NON-MSF A-WAVELETS FROM A-WAVELET SETS

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Generalizing the result of Bownik and Speegle [Approximation Theory X: Wavelets, Splines and Applications, Vanderbilt University Press, pp. 63–85, 2002], we provide plenty of non-MSF A-wavelets with the help of a given A-wavelet set. Further, by showing that the dimension function of the non-MSF A-wavelet constructed through an A-wavelet set W coincides with the dimension function of W, we conclude that the non-MSF A-wavelet and the A-wavelet set through which it is constructed possess the same nature as far as the multiresolution analysis is concerned. Some examples of non-MSF d-wavelets and non-MSF A-wavelets are also provided. As an illustration we exhibit a pathwise connected class of non-MSF non-MRA wavelets sharing the same wavelet dimension function.

Keywords: MSF wavelets; non-MSF wavelets; dimension function; path-connectivity.

AMS Subject Classification: 42C40

1. Introduction

Wavelet sets have been extensively studied taking the aspect of multiresolution analysis (MRA) into account by several workers in this field. From the result given by Chui and Shi according to which for a dilation a such that \( a^j \not\in \mathbb{Q} \), for all \( j \in \mathbb{N} \), the only wavelets that exist are MSF, the question of existence or otherwise of a non-MSF wavelet for dilation \( a \) other than the one considered by them arose. Bownik and Speegle while discussing the wavelet dimension function for a real dilation showed the existence of a non-MSF wavelet for a dilation \( a > 1 \), for which there exists a \( p \in \mathbb{Z} \setminus \{0\} \) such that \( a^p \mathbb{Z} \cap \mathbb{Z} \neq \{0\} \). Indeed, they showed the existence of a non-MSF A-wavelet for an \( n \times n \) expansive matrix \( A \) under analogous condition for \( \mathbb{R}^n \). Thus constructing non-MSF A-wavelets drew attention of various other contributors in this field.

In this paper, we generalize the result of Bownik and Speegle which provides plenty of non-MSF A-wavelets with the help of a given A-wavelet set. Calling the
dimension function of an $A$-wavelet set $W$ to be the dimension function of the $A$-wavelet, the modulus of whose Fourier transform is the characteristic function on $W$, we prove, in Sec. 4, that the dimension function of the non-MSF $A$-wavelet constructed through an $A$-wavelet set $W$ coincides with the dimension function of $W$. This, in turn, leads to conclude that the non-MSF $A$-wavelet and the $A$-wavelet set through which it is constructed possess the same nature as far as the MRA is concerned. An application of our construction is provided in Sec. 5, by first extending the set of six-interval wavelet sets $W_n, n \in \mathbb{N}\{1\}$ as obtained by Bakić$^2$ to a larger set $W$ of six-interval wavelet sets. It is found that every member of $W$ is associated with a generalized scaling set which is not a scaling set. Therefore, each $W \in W$ is a non-MRA wavelet set. We use these wavelet sets to produce countably infinitely many classes of non-MSF non-MRA wavelets each consisting of uncountably many members. Some examples of non-MSF $d$-wavelets and non-MSF $A$-wavelets are also provided. As an illustration we exhibit a pathwise connected class of non-MSF non-MRA wavelets sharing the same wavelet dimension function.

2. Prerequisites

Throughout, $A$ denotes an $n \times n$ expansive matrix and $A^*$, the transpose of $A$. By an expansive matrix, we mean a matrix for which the modulus of each eigenvalue is greater than one.

A function $\psi \in L^2(\mathbb{R}^n)$ is called an $A$-wavelet if the system

$$\{ |\det A|^{1/2} \psi (A^n \cdot - k) \}_{(n,k) \in \mathbb{Z} \times \mathbb{Z}^n}$$

forms an orthonormal basis for $L^2(\mathbb{R}^n)$.

Following is a useful criterion for an $A$-wavelet $\psi^6$:

**Theorem 2.1.** A unit vector $\psi \in L^2(\mathbb{R}^n)$ is an $A$-wavelet if and only if:

(i) $\rho(\xi) \equiv \sum_{j \in \mathbb{Z}} |\hat{\psi}(A^j \xi)|^2 = 1, \ a.e. \ \xi \in \mathbb{R}^n$,

(ii) For $\alpha \neq 0$ in $\mathbb{R}^n$,

$$t_\alpha(\xi) \equiv \sum_{(j,m) \in \mathbb{Z} \times \mathbb{Z}^n: \alpha = A^j - m} \hat{\psi}(A^j \xi) \hat{\psi}(A^\alpha (\xi + 2A^j m \pi)) = 0, \ \ a.e. \ \xi \in \mathbb{R}^n.$$

Perhaps, the most elegant method to construct $A$-wavelets is based on multiresolution analysis which is a family of closed subspaces of $L^2(\mathbb{R}^n)$ satisfying certain properties. A wavelet arose through an MRA is said to be an MRA wavelet.

**Definition 2.1.** A pair $\{(V_j)_{j \in \mathbb{Z}}, \varphi\}$ consisting of a family $\{V_j\}_{j \in \mathbb{Z}}$ of closed subspaces of $L^2(\mathbb{R}^n)$ together with a function $\varphi \in V_0$ is called a multiresolution analysis (MRA) if it satisfies the following conditions:

(i) $V_j \subset V_{j+1}$,

(ii) $\bigcap_{j \in \mathbb{Z}} V_j = \{0\}$,

(iii) $\bigcup_{j \in \mathbb{Z}} V_j = L^2(\mathbb{R}^n)$,
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(iv) \( f \in V_j \) if and only if \( f(A(\cdot)) \in V_{j+1} \), for all \( j \in \mathbb{Z} \), and
(v) \( \{ \varphi(\cdot - k) : k \in \mathbb{Z}^n \} \) is an orthonormal basis for \( V_0 \).

**Definition 2.2.** In case, (v) in Definition 2.1, is replaced by (v′):
(v′) \( \{ \varphi(\cdot - k) : k \in \mathbb{Z}^n \} \subset V_0, \) for \( \varphi \in V_0 \), the family \( \{ (V_j)_j \in \mathbb{Z}, \varphi \} \) is called a generalized multiresolution analysis (GMRA).

An \( A \)-wavelet \( \psi \) for which \( |\text{supp} \hat{\psi}| \) has a minimal Lebesgue measure, is said to be a *minimally supported frequency* (MSF) *wavelet*.\(^3\)\(^8\) For an MSF wavelet \( \psi \), there is a measurable set \( W \) of minimal Lebesgue measure such that \( |\hat{\psi}| = \chi_W \). The set \( W \) is called an \( A \)-wavelet set.\(^3\)\(^8\) If \( \psi \) is an MRA wavelet, then we say that \( W \) is an MRA wavelet set.

The following theorem characterizes an \( A \)-wavelet set.

**Theorem 2.2.**\(^3\)\(^8\) A measurable set \( W \) of \( \mathbb{R}^n \) is an \( A \)-wavelet set if and only if:
(i) \( \{ W + 2k\pi : k \in \mathbb{Z}^n \} \) is a measurable partition of \( \mathbb{R}^n \), and
(ii) \( \{ A^jW : j \in \mathbb{Z} \} \) is a measurable partition of \( \mathbb{R}^n \).

