A Characterization of Reflexive Spaces by Means of Continuous Approximate Selections for Metric Projections

N. V. Zhivkov

Institute of Mathematics, Bulgarian Academy of Sciences, Sofia 1090, Bulgaria

Communicated by Frank Deutsch

Received July 8, 1986

Reflexive spaces are characterized with the help of metric projections which possess a continuity property similar to \(n\)-lower semi-continuity and admit continuous \(\varepsilon\)-approximate selections. An example showing that almost lower semi-continuity of a metric projection is not sufficient for the existence of a continuous selection is constructed. © 1989 Academic Press, Inc.

1 Introduction

Let \((X, \tau)\) be a topological space, and \((Y, d)\) a metric space. A mapping \(F: X \to 2^Y\) which associates with every \(x \in X\) a non-empty subset \(F(x)\) of \(Y\) is said to be lower semi-continuous (l.s.c.) (respectively, upper semi-continuous (u.s.c.)) if, for each open set \(U\) in \(Y\), the set \(\{x \in X : F(x) \cap U \neq \emptyset\}\) (respectively, the set \(\{x \in X : F(x) \subseteq U\}\)) is open in \(X\). A mapping \(f: X \to Y\) is a selection for \(F\) if, for each \(x \in X\), \(f(x) \in F(x)\).

One of the most celebrated results on the existence of continuous selections is the following theorem of Michael [11]: If \(X\) is a paracompact (e.g., metric) space and \(F: X \to 2^Y\) is l.s.c. and has closed convex images, then \(F\) admits a continuous selection. The key step in the proof of this theorem is the construction of continuous \(\varepsilon\)-approximate selections. For an arbitrary non-empty set \(A \subseteq Y\) and \(\varepsilon > 0\), let \(B_\varepsilon(A)\) denote the union of open balls with radii equal to \(\varepsilon\) and centers running over \(A\). A mapping \(f: X \to Y\) is called an \(\varepsilon\)-approximate selection for \(F: X \to 2^Y\) if for each \(x\) in \(X\) \(f(x) \in B_\varepsilon(F(x))\).

In [7] Deutsch and Kenderov introduced two continuity properties for multivalued mappings and identified topologically those mappings which admit continuous \(\varepsilon\)-approximate selections.
DEFINITION (Deutsch and Kenderov). A multivalued mapping \( F: X \to 2^Y \) is said to be almost lower semi-continuous (a.l.s.c.) (resp. \( n \)-lower semi-continuous (\( n \)-l.s.c.)) at \( x_0 \in X \) if for each \( \varepsilon > 0 \) there is a neighbourhood \( \mathcal{U} \) of \( x_0 \) such that \( \bigcap_{x \in \mathcal{U}} B_\varepsilon(F(x)) \neq \emptyset \) (resp. \( \bigcap_{i=1}^n B_\varepsilon(F(x_i)) \neq \emptyset \) for each choice of \( n \) points \( x_1, x_2, \ldots, x_n \) in \( \mathcal{U} \)). \( F \) is a.l.s.c. (resp. \( n \)-l.s.c.) if \( F \) is a.l.s.c. (resp. \( n \)-l.s.c.) at each point \( x \) of \( X \).

For our purposes we give a slightly different

DEFINITION. A multivalued mapping \( F: X \to 2^Y \) is said to be finite lower semi-continuous (f.l.s.c.) at \( x_0 \) if for each \( \varepsilon > 0 \) there is a neighbourhood \( \mathcal{U} \) of \( x_0 \) such that for each finite set of points \( A \) in \( \mathcal{U} \cap x \in A B_\varepsilon(F(x)) \neq \emptyset \). \( F \) is f.l.s.c. if \( F \) is f.l.s.c. at each point \( x \) of \( X \).

One of the main results in [7] is the following

THEOREM (Deutsch and Kenderov). Let \( X \) be a paracompact space and let \( Y \) be a normed linear space. Suppose \( F: X \to 2^Y \) has convex images. Then \( F \) is a.l.s.c. if, and only if, for each \( \varepsilon > 0 \) \( F \) admits a continuous \( \varepsilon \)-approximate selection.

