mu^n_k = \sum_{j \in J[i]} \lambda^n_j a^{j,i}_k\). Since

\[
\sum_{j=1}^J \lambda^n_j y^{jn} = \sum_{k \in K'} \mu^n_{k} z_k \quad \text{and} \quad \mu^n_k \leq \mu_{k} \quad \text{for all} \quad n
\]

and \(z_k \in -\mathbb{R}^L_{+}\), it follows that

\[
\sum_{j=1}^J \lambda^n_j y^{jn} \leq \sum_{k \in K'} \mu^n_{k} z_k
\]
Let $r'$ be a replication number such that $r'\lambda^n_j$ is an integer for all $j$. Let $\delta_j = r'\lambda^n_j$ and $\gamma_k = \sum_{j \in J[k]} \delta_j$. It holds that

$$\sum_{j=1}^J \delta_j y_j^n \leq \sum_{k \in K'} \gamma_k z_k.$$ 

Let $\tilde{r}$ be an integer sufficiently large so that there are $\gamma_k$ copies of the club $S_k$, for each $k$, contained in the set $\tilde{r}^{th}$ replication $N_0^\tilde{r}$ of $N_0$ and so that this does not hold for any $r < \tilde{r}$, that is, $\tilde{r}$ is minimal. This implies that there is a state of the economy for a coalition $S \subset N_0^{\tilde{r}}$ that can $c(\varepsilon_0)$-improve upon the initially given state of the economy. The state of the economy for $S$ described by the consumption plans $(y_j^n, a^i)$, for $\delta_j$ consumers, for each $j$ is $c(\varepsilon_0)$-feasible and preferred by all members of the replication of the initially given state of the economy. Consequently, $S$ can $c(\varepsilon_0)$-improve upon the $\tilde{r}^{th}$ replication of $(x_N^N, C(N^0))$, which is a contradiction. Therefore $\Gamma \cap Y = \emptyset$.

**Step 3: Prices.** From the Minkowski Separating Hyperplane Theorem, there is a hyperplane with normal $(p, \pi) \neq 0$, where $p$ is in the private goods price space, and $\pi \in \mathbb{R}^{N_0 K}$ such that, for some constant $C$,

$$p \cdot x + \pi \cdot a \geq C \text{ for all } (x, a) \in \Gamma \text{ and }$$

$$p \cdot z + \pi \cdot b \leq C \text{ for all } (z, b) \in Y.$$ 

Since $Y$ is a closed convex cone with vertex zero, it follows that we can choose $C = 0$. Then, in particular, for each $(y^j, a^i)$ such that $u^i(y^j, a^i) > u^i(x^i, \bar{a}^i)$, it follows that

$$p \cdot (y^j - e^i + \varepsilon_0 \bar{z}) + \sum_{\{k | a_k^i = 1\}} \pi^i(S_k) \geq 0,$$

and for each club $S_k \subset N_0$ we have

$$p \cdot z_{S_k} + \sum_{i \in S_k} \pi^i(S_k) \leq 0.$$ 

Recall that $(x_N^N, C(N^0))$ is a $c(\varepsilon_0)$-core state of the economy relative to the club structure $C(N^0) = \{J_1, ..., J_G\}$ of $N^0$.

From monotonicity it follows that $p \geq 0$. Suppose that $p = 0$. Therefore, from the separating hyperplane it follows that for each $S_k$ we have

$$\sum_{i \in S_k} \pi^i(S_k) \leq 0,$$ 

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and for each \( i \in S_k \) we have \( \pi^i(S_k) \geq 0 \). Thus \( \pi^i(S_k) = 0 \), for each \( S_k \) and each \( i \in S_k \), which is a contradiction to the fact that \( (p, \pi) \neq 0 \).

Since, for each \( i \), \((x^i - e^i + \varepsilon_0 \bar{z}, \tilde{a}^i)\) is in the closure of \( \Gamma_i \), it holds that

\[
p \cdot (x^i - e^i + \varepsilon_0 \bar{z}) + \sum_{\{k: \bar{a}_k^i = 1\}} \pi^i(S_k) \geq 0.
\]

Moreover, for each club \( J_g \) we have

\[
p \cdot z_{J_g} + \sum_{\{i \in J_g\}} \pi^i(J_g) \leq 0.
\]

Summing the above inequalities over consumers one obtains

\[
p \cdot \sum_{i \in \mathbb{N}^0} (x^i - e^i + \varepsilon_0 \bar{z}) + \sum_g \sum_{\{i \in J_g\}} \pi^i(J_g) \geq 0,
\]

and summing over clubs one obtains

\[
\sum_g p \cdot z_{J_g} + \sum_{\{i \in J_g\}} \sum_g \pi^i(J_g) \leq 0.
\]

Since \( p \in \mathbb{R}_+^L \setminus \{0\} \) and \( \sum_{i \in \mathbb{N}^0} (x^i - e^i) \leq \sum_g z_{J_g} \) it follows that

\[
p \cdot \sum_{i \in \mathbb{N}^0} (x^i - e^i) \leq p \cdot \sum_g z_{J_g}.
\]

Then from the above inequalities it follows that

\[
\sum_{i \in \mathbb{N}^0} p \cdot x^i + \sum_g \sum_{\{i \in J_g\}} \pi^i(J_g) \leq \sum_{i \in \mathbb{N}^0} p \cdot e^i,
\]

and

\[
\sum_g p \cdot z_{J_g} + \sum_g \sum_{\{i \in J_g\}} \pi^i(J_g) \geq -\sum_{i \in \mathbb{N}^0} \varepsilon_0 p \cdot \bar{z}.
\]

Now we claim that \(((x^{N^0}, C(N^0)), p, \Pi)\) is a \( c(\varepsilon_0)\)-equilibrium. Checking the proof so far, it remains only to show that individual consumers are optimizing, i.e., that the prices \( p, \Pi \) and the state \((x^{N^0}, C(N^0))\) satisfy condition \((iii)\) of the definition of an equilibrium.

Suppose that for some consumer \( i \), and some consumption \((y^i, a^i)\),

\[
u^i(y^i, a^i) > u^i(x^i, \tilde{a}^i) \quad \text{and}
\]
\[ p \cdot (y^i - e^i + \varepsilon_0 \bar{z}) + \sum_{\{k | a^i_k = 1\}} \pi^i(S_k) \leq 0. \]

From our desirability of endowment assumption, there is a consumption \( y^0 \in \mathbb{R}^L_+ \) such that
\[ p \cdot (y^0 - e^i + \varepsilon_0 \bar{z}) + \sum_{\{k | a^i_k = 1\}} \pi^i(S_k) < 0. \]
It follows that for some \( y'^i \) in the segment \([y^0, y^i]\)
\[ u^i(y'^i, a^i) > u^i(x^i, \tilde{a}^i) \]
and
\[ p \cdot (y'^i - e^i + \varepsilon_0 \bar{z}) + \sum_{\{k | a^i_k = 1\}} \pi^i(S_k) < 0, \]
which is a contradiction. ■
References