To identify an \( A \)-wavelet \( \psi \) to be an MRA wavelet and also an \( A \)-wavelet set to be an MRA wavelet set, the notion of “dimension function” plays a decisive role. For an \( n \times n \) expansive matrix \( A \) such that \( AZ^n \subset \mathbb{Z}^n \) and an \( A \)-wavelet \( \psi \), the following
\[
D_\psi(\xi) = \sum_{(j,k) \in \mathbb{N} \times \mathbb{Z}^n} |\hat{\psi}(A^{*j}(\xi + 2k\pi))|^2
\]
describes the *dimension function* \( D_\psi \) for an \( A \)-wavelet \( \psi \).\(^3\) Let \( W \) be an \( A \)-wavelet set and \( \psi \) be the MSF wavelet such that \( |\hat{\psi}| = \chi_W \). Then the dimension function of \( \psi \) is called the *dimension function of the \( A \)-wavelet set \( W \)*, denoted by \( D(W) \). We have
\[
D(W) = \sum_{(j,k) \in \mathbb{N} \times \mathbb{Z}^n} \chi_W(A^{*j}(\xi + 2k\pi)).
\]
It is well defined because if \( \psi_1 \) and \( \psi_2 \) are MSF wavelets for which \( |\hat{\psi}_1| = |\hat{\psi}_2| = \chi_W \), then the dimension function of \( \psi_1 \) is the same as that of \( \psi_2 \).

**Theorem 2.3.** (a) An \( A \)-wavelet \( \psi \) is an MRA wavelet if and only if \( D_{\psi(\xi)} = 1, \text{a.e.} \)
(b) An \( A \)-wavelet set \( W \) is an MRA wavelet set if and only if \( D(W) = 1, \text{a.e.} \)

Suppose \( AZ^n \subset \mathbb{Z}^n \) and \( |\det A| = 2 \). Having observed that a minimally supported frequency (MSF) wavelet \( \psi \) arises from an MRA with scaling function \( \varphi \) if and only if there is a measurable set \( S \) in \( \mathbb{R} \) such that \( |\hat{\varphi}| = \chi_S \), the notion of a scaling set has been developed.\(^{12}\) In case an MSF wavelet arises from a GMRA with a function \( \varphi \) and there is a measurable set \( S \) in \( \mathbb{R} \) such that \( |\hat{\varphi}| = \chi_S \), \( S \) has been called to be a *generalized scaling set*.\(^3\) Generalized scaling sets determine all wavelet sets while
those arising from scaling sets are precisely MRA wavelet sets. Following is given the precise definition of a generalized scaling set.

**Definition 2.3.** A measurable set \( S \) in \( \mathbb{R}^n \) is called a generalized scaling set (of order \( L \)) for a dilation matrix \( A \) with integer entries if \( S = \bigcup_{j \in \mathbb{Z}} A^j W \), for some multiwavelet set \( W \) (of order \( L \)), where \( L = |S|(|\det A| - 1) \).

**Theorem 2.4 (3; Corollary 3.4).** A measurable set \( S \) of \( \mathbb{R}^n \) is an \( A \)-scaling set if and only if it is a generalized scaling set of order \( (|\det A| - 1) \) and \( \sum_{k \in \mathbb{Z}} \chi_S(\xi + 2k\pi) = 1 \).

In case of \( |\det A| = 2 \), an \( A \)-scaling set provides an \( A \)-wavelet set associated with an MRA by \( W = A^* S \setminus S \) of \( \mathbb{R}^n \). Otherwise, it provides multiwavelet sets (of order \( (|\det A| - 1) \)) associated with an MRA.

### 3. Non-MSF \( A \)-Wavelets from \( A \)-Wavelet Sets

Bowenik and Speegle\(^5\) established the following result.

**Theorem 3.1.** Let \( A \) be an \( n \times n \) expansive matrix. Then there is an \( A \)-wavelet \( \psi \) which is not MSF if there exists a \( p \in \mathbb{Z}\setminus\{0\} = \mathbb{Z}^* \) such that \( A^p Z^n \cap Z^n \neq \{0\} \).

Bowenik\(^4\) established the following result.

**Theorem 3.2.** Let \( A \) be an \( n \times n \) expansive matrix. Then there is an \( A \)-wavelet \( \psi \) which is MSF if for each \( p \in \mathbb{Z}^* \), \( A^p Z^n \cap Z^n = \{0\} \).

In this section, we put Theorem 3.1 in a more general setting.

**Theorem 3.3.** Let \( W \) be an \( A \)-wavelet set and \( p, q \in \mathbb{Z}^* \) be such that \( A^p Z^n \cap Z^n \neq \{0\} \) and \( A^q Z^n \cap Z^n \neq \{0\} \), where \( A \) is an \( n \times n \) expansive matrix. Then there is an \( A \)-wavelet \( \psi \) which is not MSF.

First, we prove the following. **Note that**, \( \cup \) denotes the disjoint union of two sets:

**Theorem 3.4.** Let \( W \) be an \( A \)-wavelet set. Then there is a measurable set \( I \) having positive measure of \( \mathbb{R}^n \) and \( (p, k) \in \mathbb{Z}^* \times (\mathbb{Z}^n \setminus \{0\}) \) such that \( I \cup (A^{* p} I + 2k\pi) \subset W \), and hence there is an \( A \)-wavelet \( \psi \) which is not MSF, whenever \( A^{* p} Z^n \cap Z^n \neq \{0\} \).

**Proof.** Suppose \( I \) is a measurable set in \( W \) having positive measure. Then

\[
A^{-p} I \cap W = \phi \text{ a.e.}, \quad \text{for } p \in \mathbb{Z}^* \text{ as } \bigcup_{p \in \mathbb{Z}} (A^{-p} W) = \mathbb{R}^n \text{ a.e.}
\]

Fix \( p \in \mathbb{Z}^* \). As \( \bigcup_{k \in \mathbb{Z}^*} (W + 2k\pi) = \mathbb{R}^n \text{ a.e.} \), there is a \( k \in (\mathbb{Z}^n \setminus \{0\}) \) such that

\[
A^{-p} I \cap (W - 2k\pi) \neq \phi \text{ a.e.}, \quad \text{i.e. } (A^{* -p} I + 2k\pi) \cap W \neq \phi.
\]

As \( I \subset W \), we have either of two possibilities:

**Case I:** \( (A^{* -p} I + 2k\pi) \cap I = \phi \text{ a.e.} \)

**Case II:** \( (A^{* -p} I + 2k\pi) \cap I \neq \phi \text{ a.e.} \)
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Since $A$ is an $n \times n$-expansive matrix, $(A^{*}^{-p}I + 2k\pi) \neq I$, otherwise

$$|I| = |(A^{*}^{-p}I + 2k\pi)| = |\det(A^{*}^{-p})||I|,$$

i.e. $|\det(A)| = 1,$

which is not possible.

First, we take up Case I with $(A^{*}^{-p}I + 2k\pi) \cap W \neq \phi$. In case $(A^{*}^{-p}I + 2k\pi) \subset W$, we have a measurable set $I$ and $(p,k) \in \mathbb{Z}^{*} \times (\mathbb{Z}^{n}\backslash\{0\})$ such that

$$I \cup (A^{*}^{-p}I + 2k\pi) \subset W.$$

In case $(A^{*}^{-p}I + 2k\pi) \not\subset W$, we can choose a measurable set $I' \subset I$ with positive measure such that

$$(A^{*}^{-p}I' + 2k\pi) \subset W,$$

and hence, we have $I'$ and $(p,k) \in \mathbb{Z}^{*} \times (\mathbb{Z}^{n}\backslash\{0\})$ such that $I' \cup (A^{*}^{-p}I' + 2k\pi) \subset W$.

