Extremal polynomials of degree $\tau + 2$ and $\tau + 3$, which improve the Delsarte bound for $\tau$-designs

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Abstract

We investigate bounds for $\tau$-designs in infinite polynomial metric spaces. When the necessary and sufficient conditions for improving the Delsarte bound are satisfied we derive extremal polynomials of certain degree and obtain new bounds.

1 Introduction

Let $\mathcal{M}$ be a polynomial metric space (PMS). They are finite metric spaces represented by $P$- and $Q$- polynomial association schemes as well as infinite metric spaces, which are completely classified as the real sphere, a real, complex or quaternions projective space and the Cayley projective plane. Hamming $H(n, r)$, Johnson $J(n, w)$ and Grassmann spaces are the most important examples of finite polynomial metric spaces.

Every polynomial metric space $\mathcal{M}$ is characterized by its metric $d(x, y)$, and normalized measure $\mu_{\mathcal{M}}(.)$.

A basic property of a polynomial metric space $\mathcal{M}$ is the existence of a decomposition of the Hilbert space $L_2(\mathcal{M}, \mu)$ of complex-valued quadratic-integrable functions with the usual inner product, into a direct sum of mutually orthogonal subspaces $V_i$ of dimension $r_i$. Besides, there exist real polynomials $\{Q_i(t)\}_{i=0}^{\infty}$, ($Q_i(t)$ of degree $i$), called zonal spherical functions.

Any finite nonempty subset $C$ of $\mathcal{M}$ is called a code.

**Definition 1.1** A code $C \subset \mathcal{M}$ is called a $\tau$-design if $\sum_{x \in C} v(x) = 0$  
for all $v(x) \in V_1 \oplus \cdots \oplus V_\tau$,

The designs in $H(n, r)$ are known as orthogonal arrays. The designs in the Johnson space are nothing but the classical $t - (v, k, \lambda)$ designs.
For each $a$ and $b \in \mathbb{N}$, one can associate the zonal spherical functions (ZSF) \( \{Q_i(t)\}_{i=0}^{\infty} \) with their adjacent systems of orthogonal polynomials \( \{Q_{a,b}^i(t)\}_{i=0}^{\infty} \) [6]. These polynomials are orthogonal with respect to the measure \( \nu^{a,b}(t) \) defined by
\[
d
\nu^{a,b}(t) = c^{a,b}(1-t)^a(1+t)^b \, dv(t) \quad (c^{a,b} \text{ is a constant}),
\]
i.e.
\[
r_{i}^{a,b} \int_{-1}^{1} Q_{i}^{a,b}(t) Q_{j}^{a,b}(t) \, dv^{a,b}(t) = \delta_{ij},
\]
for \( i, j \geq 0 \), where \( Q_{i}^{a,b}(1) = 1, Q_{0}^{a,b}(t) \equiv 1, r_{0}^{a,b} = 1 \).

A polynomial metric space \( \mathcal{M} \) is called antipodal if for every point \( x \in \mathcal{M} \) there exists a point \( \overline{x} \in \mathcal{M} \) such that for any point \( y \in \mathcal{M} \) we have \( \sigma(d(x,y)) + \sigma(d(\overline{x},y)) = 0 \).

The universal lower bound \( D(\mathcal{M}, \tau) \), so called Delsarte bound, for the cardinality of a \( \tau \)-design can be presented in the following form [5]:
\[
|C| \geq D(\mathcal{M}, \tau) = 2^\theta c^{0,0} \sum_{i=0}^{k} r_{i}^{0,0},
\]
where \( \theta \in \{0, 1\} \) and \( \tau = 2k + \theta \).

The bound (1) can be obtained by using the polynomial \( f^{(\tau)}(t) = (t+1)^\theta (Q_k^{1,\theta}(t))^2 \) in the following Theorem.

**Theorem 1.2** Let \( C \subset \mathcal{M} \) \( \tau \)-design and let \( f(t) \) be a real nonzero polynomial such that

**(B1)** \( f(t) \geq 0 \), for \( -1 \leq t \leq 1 \),

**(B2)** the coefficients in the ZSF expansion \( f(t) = \sum_{i=0}^{k} f_i Q_i(t) \)

satisfy \( f_0 > 0, f_i \leq 0 \) for \( i = \tau + 1, \ldots, k \).

Then, \( |C| \geq f(1)/f_0 = \Omega(f) \).

We denote by \( B_{M,\tau} \) the set of real polynomials which satisfy the conditions *(B1)* and *(B2)* and \( B(\mathcal{M}, \tau) = \max\{\Omega(f) : f(t) \in B_{M,\tau}\} \).

**Definition 1.3** A polynomial \( f(t) \in B_{M,\tau} \) is called \( B_{M,\tau} \)-extremal if
\[
\Omega(f) = \max\{\Omega(g) : g(t) \in B_{M,\tau}, \deg(g) \leq \deg(f)\}.
\]

The coefficient \( f_0 \), which is very important for our investigations, can be expressed as follows
\[
f_0 = \int_{-1}^{1} f(t) dv(t).
\]
The conditions of the Theorem 1.2 are independent from the multiplication of $f(t)$ with a positive constant. Thus we shall not distinguish proportional polynomials from $B_{M,\tau}$.