The above theorem, as well as other topological results in [7], Deutsch and Kenderov apply to metric projections. Recall that a map \( P_M: X \to 2^M \), where \( M \subseteq X \) and \( X \) is normed, is referred to as the metric projection generated by \( M \) provided that for each \( x \in X \)

\[
P_M(x) = \{ y \in M : \| y - x \| = d(x, M) \},
\]

where

\[
d(x, M) = \inf\{ \| x - z \| : z \in M \}
\]

is the distance function generated by \( M \). A set \( M \) is called proximinal if \( P_M(x) \neq \emptyset \) for all \( x \) in \( X \). It is well known that the proximinal sets are closed.

Various problems concerning existence or non-existence of continuous selections for metric projections are studied in [1–3, 7, 10, 12–15, 18, 19] and others. Closely related to [7] is the work of Beer [1]. We note that the notion of approximate selection in [2, 4, 5, 16, 17] bears a different meaning.

This paper is motivated by the work of Deutsch and Kenderov [7]. It contains two results. The first one gives a characterization of reflexivity: A Banach space \( X \) is reflexive if, and only if, for every equivalent norm in \( X \) every f.l.s.c. metric projection generated by a proximinal subset of \( X \) has continuous \( \varepsilon \)-approximate selections for each \( \varepsilon > 0 \). The second result
shows that almost lower semi-continuity of a metric projection does not imply existence of a continuous selection, even for finite dimensions: In a five-dimensional Minkowskian space there is an a.l.s.c. metric projection, generated by a three-dimensional subspace, which fails to possess a continuous selection.

2. The Main Result

THEOREM 1. Let $X$ be a non-reflexive Banach space and $M \subseteq X$ be a closed subspace with $\text{codim}(M) = 2$. Then there is an equivalent renorming of $X$ such that $M$ is proximinal and the metric projection $P_M: X \rightarrow 2^M$ is finite lower semi-continuous but not almost lower semi-continuous.

Proof. Since $M$ is closed and $\text{codim}(M) = 2$, then $M$ is non-reflexive itself, and $X$ is isomorphic to $\mathbb{R}^2 \times M$. We will define an equivalent norm in the space $Z := \mathbb{R}^2 \times M$. Suppose $f \in M^*$ is a bounded linear functional with $\|f\| = 1$ which does not achieve its supremum on the closed unit ball $U(M)$. The existence of such a functional is ensured by the theorem of James [9].

Consider the sets

\[ C = \{ (r, s, \eta) \in \mathbb{R} \times \mathbb{R} \times M : s = \langle f, \eta \rangle, r^2 + s^2 \leq 1, \|\eta\|_M \leq 1 \} \]

\[ D = \{ (0, t, \eta) \in \mathbb{R} \times \mathbb{R} \times M : |t| \leq 1, \|\eta\|_M \leq 1 \}, \]

where $\langle \cdot, \cdot \rangle$ is the dual pairing between $M$ and $M^*$. Obviously, $C$ and $D$ are closed convex bounded and symmetric. Designate by $V$ the closed convex hull of $C \cup D$, i.e., $V = \overline{\text{co}}(C \cup D)$. Then $V$ is a closed convex bounded and symmetric set. Also, it has non-empty interior: If $C_1$ is the set $\{ (r, 0, 0) : |r| \leq 1 \}$, then $2^{-1}(C_1 + D) = \{ (r/2, t/2, \eta/2) : |r| \leq 1, |t| \leq 1, \|\eta\|_M \leq 1 \}$ has non-empty interior. On the other hand the latter set is properly contained in $V$.

Now $V$ viewed as a unit ball defines an equivalent norm $\| \cdot \|$ in $Z$. Let $P_M: Z \rightarrow 2^M$ be the metric projection generated by $M$ with respect to the $V$-norm.

For arbitrary $q \in [0, 2\pi)$, let $a_q = (\cos q, \sin q, 0) \in Z$. Our next goal is to determine the set $P_M(a_q)$. Notice that the orthogonal projections of $C$ and $D$ over $\mathbb{R}^2$ are both contained in the circle $\{ (r, s, 0) \in \mathbb{R}^2 \times M : r^2 + s^2 \leq 1 \}$. Since it is closed, the orthogonal projection of $V$ is in the same circle too. Therefore

\[ d(a_q, M) \geq 1. \] (1)