Next, we take up Case II with $(A^{*}^{-p}I + 2k\pi) \cap W \neq \phi$. Then, we have three possibilities:

(i) $(A^{*}^{-p}I + 2k\pi) \subset I$,

(ii) $(A^{*}^{-p}I + 2k\pi) \supset I$, and

(iii) neither (i) nor (ii).

If $(A^{*}^{-p}I + 2k\pi) \subset I$, choose a measurable set $I'$ with positive measure such that

$$I' \subset I \backslash (A^{*}^{-p}I + 2k\pi).$$

Then $I' \cap (A^{*}^{-p}I' + 2k\pi) = \phi$, a.e. which is Case I.

If $I \subset (A^{*}^{-p}I + 2k\pi)$, choose a measurable set $I' \subset I$ such that

$$(A^{*}^{-p}I' + 2k\pi) \subset (A^{*}^{-p}I + 2k\pi) \backslash I.$$

Then $I' \cap (A^{*}^{-p}I' + 2k\pi) = \phi$, a.e. which is again Case I.

Finally, in Case III, we choose a measurable set $I' \subset I \backslash (A^{*}^{-p}I + 2k\pi)$. Then

$I' \cap (A^{*}^{-p}I' + 2k\pi) = \phi$, a.e. Hence, we get a proof of first part.

For second part, since $I$ and $(A^{*}^{-p}I + 2k\pi)$ are disjoint subsets of $W$ and

$$\bigcup_{p \in \mathbb{Z}}(A^{*}^{-p}W) = \mathbb{R}^{n} \quad \text{and} \quad \bigcup_{m \in \mathbb{Z}}(W + 2m\pi) = \mathbb{R}^{n},$$

therefore for $m, n \in \mathbb{Z}$, we have

$$A^{m}I \cap A^{n}(A^{*}^{-p}I + 2k\pi) = \phi \quad \text{a.e.}, \quad d(I) \cap d(A^{*}^{-p}I + 2k\pi) = \phi,$$

where $d$ is the dilation map defined on $\mathbb{R}^{n}$ by $d(I) = \bigcup_{j \in \mathbb{Z}}(A^{*}^{j}(I) \cap O)$, where $O$ is a bounded and bounded away from the origin set that tiles $\mathbb{R}^{n}$ by $A^{*}$-dilations, and for $p, q \in \mathbb{Z}^{n}$,

$$(I + 2p\pi) \cap (A^{*}^{-p}I + 2(k + q)\pi) = \phi \quad \text{a.e.}, \quad \tau(I) \cap \tau(A^{*}^{-p}I + 2k\pi) = \phi,$$

where $\tau$ is the translation map defined on $\mathbb{R}^{n}$ by $\tau(\xi) = \eta$, where $\eta \in [-\pi, \pi]^{n}$ and $\xi - \eta = 2l\pi$ for some $l \in \mathbb{Z}$.
Suppose $k_1 = A^p k$. Then $k_1 \in A^p \mathbb{Z}^n$ as $k \in \mathbb{Z}^n$. If $I$ is an open ball in $\mathbb{R}^n$ which is a subset of $W$, we have

$$\tau (I + 2k_1 \pi) \cap \tau (W \setminus I) = \phi \quad \text{and} \quad I \cap (I + 2(k_1 + l)\pi) = \phi,$$

for $l \in \mathbb{Z}^n \setminus \{-k_1\}$, whenever $k_1 \in \mathbb{Z}^n$, i.e., $k_1 \in \mathbb{Z}^n \cap A^p \mathbb{Z}^n \neq \{0\}$.

Now, the function $\psi$ in $L^2(\mathbb{R}^n)$ for which the Fourier transform is defined as follows:

$$\hat{\psi}(\xi) = \begin{cases} \frac{1}{\sqrt{2}} & \text{if } \xi \in I \cup A^{-p} I \cup (I + 2k_1 \pi), \\ -\frac{1}{\sqrt{2}} & \text{if } \xi \in (A^{-p} I + 2k_1 \pi), \\ 1 & \text{if } \xi \in W \setminus (I \cup (A^{-p} I + 2k_1 \pi)), \\ 0 & \text{otherwise}, \end{cases}$$

(a)

is a non-MSF $A$-wavelet. That the norm of $\psi$ is one that follows by noting

$$\|\hat{\psi}\|_2^2 = \int_{\mathbb{R}^n} |\hat{\psi}(\xi)|^2 d\xi = \frac{1}{2} |I| + |A^{-p} I| + |(I + 2k_1 \pi)| + |(A^{-p} I + 2k_1 \pi)|$$

$$+ |W \setminus (I \cup (A^{-p} I + 2k_1 \pi))| = |W| = (2^n)^n.$$

Below is shown that $\psi$ satisfies (i) and (ii) of Theorem 2.1.

(i) In view of the fact that $\mathbb{R}^n = \bigcup_{j \in \mathbb{Z}} A^j W$, a.e. it suffices to show that

$$\rho(\xi) \equiv \sum_{j \in \mathbb{Z}} |\hat{\psi}(A^j \xi)|^2 = 1 \quad \text{for } \xi \in W.$$

If $\xi \in W \setminus (I \cup (A^{-p} I + 2k_1 \pi))$, then $\rho(\xi) = 1$.

If $\xi \in I$, then $A^{-p} \xi \in A^{-p} I$, and hence $\rho(\xi) = \frac{1}{2} + \frac{1}{2} = 1$.

Similarly, we see that for $\xi \in (A^{-p} I + 2k_1 \pi)$, $\rho(\xi) = 1$.

(ii) Next, we show that

$$t_\alpha(\xi) \equiv \sum_{(j,m)\in \mathbb{Z}\times\mathbb{Z}^n: \alpha=A^j m} \hat{\psi}(A^j \xi) \psi(A^{-j} (\xi + 2A^{-j} m \pi)) = 0, \quad \text{a.e. } \xi \in \mathbb{R}^n,$$

for all $\alpha$ other than $0$ in $\mathbb{R}^n$. The term

$$\hat{\psi}(A^j \xi) \psi(A^{-j} (\xi + 2A^{-j} m \pi))$$

is nonzero only when both $A^j \xi$ and $A^{-j} (\xi + 2A^{-j} m \pi)$ are in the support of $\hat{\psi}$. Since $I \cup (A^{-p} I + 2k_1 \pi) \subset W$, $\tau(I_0) \cap \tau(A^{-p} I_0) = \phi$. Hence, by the definition of $\psi$, $m$ can assume either of the values $k$ or $k_1$.

If $A^j \xi \in I$ and $A^j \xi + 2k_1 \pi \in I + 2k_1 \pi$, then $A^{j-p} \xi \in A^{-p} I$ and $A^{j-p} \xi + 2k_1 \pi \in A^{-p} I + 2k_1 \pi$, and hence $t_{A^{-j-p} k_1}(\xi) = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = 0$.

If $A^j \xi \in A^{-p} I$ and $A^j \xi + 2k_1 \pi \in A^{-p} I + 2k_1 \pi$, then $A^{j+p} \xi \in I_0$ and $A^{j+p} \xi + 2k_1 \pi \in I + 2k_1 \pi$, and hence $t_{A^{j+p} k}(\xi) = 0$.