In this paper we apply the necessary and sufficient conditions for the optimality of the Delsarte bound and when it is not the best bound possible we give an analytical form of the $B_{M,\tau}$-extremal polynomials.

2 Preliminary results

In this paper we consider only infinite polynomial metric spaces. We define the following linear functional, which was introduced and investigated in [7, 8].

$$G_r(M,f) = \frac{f(1)}{D(M,\tau)} + \sum_{i=1}^{k+\theta} \rho_i^{(\tau)} f(\alpha_i)$$

(3)

where $\alpha_i, \rho_i^{(\tau)}$ are defined in [Theorem 2.1 [7]].

We will call it briefly “test” functions. Similar test functions for codes were introduced and investigated by Boyvalenkov, Danev and Bumova in [1] (for $M = S^{n-1}$) and by Boyvalenkov and Danev [2] (in the general case).

**Theorem 2.1** The bound $D(M,\tau)$ can be improved by a polynomial $f(t) \in B_{M,\tau}$ of degree at least $\tau + 1$, if and only if $G_r(M,Q_{j+1}) < 0$ for some $j \geq \tau + 1$. Moreover, if $G_r(M,Q_{j+1}) < 0$ for some $j \geq \tau + 1$, then $D(M,\tau)$ can be improved by a polynomial in $B_{M,\tau}$ of degree $j$.

This theorem gives us necessary and sufficient conditions for the optimality of the Delsarte bound.

**Lemma 2.2**

a) Let $M$ be PMS, then $G_r(M,Q_{\tau+1}) > 0$

b) For $M$ antipodal

$$G_r(M,Q_{\tau+2}) \begin{cases} > 0, & \text{for } \tau = 2k \\
0, & \text{for } \tau = 2k + 1. \end{cases}$$

The “test” functions $G_r(M,Q_{\tau+2}), G_r(M,Q_{\tau+3})$ are negative for $3 \leq n \leq N(\tau,M)$. The exact values of $N(\tau,M)$ are given in [8].

3 Extremal polynomials of degree $\tau + 2$ and $\tau + 3$

From the previous section we have necessary and sufficient conditions for improving the Delsarte bound by using linear programming. The investigations of the test functions for designs show that the smallest possible degree of an improving polynomial in non-antipodal PMS is $\tau + 2$ and for antipodal and is $\tau + 3$ (see Lemma 2.2).
Theorem 3.1 Let $\mathcal{M}$ be non-antipodal PMS. Then, any $B_{\mathcal{M},\tau}$-extremal polynomial of degree $\tau + 2$ ($\tau = 2k + \theta$) has the form

$$f^{(\tau)}(t; \tau + 2) = (1 + t)^{1-\theta}[q(t + 1) + (1 - t)][\eta Q_{k-1+\theta}^{1,1}(t) + Q_{k+\theta}^{1,1}(t)]^2$$  (4)

Corollary 3.2 Let $\mathcal{M}$ be a non-antipodal PMS and let $\tau$ be an integer. Then

$$B(\mathcal{M}, \tau) \geq S(\mathcal{M}, \tau) = \Omega(f^{(\tau)}(t; \tau + 2)).$$

Corollary 3.3 Let $\mathcal{M}$ be a non-antipodal PMS and let $\tau$ be an integer. Then $S(\mathcal{M}, \tau) > D(\mathcal{M}, \tau)$ if and only if $G(\mathcal{M}, Q_{\tau+2}) < 0$.

As we mentioned before, for antipodal PMS the corresponding $B_{\mathcal{M},\tau}$-extremal polynomial is of degree $\tau + 3$. We can prove in a similar way analogous theorem for the form of this polynomials.

Theorem 3.4 Let $\mathcal{M}$ be antipodal PMS. Then, any $B_{\mathcal{M},\tau}$-extremal polynomial of degree $\tau + 3$ ($\tau = 2k + \theta$) has the form

$$f^{(\tau)}(t; \tau + 3) = (1 + t)^{\theta}[q(t + 1) + (1 - t)][\eta_1 Q_{k-1}^{1,\theta}(t) + \eta_2 Q_k^{1,\theta}(t) + Q_{k+1}^{1,\theta}(t)]^2$$  (5)

Corollary 3.5 Let $\mathcal{M}$ be an antipodal PMS and let $\tau$ be an integer. Then

$$B(\mathcal{M}, \tau) \geq S(\mathcal{M}, \tau) = \Omega(f^{(\tau)}(t; \tau + 3)).$$

We compared by computer the results in [3, 4] and the bounds from Corollary 3.2 and 3.5. We also compared the polynomials, which we use for obtaining new bounds in [3] and [4], and the polynomials described in (4) and (5). This investigation showed the coincidence between the corresponding polynomials and coincidence between the corresponding bounds.

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References