Denote by $m_q$ the affine set $\{ (\cos q, \sin q, \eta) \in Z : \eta \in M \}, q \in [0, 2\pi)$. It follows from (1) that $m_q$ does not intersect the interior of $V$. If we show
that \( V \cap m_q \neq \emptyset \), then the formula \( P_M(a_q) = (a_q + V) \cap M \) will take place. Towards this end, suppose first that \( q \neq \pi/2 \) and that \( q \neq 3\pi/2 \). In this situation \( m_q \cap D = \emptyset \). Moreover, both sets are separated by the functional \((\cos q, \sin q, 0) \in \mathbb{Z}^*\). So are \( m_q \) and \( C \). In order to prove that \( V \cap m_q = C \cap m_q \), we need the following

**Lemma 1.** Let \( C, D, \) and \( H \) be closed convex subsets of a normed space \( X \), and let \( g \in X^* \) be a bounded linear functional such that

\[
\sup \{ \langle g, y \rangle : y \in D \} = \alpha < \beta = \inf \{ \langle g, z \rangle : z \in H \},
\]

and

\[
\sup \{ \langle g, x \rangle : x \in C \} \leq \beta.
\]

Then \( H \cap \overline{\partial}(C \cup D) = H \cap C \).

**Proof.** Obviously \( H \cap C \subseteq H \cap \overline{\partial}(C \cup D) \). Let \( z \in H \), \( z = \lim z_n, z_n = \lambda_n x_n + (1 - \lambda_n) y_n \) where \( (x_n) \subseteq C, (y_n) \subseteq D, (\lambda_n) \subseteq [0,1] \). Choose a convergent subsequence of \((\lambda_n)\). With abuse of notation, let \( \lambda_n \to \lambda_0 \). Then we have

\[
\beta \leq \langle g, z \rangle \leq \lambda_0 \lim \langle g, x_n \rangle + (1 - \lambda_0) \lim \langle g, y_n \rangle
\]

\[
\leq \lambda_0 \beta + (1 - \lambda_0) \alpha \leq \lambda_0 \beta + (1 - \lambda_0) \beta = \beta.
\]

This implies \( \lambda_0 = 1 \), whence \( z = \lim x_n \). Therefore \( z \in C \) because \( C \) is closed. The proof is completed.

By Lemma 1 \( V \cap m_q = C \cap m_q \). So

\[
V \cap m_q = \{ (\cos q, \sin q, 0) \in Z : \langle f, \eta \rangle = -\sin q, \| \eta \|_M \leq 1 \}, \quad q \neq \pi/2, 3\pi/2
\]

(2)

The explicit form of \( V \cap m_q \) convinces us that \( V \) and \( m_q \) have a nonempty intersection.

For \( q = \pi/2 \) we obtain

\[
V \cap m_{\pi/2} = D \cap m_{\pi/2} = \{ (0, 1, \eta) \in Z : \| \eta \|_M \leq 1 \}.
\]

(3)

Analogously, for \( q = 3\pi/2 \)

\[
V \cap m_{3\pi/2} = D \cap m_{3\pi/2} = \{ (0, -1, \eta) \in Z : \| \eta \| \leq 1 \}.
\]

(4)

It is a routine matter to verify that \( P_M(a_q) = (a_q + V) \cap M = a_q + V \cap (-m_q) \), whence by (2)–(4) we have

\[
P_M(a_q) = \{ (0, 0, \eta) \in \mathbb{R} \times \mathbb{R} \times M : \langle f, \eta \rangle = -\sin q, \| \eta \|_M \leq 1 \}, \quad q \notin \{\pi/2, 3\pi/2\}
\]

(5)
as well as

\[ P_M(a_{\pi/2}) \supseteq \{(0, 0, \eta) \in Z : \|\eta\|_M \leq 1\} \]  

(6)

and

\[ P_M(a_{3\pi/2}) \supseteq \{(0, 0, \eta) \in Z : \|\eta\|_M \leq 1\}. \]  

(7)

In this way, for the points of the circumference \( E = \{(\cos q, \sin q, 0) \in Z : q \in [0, 2\pi] \} \), \( q \notin \{\pi/2, 3\pi/2\} \), the images of \( P_M \) correspond to the level-sets of \( f \) intersected by the closed unit ball \( U(M) \), while for \( q = \pi/2 \) or \( q = 3\pi/2 \), \( U(M) \) is contained in \( P_M(a_q) \).

We claim now that the restriction of \( P_M \) over \( E \) is finite lower semi-continuous. The claim is almost obvious; however, for the sake of completeness, we give a demonstration in the particular case \( q_0 = \pi/2 \) (for arbitrary \( q \) the proof is similar).