In other cases also, the situation remains the same.
Proof of Theorem 3.3. Suppose that for nonzeros \( k_0 \) and \( l_0 \) in \( A^+ \mathbb{Z}^n \cap \mathbb{Z}^n \) and \( A^q \mathbb{Z}^n \cap \mathbb{Z}^n \), respectively, there exist measurable sets \( I_0 \) and \( I_1 \) in \( \mathbb{R}^n \) such that:

(i) \( A \) \( I_0 \cup (A^{-p} I_0 + 2k_1 \pi) \subset W \), where \( k_1 = A^{-p} k_0 \),

(ii) \( I_1 \cup (A^{-q} I_1 + 2l_1 \pi) \subset W \), where \( l_1 = A^{-q} l_0 \), and

(iii) \( I_0, I_1, (A^{-p} I_0 + 2k_1 \pi) \) and \( (A^{-q} I_1 + 2l_1 \pi) \) are pairwise disjoint.

Then the function \( \psi \) in \( L^2(\mathbb{R}^n) \) for which the Fourier transform is defined as follows:

\[
\hat{\psi}(\xi) = \begin{cases} 
1 & \text{if } \xi \in I_0 \cup A^{-p} I_0 \cup I_1 \cup A^{-q} I_1 \cup (I_0 + 2k_0 \pi) \cup (I_1 + 2l_0 \pi), \\
-1 & \text{if } \xi \in (A^{-p} I_0 + 2k_1 \pi) \cup (A^{-q} I_1 + 2l_1 \pi), \\
0 & \text{otherwise,}
\end{cases}
\]

is a non-MSF \( A \)-wavelet.

That the norm of \( \psi \) is one that follows by noting

\[
\|\hat{\psi}\|^2 = \int_{\mathbb{R}^n} |\hat{\psi}(\xi)|^2 d\xi
\]

\[
= \frac{1}{2} |I_0| + |A^{-p} I_0| + |I_1| + |A^{-q} I_1| + |I_0 + 2k_0 \pi| + |I_1 + 2l_0 \pi|
\]

\[
+ |(A^{-p} I_0 + 2k_1 \pi)| + |(A^{-q} I_1 + 2l_1 \pi)|
\]

\[
+ |W| = (2\pi)^n.
\]

Below is shown that \( \psi \) satisfies (i) and (ii) of Theorem 2.1.

(i) In view of the fact that \( \mathbb{R}^n = \bigcup_{j \in \mathbb{Z}} A^j W \), a.e. it suffices to show that

\[
\rho(\xi) \equiv \sum_{j \in \mathbb{Z}} |\hat{\psi}(A^j \xi)|^2 = 1, \quad \text{for } \xi \in W.
\]

If \( \xi \in W \setminus [I_0 \cup (A^{-p} I_0 + 2k_1 \pi) \cup I_1 \cup (A^{-q} I_1 + 2l_1 \pi)] \), then \( \rho(\xi) = 1 \).

If \( \xi \in I_0 \), then \( A^{-p} \xi \in A^{-p} I_0 \), and hence \( \rho(\xi) = \frac{1}{2} + \frac{1}{2} = 1 \).

Similarly, we see that for \( \xi \in (A^{-p} I_0 + 2k_1 \pi) \cup I_1 \cup (A^{-q} I_1 + 2l_1 \pi) \), \( \rho(\xi) = 1 \).

(ii) Next, we show that

\[
t_\alpha(\xi) \equiv \sum_{(j,m) \in \mathbb{Z} \times \mathbb{Z}: \alpha = A^{-j} m} \hat{\psi}(A^j \xi) \hat{\bar{\psi}}(A^j(\xi + 2A^{-j} m \pi)) = 0, \quad \text{a.e. } \xi \in \mathbb{R}^n,
\]

for all \( \alpha \) other than 0 in \( \mathbb{R}^n \). The term

\[
\hat{\psi}(A^j \xi) \hat{\bar{\psi}}(A^j(\xi + 2A^{-j} m \pi))
\]
Remark 3.1. Consider the Shannon wavelet set $W = [-2\pi, -\pi) \cup [\pi, 2\pi)$. Then $I_0, I_1, 2^{-2}I_0 - 2\pi = [-\frac{7\pi}{2}, -\frac{3\pi}{2})$ and $2^{-1}I_1 + 2\pi = (\frac{5\pi}{2}, \frac{7\pi}{2})$ are contained in $W$, where $I_0 = [\pi, \frac{3\pi}{2})$ and $I_1 = [-\frac{3\pi}{2}, -\pi)$. Since $|I_0 \cap (2^{-1}I_1 + 2\pi)| > 0$, $I_0, I_1, 2^{-2}I_0 - 2\pi$ and $2^{-1}I_1 + 2\pi$ are not pairwise disjoint.

Theorem 3.3 can be put in a more general form as follows:

Theorem 3.5. Let $W$ be an A-wavelet set. Let $p_i \in \mathbb{Z}^*$, $i = 1, 2, \ldots, m$, and $m \in \mathbb{N}$ be such that $A^{p_i} \mathbb{Z}^* \cap \mathbb{Z}^* \neq \{0\}$. Then there is an A-wavelet $\psi$ which is not MSF.

Proof. Suppose that for nonzero $k_i$'s in $A^{p_i} \mathbb{Z}^* \cap \mathbb{Z}^*$, there exist measurable sets $I_i$ in $\mathbb{R}^n$ such that:

(i) $I_i \cup (A^{-p_i}I_i + 2k_i \pi) \subset W$,

(ii) $I_i, (A^{-p_i}I_i + 2k_i \pi)$, for $i = 1, 2, \ldots, m$ are pairwise disjoint.

Then the function $\tilde{\psi}$ in $L^2(\mathbb{R}^n)$ for which the Fourier transform is defined as follows:

$$
\tilde{\psi}(\xi) = \begin{cases} 
\frac{1}{\sqrt{2}} & \text{if } \xi \in \bigcup_{i=1}^{m} I_i \cup \bigcup_{i=1}^{m} A^{-p_i}I_i \cup \bigcup_{i=1}^{m} (I_i + 2k_i \pi), \\
-\frac{1}{\sqrt{2}} & \text{if } \xi \in \bigcup_{i=1}^{m} (A^{-p_i}I_i + 2k_i \pi), \\
1 & \text{if } \xi \notin \bigcup_{i=1}^{m} (I_i \cup (A^{-p_i}I_i + 2k_i \pi)), \\
0 & \text{otherwise},
\end{cases}
$$

is a non-MSF A-wavelet.
Remark 3.2. Gu and Han have provided some examples of non-MSF wavelets for dilation by two which are different from above constructions.

4. Dimension Function

In this section, we show that the dimension function of the non-MSF $A$-wavelet $\psi$ constructed from the given $A$-wavelet set $W$ of $\mathbb{R}^n$ in Theorem 3.3 is the same as that of $W$. From this, we conclude that the nature of $\psi$ remains the same as that of $W$ in respect of multiresolution analysis. In other words, the non-MSF $A$-wavelet $\psi$ developed from an MRA $A$-wavelet set is an MRA wavelet, while the one developed from a non-MRA $A$-wavelet set is a non-MRA $A$-wavelet.

Theorem 4.1. Let $A$ be an $n \times n$ expansive matrix such that $A\mathbb{Z}^n \subset \mathbb{Z}^n$. Then for a.e. $\xi \in \mathbb{R}^n$, $D_\psi(\xi) = D(W)$, where $\psi$ is a non-MSF $A$-wavelet whose Fourier transform is defined by $(\beta)$.