Fix \( \varepsilon > 0 \) (\( \varepsilon < \pi/2 \)) and take an open neighbourhood \( \mathcal{U} \) in \( Z \), \( a_{q_0} \in \mathcal{U} \), such that for arbitrary \( a_q = (\cos q, \sin q, 0), a_q \in \mathcal{U} \), it follows that \( |q - \pi/2| < \varepsilon \). Let \( (a_q)_{q=1}^n \in \mathcal{U} \) and \( k \) is an index satisfying \( |q_k - \pi/2| = \min\{|q_i - \pi/2| : i = 1, 2, \ldots, n\} \). Suppose \( q_k \neq \pi/2 \) (the case \( q_k = q_0 \) is trivial) and take \( y \in P_M(a_{q_k}) \). Then \( \langle f, y \rangle = -\sin q_k \). For each \( i \) choose \( \lambda_i, 0 < \lambda_i \leq 1 \), such that \( \langle f, \lambda_i y \rangle = -\sin q_i \), and define \( y_i = \lambda_i y \). Since \( y_i \) belongs to \( P_M(a_{q_i}) \), we have the estimation

\[
\|y - y_i\| = (1 - \lambda_i) \|y\| \leq 1 - \lambda_i
\]

\[
= 1 - \frac{\sin q_i}{\sin q_k} < 1 - \sin q_i \leq |\pi/2 - q_i| < \varepsilon.
\]

Therefore \( \bigcap_{i=1}^n B_\varepsilon(P_M(a_{q_i})) \neq \emptyset \), i.e., \( P_M|E \) is f.l.s.c. at \( a_{q_0} \). Our next lemma implies that \( P_M \) is everywhere f.l.s.c.

**Lemma 2.** Let \( Z = (Y \times M, \|\cdot\|) \) be a product space of two Banach spaces \( Y \) and \( M \), and let \( P_M \) be the metric projection generated by \( M \) (i.e., by \( \{0\} \times M \)). If for \( E = \{(y, 0) \in Z : \|y\|_Y = 1\} \) the restriction map \( P_M|E \) is a.l.s.c. (respectively f.l.s.c.) and has non-empty images, then so is \( P_M \).

**Proof.** For arbitrary \( z \in Z \) the representation \( z = \lambda y + m \) holds, where \( \lambda \geq 0, y \in Y, \|y\|_Y = 1, m \in M \). We claim that

\[ P_M(z) = m + \lambda \cdot P_M(y). \]  

(8)

Designate the closed unit ball of \( Z \) by \( V \) and suppose \( d(y, M) = r > 0 \). Then \( m + \lambda P_M(y) = m + \lambda (M \cap (y + rV)) = m + M \cap (\lambda y + \lambda rV) = M \cap (z + \lambda rV) \). Now since for any \( k \in (0, r) \) \( M \cap (y + kV) = \emptyset \), then \( M \cap (z + \lambda kV) = \emptyset \). Therefore \( d(z, M) = \lambda r \), which establishes the claim. In particular, (8) implies \( P_M(z) \neq \emptyset \).
We next prove that $P_M$ is a.1.s.c. at $z_0$ where $z_0$ is an arbitrary point in $Z$ (the case of f.l.s.c. is treated analogously). If $z_0 = m_0 \in M$, then for each $z \in B_{\sqrt{2}}(m_0)$ and each $m \in P_M(z)$
\[ \|m_0 - m\| \leq \|m_0 - z\| + \|z - m\| \leq 2 \cdot \|m_0 - z\| < \epsilon, \]
whence \( \bigcap \{ B_{\epsilon/2}(P_M(z)): \|z - z_0\| < \epsilon/2 \} \neq \emptyset \). So let $z_0 = \lambda y_0 + m_0$, $\lambda_0 > 0$, $\|y_0\| = 1$, $m_0 \in M$. Since $P_{M/E}$ is a.l.s.c. at $y_0$, there exist $\delta > 0$ and $u_0 \in Z$ such that
\[ u_0 \in \bigcap \{ B_{\sqrt{3}\lambda_0}(P_M(y)): y \in E, \|y - y_0\| < \delta \}. \]
(9)

Obviously, we can always assume that $\|u_0\| > 0$. Consider the open neighbourhood of $z_0$
\[ \mathcal{U} = \{\lambda y + m: |\lambda - \lambda_0| < \min \{\epsilon/4 \|u_0\|, \lambda_0/2\}, \]
$y \in E$, $\|y - y_0\| < \delta$, $m \in M$, $\|m - m_0\| < \epsilon/4 \}$.\n
Suppose $z \in \mathcal{U}$, $z = \lambda y + m$, and take in (9) a point $u \in P_M(y)$ such that $\|u - u_0\| < \epsilon/3\lambda_0$. According to (8) $\lambda u + m \in P_M(z)$. It follows from
\[ \|\lambda u + m - \lambda_0 u_0 - m_0\| \leq \lambda \|u - u_0\| + |\lambda - \lambda_0| \cdot \|u_0\| + \|m - m_0\| < \epsilon/3\lambda_0 + \epsilon/2 < \epsilon \]
that $\lambda_0 u_0 + m_0 \in \bigcap \{ B_{\epsilon/2}(P_M(z)): z \in \mathcal{U} \}$, and this completes the proof.

In this way, for an arbitrary bounded linear functional $f \in M^*$ not achieving its norm, we defined an equivalent norm in $Z$ with respect to which $M$ is proximinal and $P_M$ is f.l.s.c. Now $f$ is chosen in a more sophisticated manner so that $P_M$ fails to be a.l.s.c. In doing so we employ a theorem of James. But, before that, we make some explanatory remarks.

Suppose $(g_n) \subset M^*$ is a sequence of bounded linear functionals. Denote by $L(g_n)$ the set
\[ \{ w \in M^*: \lim \langle g_n, x \rangle \leq \langle w, x \rangle \leq \lim \langle g_n, x \rangle, \forall x \in M \} \]
and observe that $L(g_n)$ is non-empty. Indeed, the mapping $T: M \to l_\infty$, $T(x) = \langle g_n, x \rangle$, associates with each $x \in M$ a bounded sequence. If $\varphi \in l_\infty^*$ is a Banach limit, then $\lim \langle g_n, x \rangle \leq \varphi(T(x)) < \lim \langle g_n, x \rangle$ whence $w(\cdot) = \varphi(T(\cdot)) \in M^*$.

For arbitrary $f \in M^*$, $\|f\| = 1$, denote
\[ S(f, \gamma) = \{ x \in U(M): \langle f, x \rangle = \gamma \}, \quad 0 < \gamma < 1. \]
It follows from (5–7) and Lemma 2 that $P_{M/E}$ is a.l.s.c. at $(0, -1, 0) \in Z$ if, and only if,
\[ \forall \epsilon > 0 \exists \gamma_0 \in (0, 1): \bigcap_{\gamma > \gamma_0} B_{\epsilon}(S(f, \gamma)) \neq \emptyset. \]
(10)
Using the next theorem we show that in a non-reflexive Banach space there exists a functional \( f \) which does not satisfy (10).

**Theorem (James [9], [8, p. 12]).** Suppose \( M \) is a non-reflexive Banach space. Then for \( \theta \in (0, 1) \) and \( \lambda > 0 \), \( \sum_{n=1}^{\infty} \lambda_n = 1 \), there exist \( \alpha, \theta \leq \alpha \leq 2 \), and \( (g_n) \subset M^* \), \( \|g_n\| \leq 1 \), such that each \( w \in L(g_n) \) satisfies

\[
\sum_{n=1}^{\infty} \lambda_n (g_n - w) \leq \alpha \tag{11}
\]

and

\[
\sum_{n=1}^{k} \lambda_n (g_n - w) \leq \alpha \left( 1 - \theta \cdot \sum_{n=k+1}^{\infty} \lambda_n \right), \quad k \in \mathbb{N}. \tag{12}
\]

Pick \( \theta \in (0, 1) \) and choose \( \delta > 0 \) so that \( \delta < \theta^2/2 \). If \( \lambda_1 = 1 - \delta \), \( \lambda_{n+1} = \delta \lambda_n \), then \( \lambda_n > 0 \), \( \sum_{n=1}^{\infty} \lambda_n = 1 \), and according to the theorem of James there exist \( \alpha, \theta \leq \alpha \leq 2 \), and \( (g_n) \subset M^* \), \( \|g_n\| \leq 1 \), such that each \( w \), \( w \in L(g_n) \), satisfies (11) and (12).