Proof. By writing

\[ X \equiv I_0 \cup A^{-p}I_0 \cup I_1 \cup A^{-q}I_1 \cup (I_0 + 2k_0 \pi) \cup (I_1 + 2l_0 \pi), \]
\[ Y \equiv (A^{-p}I_0 + 2k_1 \pi) \cup (A^{-q}I_1 + 2l_1 \pi), \]
\[ Z \equiv W \setminus [I_0 \cup (A^{-p}I_0 + 2k_1 \pi) \cup I_1 \cup (A^{-q}I_1 + 2l_1 \pi)], \]

we have $\hat{\psi}(\xi) = \frac{1}{\sqrt{2}} \chi_X(\xi) - \frac{1}{\sqrt{2}} \chi_Y(\xi) + \chi_Z(\xi)$. Since $X, Y$ and $Z$ are disjoint

\[ |\hat{\psi}(\xi)|^2 = \frac{1}{2} \chi_X(\xi) + \frac{1}{2} \chi_Y(\xi) + \chi_Z(\xi) \]
\[ = \chi_W(\xi) - \frac{1}{2} \chi_{[I_0 \cup (A^{-p}I_0 + 2k_1 \pi) \cup I_1 \cup (A^{-q}I_1 + 2l_1 \pi)]}(\xi) \]
\[ + \frac{1}{2} \chi_{[A^{-p}I_0 \cup A^{-q}I_1 \cup (I_0 + 2k_0 \pi) \cup (I_1 + 2l_0 \pi)]}(\xi) \]

and hence

\[ D_\psi(\xi) = \sum_{(j,k) \in \mathbb{N} \times \mathbb{Z}^n} \chi_W(A^{*j}(\xi + 2k \pi)) \]
\[ - \frac{1}{2} \sum_{(j,k) \in \mathbb{N} \times \mathbb{Z}^n} \chi_{[I_0 \cup (A^{-p}I_0 + 2k_1 \pi) \cup I_1 \cup (A^{-q}I_1 + 2l_1 \pi)]}(A^{*j}(\xi + 2k \pi)) \]
\[ + \frac{1}{2} \sum_{(j,k) \in \mathbb{N} \times \mathbb{Z}^n} \chi_{[A^{-p}I_0 \cup A^{-q}I_1 \cup (I_0 + 2k_0 \pi) \cup (I_1 + 2l_0 \pi)]}(A^{*j}(\xi + 2k \pi)). \]

Claiming the cancellation of the last two terms on the right-hand side of the above identity, we obtain

\[ D_\psi(\xi) \equiv \sum_{(j,k) \in \mathbb{N} \times \mathbb{Z}^n} |\hat{\psi}(A^{*j}(\xi + 2k \pi))|^2 = \sum_{(j,k) \in \mathbb{N} \times \mathbb{Z}^n} \chi_W(A^{*j}(\xi + 2k \pi)). \]
Finally, the claim follows by noting that \( A^*Z^n \subset Z^n \), for \( j \in \mathbb{N} \), \( k_1 = A^{*p}k_0 \), \( l_1 = A^{*-q}l_0 \) and

(i) \( A^j(\xi + 2k\pi) \in I_0 \) if and only if \( A^{*j-p}(\xi + 2k\pi) \in A^{*p}I_0 \) for \( j > p \)
or \( A^j(\xi + 2(k + A^{*p-j}k_1)\pi) \in I_0 + 2k_0\pi \) for \( p \geq j \),
(ii) \( A^j(\xi + 2k\pi) \in I_1 \) if and only if \( A^{*j-q}(\xi + 2k\pi) \in A^{*-q}I_0 \) for \( j > q \)
or \( A^j(\xi + 2(k + A^{*-q-j}l_1)\pi) \in I_1 + 2l_0\pi \) for \( q \geq j \),
(iii) \( A^j(\xi + 2k\pi) \in (A^{*p}I_0 + 2k_1\pi) \) if and only if \( A^{*j+p}(\xi + 2k\pi) \in I_0 + 2k_0\pi \) for \( j + p > 0 \)
or \( A^j(\xi + 2(k - A^{*-p-j}k_0)\pi) \in A^{*-p}I_0 \) for \( j + p \leq 0 \),
(iv) \( A^j(\xi + 2k\pi) \in (A^{*-q}I_1 + 2l_1\pi) \) if and only if \( A^{*j+q}(\xi + 2k\pi) \in I_1 + 2l_0\pi \) for \( j + q > 0 \)
or \( A^j(\xi + 2(k - A^{*-q-j}l_0)\pi) \in A^{*-q}I_1 \) for \( j + q \leq 0 \).

**Remark 4.1.** The technique employed in the proof of Theorem 4.1 works for non-MSF A-wavelets obtained from A-wavelet sets through the construction given by equation (α) or equation (γ).

5. An Application

For a set \( W \) of the real line \( \mathbb{R} \), denoting \( W \cap (0, \infty) \) and \( W \cap (-\infty, 0) \) by \( W^+ \) and \( W^- \), respectively, we have the following characterization of a wavelet set.

**Theorem 5.1.** A measurable set \( W \) of \( \mathbb{R} \) is a wavelet set by dilation two if and only if \( \tau_a, \delta_b \) and \( \delta_c \), defined below, are measurable bijections for some \( a \in \mathbb{R}, b \in \mathbb{R}^+ \) and \( c \in \mathbb{R}^- \), where

(i) \( \tau_a : W \rightarrow [a, a + 2\pi) \), defined by \( \tau_a(x) = x + 2k\pi \) for \( k \in \mathbb{Z} \),
(ii) \( \delta_b : W^+ \rightarrow [b, 2b) \), defined by \( \delta_b(x) = 2^jx \) for \( j \in \mathbb{Z} \), and
(iii) \( \delta_c : W^- \rightarrow [2c, c) \), defined by \( \delta_c(x) = -\delta_c(-x) \).

Following are examples of non-MSF wavelets in one dimension and also in higher dimension:

(A) **Non-MSF wavelets by dilation 2:** In this section, we consider the wavelet sets \( W_n \), where \( n \in \mathbb{N} \setminus \{1\} \) as obtained in Ref. 2. These are precisely given by

\[
W_n \equiv \left[ -2^n\pi, -2^n\pi + \frac{2^n\pi}{2^{n+1} - 1} \right] \cup \left[ -\frac{2^{n+1}\pi}{2^{n+1} - 1}, -\frac{2^n\pi}{2^{n+1} - 1} \right] \cup \left[ -\frac{2^n\pi}{2^{n+1} - 1}, \frac{2^n\pi}{2^{n+1} - 1} \right] \cup \left[ \frac{2^n\pi}{2^{n+1} - 1}, \frac{2^{n+1}\pi}{2^{n+1} - 1} \right] \cup \left[ \frac{2^{n+1}\pi}{2^{n+1} - 1}, \frac{2^{n+2}\pi}{2^{n+2} - 1} \right],
\]

(δ)
For $j \in \mathbb{N}\{1\}$, we suitably choose an interval $I_j$ such that each $\beta \in I_j$ determines a six-interval wavelet set $W^j_\beta$. The family $\mathcal{W}$ consisting of $W^j_\beta$, where $j \in \mathbb{N}\{1\}$, $\beta \in I_j$ includes wavelet sets $W_n$ as given in (\(\delta\)). In fact, $W^j_0 = W_n$, for $j = n$.

**Theorem 5.2.** For $j \in \mathbb{N}\{1\}$, and $\beta \in (\frac{-\pi}{2j+1}, \frac{-\pi}{2j+1})$,

$$W^j_\beta \equiv \left[ -2^j \pi + 2^j \beta, -2^j \pi + 2^j \pi \frac{2^j+1}{2^j+1-1} \right] \cup \left[ -2^j \pi + 2^j \pi \frac{2^j+1}{2^j+1-1}, -\pi + \beta \right] \cup$$

$$\left[ (2^j+1-2)\pi \frac{2^j+1}{2^j+1-1}, 2^j \pi \frac{2^j+1}{2^j+1-1} \right] \cup \left[ 2^j \pi \frac{2^j+1}{2^j+1-1}, (2^j+1-2)\pi \frac{2^j+1}{2^j+1-1} \right] \cup$$

$$\left[ \pi + \beta, 2^j+1 \frac{2^j+1}{2^j+1-1} \pi \frac{2^j+1}{2^j+1-1} \right] \cup \left[ 2^j \pi - 2^j \pi \frac{2^j+1}{2^j+1-1}, 2^j \pi + 2^j \beta \right]$$

$$\equiv I^j_{1, \beta} \cup I^j_{2, \beta} \cup I^j_{3, \beta} \cup I^j_{4, \beta} \cup I^j_{5, \beta} \cup I^j_{6, \beta} \quad (\text{say})$$

is a six-interval non-MSF wavelet set.

**Proof.** Since the following maps

(i) $\delta_{\frac{2^j \pi}{2^j+1}} : I^j_{1, \beta} \cup I^j_{2, \beta} \cup I^j_{3, \beta} \to \left[ -\frac{2^j+1}{2^j+1}, -\frac{2^j+1}{2^j+1} \right]$ defined by

$$\delta_{\frac{2^j \pi}{2^j+1}}(x) = \begin{cases} x & \text{if } x \in I^j_{2, \beta} \cup I^j_{3, \beta}, \\ 2^{-j}x & \text{if } x \in I^j_{1, \beta}, \end{cases}$$

(ii) $\delta_{\frac{2^j \pi}{2^j+1}} : I^j_{4, \beta} \cup I^j_{5, \beta} \cup I^j_{6, \beta} \to \left[ \frac{2^j+1}{2^j+1}, \frac{2^j+1}{2^j+1} \right]$ defined by

$$\delta_{\frac{2^j \pi}{2^j+1}}(x) = \begin{cases} x & \text{if } x \in \bigcup I^j_{4, \beta} \cup I^j_{5, \beta}, \\ 2^{-j}x & \text{if } x \in I^j_{6, \beta}, \end{cases}$$

(iii) $\tau_{(\pi+\beta)} : W^j_\beta \to [-\pi + \beta, \pi + \beta]$ defined by

$$\tau_{(\pi+\beta)}(x) = \begin{cases} x + 2^j \pi & \text{if } x \in I^j_{1, \beta}, \\ x + 2^j \pi & \text{if } x \in I^j_{2, \beta}, \\ x & \text{if } x \in I^j_{4, \beta}, \\ x - 2^j \pi & \text{if } x \in I^j_{5, \beta}, \\ x - 2^j \pi & \text{if } x \in I^j_{6, \beta}, \end{cases}$$
are bijective, the set $W^j_\beta$ is a wavelet set from Theorem 5.1. For $j \in \mathbb{N}\setminus\{1\}$ and
\[ \beta \in \left( -\frac{\pi}{2^{j+1}}, \frac{\pi}{2^{j+1}} \right) , \]
the set
\[ S^j_\beta \equiv \bigcup_{n=1}^j 2^{-n} \left[ -2^j \pi + 2^j \beta, -2^j \pi + \frac{2^j \pi}{2^{j+1} - 1} \right] \cup \left[ \frac{2^j \pi}{2^{j+1} - 1}, \frac{2^j \pi}{2^{j+1} - 1} \right] \cup \left[ 2^j \pi - \frac{2^j \pi}{2^{j+1} - 1}, 2^j \pi + 2^j \beta \right] \]
\[ \equiv \bigcup_{n=1}^j I^j_{1,n} \cup I^j_{2} \cup \bigcup_{n=1}^j I^j_{3,n} \quad \text{(say)} \]
is the generalized scaling set which determines the wavelet set $W^j_\beta$. Further, $S^j_\beta$ is not a scaling set. This follows by noting that:

(i) $S^j_\beta \subset 2S^j_\beta$,
(ii) $2S^j_\beta \setminus S^j_\beta \equiv W^j_\beta$, and
(iii) $\{S^j_\beta + 2k\pi : k \in \mathbb{Z}\}$ fails to be a measurable partition of $\mathbb{R}$ on account of the fact that $(I^j_{3,1} - 2^{-j+1}\pi) \cap I^j_{2}$ has positive Lebesgue measure.

**Theorem 5.3.** For $j \in \mathbb{N}\setminus\{1\}$, $m, n \in \mathbb{N}$, and $\beta \in \left( -\frac{2^{-m}}{2^{j+1} - 1}, \frac{2^{-n}}{2^{j+1} - 1} \right)$, $\psi^j_\beta$ defined by
\[ \tilde{\psi}^j_\beta(\xi) = \begin{cases} 
\frac{1}{\sqrt{2}} & \text{if } \xi \in I^j_{3} \cup 2^{-m}I^j_{3} \cup I^j_{4} \cup 2^{-n}I^j_{3} \cup (I^j_{3} + 2^{m+j}\pi) \cup (I^j_{4} - 2^{n+j}\pi), \\
\frac{1}{\sqrt{2}} & \text{if } \xi \in (2^{-m}I^j_{3} + 2^j\pi) \cup (2^{-n}I^j_{3} - 2^j\pi), \\
1 & \text{if } \xi \in I^j_{2,\beta} \cup I^j_{5,\beta} \cup (I^j_{6,\beta} \setminus (2^{-m}I^j_{3} + 2^j\pi)) \cup (I^j_{6,\beta} \setminus (2^{-n}I^j_{3} - 2^j\pi)), \\
0 & \text{otherwise},
\end{cases} \]
is a non-MSF non-MRA wavelet, where $W^j_\beta = I^j_{1,\beta} \cup I^j_{2,\beta} \cup I^j_{3} \cup I^j_{4} \cup I^j_{5,\beta} \cup I^j_{6,\beta}$ is given in Theorem 5.2.

**Proof.**

(i) assuming $A = 2$, $l_0 = I_0 \equiv I^j_{3}$, $l_1 \equiv I^j_{4}$, $p = m$, $q = n$, $k_0 = 2^{(m+j-1)}$, $l_0 = -2^{(n+j-1)}$, $k_1 = 2^j - 1$, $l_1 = -2^j - 1$, in Theorem 3.3, and
(ii) noting that $W^j_\beta$ is a non-MRA wavelet set.

It is known that the metric of $L^2(\mathbb{R})$ arose through its usual norm when restricted to its subspace of characteristic functions $\mathcal{C}$ becomes equivalent to the metric $d$ given by $d(f, g) = |\text{supp } f \Delta \text{supp } g|$, where $\Delta$ denotes the symmetric difference. We denote the collection of all measurable sets of $\mathbb{R}$ having finite measure equipped with the symmetric difference metric by $\mathcal{M}$. It is easily seen that the
convergence of functions in \( C \) amounts to that of their supports in \( M \). Also, we can identify \( M \) with \( C \) through characteristic functions.

With the simple observations, we can state that \([a_n, b_n] \to [a, b] \) in \( M \) iff \( a_n \to a \) and \( b_n \to b \) as \( n \to \infty \), where \( a,b,a_n,b_n \in \mathbb{R} \) with \( b - a > 0 \), \( b_n - a_n > 0 \), and \( n \in \mathbb{N} \).

**Theorem 5.4.** (i) For \( j \in \mathbb{N} \setminus \{1\} \), the set

\[
W^j = \left\{ W^j_\beta : \beta \in \left(-\frac{\pi}{2^{j+1} - 1}, \frac{\pi}{2^{j+1} - 1}\right) \right\}
\]

is path-connected, where \( W^j_\beta \) is given in Theorem 5.2.

(ii) For \( j \in \mathbb{N} \setminus \{1\} \) and \( m, n \in \mathbb{N} \), the set

\[
\Psi^j = \left\{ \psi^j_\beta : \beta \in \left(-\frac{2^{-m}}{2^{j+1} - 1}, \frac{2^{-m}}{2^{j+1} - 1}\right) \right\}
\]

is path-connected, where \( \psi^j_\beta \) is given in Theorem 5.3.

**Proof.** (i) If \( t \in [0, 1] \) and \( \beta_0, \beta_1 \in (-\frac{\pi}{2^{j+1} - 1}, \frac{\pi}{2^{j+1} - 1}) \), then \( \beta_t \in (-\frac{\pi}{2^{j+1} - 1}, \frac{\pi}{2^{j+1} - 1}) \), where \( \beta_t = t\beta_1 + (1 - t)\beta_0 \) and hence the set \( W^j_\beta \) defined by

\[
W^j_\beta = -I^j_1, 1 \cup I^j_2, \beta \cup I^j_3, \beta \cup I^j_4, \beta \cup I^j_5, \beta \cup I^j_6, \beta
\]

is also a member of \( W^j \), where \( I^j_1, \beta, I^j_2, \beta, I^j_3, \beta, I^j_4, \beta, I^j_5, \beta \) and \( I^j_6, \beta \) are given in Theorem 5.2. Therefore, the map \( t \mapsto W^j_\beta \) is continuous because the maps \( t \mapsto I^j_1, \beta \), \( t \mapsto I^j_2, \beta \), \( t \mapsto I^j_3, \beta \), \( t \mapsto I^j_4, \beta \), \( t \mapsto I^j_5, \beta \), \( t \mapsto I^j_6, \beta \) are continuous.

(ii) For \( t \in [0, 1] \) and \( \beta_0, \beta_1 \in (-\frac{2^{-m}}{2^{j+1} - 1}, \frac{2^{-m}}{2^{j+1} - 1}) \), \( \beta_t \in (-\frac{2^{-m}}{2^{j+1} - 1}, \frac{2^{-m}}{2^{j+1} - 1}) \), where \( \beta_t = t\beta_1 + (1 - t)\beta_0 \) and hence the set \( W^j_\beta \) defined by

\[
W^j_\beta = I^j_1, \beta \cup I^j_2, \beta \cup I^j_3, \beta \cup I^j_4, \beta \cup I^j_5, \beta \cup I^j_6, \beta
\]

is a member of \( W^j \). Then the function \( \psi^j_\beta \) defined by

\[
\psi^j_\beta (\xi) = \begin{cases} 
\frac{1}{\sqrt{2}} & \text{if } \xi \in I^j_1 \cup 2^{-m} I^j_1, \beta \cup I^j_4 \cup 2^{-n} I^j_4 \cup (I^j_4 + 2^{m+j}\pi) \cup (I^j_4 - 2^{n+j}\pi), \\
-\frac{1}{\sqrt{2}} & \text{if } \xi \in (2^{-m} I^j_1 + 2^j\pi) \cup (2^{-n} I^j_4 - 2^j\pi), \\
1 & \text{if } \xi \in I^j_2, \beta \cup (I^j_2, \beta \setminus (2^{-m} I^j_1 + 2^j\pi)) \cup (I^j_1, \beta \setminus (2^{-n} I^j_4 - 2^j\pi)), \\
0 & \text{otherwise},
\end{cases}
\]
is also a member of $\Psi^d$ and hence the map $t \mapsto \hat{\varphi}_h^d$ is continuous because the map $t \mapsto \hat{\varphi}_h^d$ is continuous, where

$$
\hat{\varphi}_h^d = \frac{1}{\sqrt{2}} \left[ \chi_{I_2} + \chi_{(2^{-n}I_2)} + \chi_{I_4} + \chi_{(2^{-n}I_4)} + \chi_{(I_4 \setminus 2^{-n}I_4)} \right]
$$

and hence the map $t \mapsto \hat{\psi}_h^d$ is defined by

$$
\hat{\psi}(\xi) = \begin{cases} 
\frac{1}{\sqrt{2}} & \text{if } \xi \in I_2 \cup d^{-m}I_2 \cup I_3 \cup d^{-n}I_3 \cup (I_2 + 2d^m\pi) \cup (I_3 + 2d^n\pi), \\
-\frac{1}{\sqrt{2}} & \text{if } \xi \in (d^{-m}I_2 + 2\pi) \cup (d^{-n}I_3 + 2\pi), \\
1 & \text{if } \xi \in W \setminus [I_2 \cup I_3 \cup (d^{-m}I_2 + 2\pi) \cup (d^{-n}I_3 + 2\pi)], \\
0 & \text{otherwise},
\end{cases}
$$

is a non-MSF $d$-wavelet, where $W = I_1 \cup I_2 \cup I_3 \cup I_4$ is defined by (7).

**Theorem 5.5.** For $d, m \in \mathbb{N} \setminus \{1, 2\}$ and $n \in \mathbb{N}$, $\psi$ defined by

$$
W = \left[ -\frac{2d^2}{d+1} \pi, \frac{2d^2}{d^2 - 1} \pi - 2d\pi \right] \cup \left[ \frac{2d-1}{d^2 - 1} \pi - 2\pi, \frac{2d}{d+1} \right] \cup \left[ \frac{2d^2}{d^2 - 1} \pi, \frac{2d^2}{d^2 - 1} \pi \right]
$$

is a non-MSF $d$-wavelet, where $W = I_1 \cup I_2 \cup I_3 \cup I_4$ is defined by (7).

**Proof.** Since $I_4 \setminus 2\pi \equiv \left[ \frac{2d}{d^2 - 1} \pi, \frac{2d^2}{d^2 - 1} \pi \right]$ and $d \geq 3$, $I_4 \setminus 2\pi$ contains a neighborhood of zero, and hence $d^{-n}I_3 + 2\pi$ is a subset of $I_4$. Also, for $m \geq 3$, $d^{-m}I_2 + 2\pi$ is a subset of $I_4$. Hence the result follows by assuming $A = d, I_0 \equiv I_2, I_1 \equiv I_3, p = m, q = n, k_0 = d^m, l_0 = d^n, k_1 = 1$ and $l_1 = 1$ in Theorem 3.3.

**C) Non-MSF wavelets by dilation:** In this section, we consider the wavelet sets $K_\beta$, where $\beta \in (-\frac{\pi}{3}, \frac{\pi}{3})$, as obtained in Refs. 14 and 15. These are precisely given by

$$
K_\beta = \left[ -\frac{8\pi}{3}, -2\pi + 2\beta \right] \cup \left[ -\pi + \beta, -\frac{2\pi}{3} \right] \cup \left[ \frac{2\pi}{3}, \frac{2\pi}{3} + \beta \right] \cup \left[ 2\pi + 2\beta, \frac{8\pi}{3} \right].
$$
Choose $A = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}$. Then $K_\beta \times S_\beta$ is an $A$-wavelet set, where

$$S_\beta = \left[\frac{-4\pi}{3}, -\pi + \beta\right) \cup \left[\frac{-2\pi}{3}, \frac{2\pi}{3}\right) \cup \left[\pi + \beta, \frac{4\pi}{3}\right).$$

This follows by noting that:

(i) $K_\beta \times S_\beta \equiv A^* (S_\beta \times S_\beta) \setminus (S_\beta \times S_\beta) = (2S_\beta \setminus S_\beta) \times S_\beta$, and

(ii) $S_\beta \times S_\beta$ is an $A$-scaling set because $(S_\beta \times S_\beta) \subset A^* (S_\beta \times S_\beta)$, $(S_\beta \times S_\beta) + 2k\pi : k \in \mathbb{Z}^2$ is a measurable partition of $\mathbb{R}^2$, and $(S_\beta \times S_\beta)$ contains a neighborhood of zero.

In general, we have the following theorem\(^9\):

**Theorem 5.6.** Let $S$ be a scaling set by dilation two. Choose a $2 \times 2$ expansive matrix $A$ having integer entries such that $S \times S \subset A^* (S \times S)$ and $|\det A| = 2$. Then $S \times S$ is an $A$-scaling set whose associated MRA wavelet set is $A^* (S \times S) \setminus (S \times S)$.

Now, the following theorem provides non-MSF MRA wavelets in $\mathbb{R}^2$:

**Theorem 5.7.** For $m, n \in \mathbb{N}$, $A = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}$ and $\beta \in (-\frac{2m}{3}, \frac{2m}{3})$, $\psi$ defined by

$$\tilde{\psi}(\xi) = \begin{cases} \frac{1}{\sqrt{2}} & \text{if } \xi \in I_1 \cup A^{*-2m}I_1 \cup I_2 \cup A^{*-2n}I_2 \cup \left(I_1 - \left(\frac{2^{m+1}\pi}{0}\right)\right) \cup \\
\left(I_2 + \left(\frac{2^{n+1}\pi}{0}\right)\right), \\
\frac{-1}{\sqrt{2}} & \text{if } \xi \in \left(A^{*-2m}I_1 - \left(\frac{2\pi}{0}\right)\right) \cup \left(A^{*-2n}I_2 + \left(\frac{2\pi}{0}\right)\right), \\
1 & \text{if } \xi \in K_\beta \times S_\beta \setminus \left[I_1 \cup I_2 \cup \left(A^{*-2m}I_1 - \left(\frac{2\pi}{0}\right)\right) \cup \\
\left(A^{*-2n}I_2 + \left(\frac{2\pi}{0}\right)\right)\right], \\
0 & \text{otherwise,} \end{cases}$$

is a non-MSF $A$-wavelet, where $K_\beta \times S_\beta$ contains $I_1 = [-\pi + \beta, -\frac{2\pi}{3}] \times [-\frac{2\pi}{3}, \frac{2\pi}{3}]$, and $I_2 = [-\frac{2\pi}{3}, -\pi + 2\beta] \times [-\frac{2\pi}{3}, \frac{2\pi}{3}]$.

**Proof.** For $m \in \mathbb{N}$, $A^{*-2m} = \begin{pmatrix} 2^{-m} & 0 \\ 0 & 2^{-m} \end{pmatrix}$ and $A^{*-2m} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2^{-m}x \\ 2^{-m}y \end{pmatrix}$, for $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$.

Consider the sets

$$I_1 = \left[-\pi + \beta, -\frac{2\pi}{3}\right] \times \left[-\frac{2\pi}{3}, \frac{2\pi}{3}\right],$$

and

$$I_2 = \left[-\frac{2\pi}{3}, -\pi + 2\beta\right] \times \left[-\frac{2\pi}{3}, \frac{2\pi}{3}\right].$$
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\[ I_2 = \left[ -\frac{8\pi}{3}, -2\pi + 2\beta \right] \times \left[ -\frac{2\pi}{3}, \frac{2\pi}{3} \right], \]

\[ I_3 = \left[ \frac{2\pi}{3}, \pi + \beta \right] \times \left[ \frac{2\pi}{3}, \frac{2\pi}{3} \right], \]

\[ I_4 = \left[ 2\pi + 2\beta, \frac{8\pi}{3} \right] \times \left[ \frac{2\pi}{3}, \frac{2\pi}{3} \right], \]

which are subsets of the A-wavelet set \( K_\beta \times S_\beta \). Then

\[ [A^{*-2m}I_1 - (2\pi, 0)^*] \subset I_2 \quad \text{and} \quad [A^{*-2n}I_3 + (2\pi, 0)^*] \subset I_4 \]

iff \( \beta \in \left( -\frac{2^{-m}\pi}{3}, \frac{2^{-n}\pi}{3} \right) \).

Therefore, the result follows by assuming \( p = 2m, q = 2n, I_0 \equiv I_1, I_1 \equiv I_3, k_0 = \left( -\frac{2^{-m}}{0} \right), l_0 = \left( \frac{2^{-n}}{0} \right), k_1 = \left( -1 \right), l_1 = \left( 1 \right) \) in Theorem 3.3.

**Remark 5.1.** Analogous to Theorem 5.4, we can discuss the path-connectivity of

\[ \left\{ K_\beta \times S_\beta : \beta \in \left( -\frac{\pi}{3}, \frac{\pi}{3} \right) \right\} \subset \mathbb{R}^2 \]

and also, its associated non-MSF wavelets as described in Theorem 5.7.

### 6. Further Discussions

The purpose of Sec. 3 is to provide plenty of non-MSF wavelets and Sec. 4 is to show that non-MSF wavelet and the wavelet set through which it is constructed possess the same nature as far as the multiresolution analysis is concerned, which raise some problems:

*Provide a non-MSF wavelet \( \psi \) such that \( \text{supp} \hat{\psi} \) does not contain any wavelet set.*

*Provide a non-MSF wavelet \( \psi \) such that \( W \subset \text{supp} \hat{\psi} \) and \( D(W) \neq D_\psi(\xi) \), for some wavelet set \( W \).*

Since Z. Rzeszotnik in his Ph.D. thesis has proved that the support of Fourier transform of an MRA wavelet contains an MRA wavelet set for dilation by two, therefore above problems will be interesting for non-MRA wavelets.

The first and second part of Theorem 5.4 show that a collection of some MSF wavelets is path-connected, and also a collection of some non-MSF non-MRA wavelets is path-connected, while the Wutam Consortium in their 1998 paper Basic Properties of Wavelets and Speegle in his 1999 paper The S-Elementary Wavelets are Path-Connected showed that all MRA wavelets by dilation two in \( \mathbb{R} \) (denoted by \( W^{\text{MRA}} \)) and all MSF wavelets by dilation two in \( \mathbb{R} \) (denoted by \( W^{\text{MSF}} \)) are path-connected. So, \( W^{\text{MRA}} \cup W^{\text{MSF}} \) is path-connected but the path-connectivity of all two-dilation wavelets (denoted by \( W \)) is still open.
On the Non-MSF Wavelets

Suppose that the mapping
\[ \mathcal{W}^{\text{MSF}} \rightarrow \mathcal{W}, \]
defined by

MSF wavelet \( \mapsto \) non-MSF wavelet

(as constructions given in Sec. 3) is continuous. Since there is a continuous map from \([0, 1]\) to \(\mathcal{W}^{\text{MSF}}\); and \(\mathcal{W}^{\text{MRA}} \cup \mathcal{W}^{\text{MSF}}\) is path-connected, therefore there is a set \(\mathcal{W}\) such that \(\mathcal{W}^{\text{MRA}} \cup \mathcal{W}^{\text{MSF}} \subset \mathcal{W} \subset \mathcal{W}\) is path-connected. So, question is: Is the above map continuous?

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References

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