For an arbitrary fixed functional \( w, w \in L(g_n) \), take \( f = \alpha^{-1} \sum_{n=1}^{\infty} \lambda_n (g_n - w) \) where \( \|f\| = 1 \). We claim that \( f \) does not satisfy (10). Assume the contrary. Then for \( 0 < \varepsilon < (\alpha \theta - 2\delta)/2 \cdot \delta/(1 - \delta) \) there are \( x_\varepsilon \in U(M) \) and \( y_\varepsilon \in (0, 1) \) such that \( x_\varepsilon \in B_\gamma(S(f, y)) \) whenever \( y \in (y_\varepsilon, 1) \). Since \( \lim \langle g_n, x_\varepsilon \rangle \leq \langle w, x_\varepsilon \rangle \), there is \( k \) so that

\[
\langle g_k - w, x_\varepsilon \rangle < \alpha \theta - 2\delta.
\]

Estimate \( \langle f, x \rangle \) for \( x \in U(M), \|x - x_\varepsilon\| \leq \varepsilon, \)

\[
\alpha \langle f, x \rangle = \left( \sum_{n=1}^{k-1} \lambda_n (g_n - w), x \right) + \lambda_k \langle g_k - w, x - x_\varepsilon \rangle
\]

\[
+ \lambda_k \langle g_k - w, x_\varepsilon \rangle + \left( \sum_{n=k+1}^{\infty} \lambda_n (g_n - w), x \right)
\]

\[
\leq \left\| \sum_{n=1}^{k-1} \lambda_n (g_n - w) \right\| + 2\varepsilon \lambda_k
\]

\[
+ (\alpha \theta - 2\delta) \lambda_k + 2 \cdot \sum_{n=k+1}^{\infty} \lambda_n
\]

\[
\leq \alpha \left( 1 - \theta \cdot \sum_{n=k}^{\infty} \lambda_n \right) + 2\varepsilon \lambda_k
\]

\[
+ (\alpha \theta - 2\delta) \lambda_k + 2\delta \cdot \sum_{n=k}^{\infty} \lambda_n
\]

\[
= \alpha - (\alpha \theta - 2\delta) \sum_{n=k+1}^{\infty} \lambda_n + 2\varepsilon \lambda_k.
\]
Hence

\[ \langle f, x \rangle \leq 1 - \epsilon, \]  

where

\[ c = a^{-1} \left[ (a\theta - 2\delta) \cdot \sum_{n=k+1}^{\infty} \lambda_n - 2\epsilon \lambda_k \right]. \]

Since \( \epsilon \) is sufficiently small, then

\[ c > a^{-1}(a\theta - 2\delta) \cdot \left[ \sum_{n=k+1}^{\infty} \lambda_n - \frac{\delta \lambda_k}{1 - \delta} \right] = 0. \]

It follows from (13) that \( B_\epsilon(x_\delta) \cap S(f, \gamma) = \emptyset \) whenever \( \gamma > 1 - c \) and this contradicts the choice of \( x_\epsilon \). Therefore \( f \) does not satisfy (10). Therefore \( P_M \) is not a.1.s.c. at \((0, -1, 0) \in Z\). The proof of Theorem 1 is completed.

With the help of Theorem 1 and the theorem of Deutsch and Kenderov we give the following criterion for reflexivity:

**Theorem 2.** A Banach space \( X \) is reflexive if, and only if, for every equivalent renorming of \( X \) every finite lower semi-continuous metric projection generated by a convex proximinal subset of \( X \) admits a continuous \( \epsilon \)-approximate selection for each \( \epsilon > 0 \).

**Proof.** Necessity. Let \((X, \| \cdot \|)\) be reflexive. Suppose \( \| \cdot \| \) is an equivalent norm and \( M \) is a convex subset of \( X \) which is proximinal with respect to \( \| \cdot \| \). Suppose also that the metric projection \( P_M: X \to 2^M \) is f.l.s.c. The Banach space \((X, \| \cdot \|)\) is reflexive and \( M \) is closed. We recall that in a reflexive space a convex set is proximinal if, and only if, it is closed. For arbitrary \( x \in (X, \| \cdot \|) \) and \( \epsilon > 0 \) there exists a neighbourhood \( \mathcal{U} \) of \( x \) such that \( \bigcap_{\gamma=1}^{\infty} B_{\epsilon/2}(P_M(x)) \neq \emptyset \) for each \( n \) and each choice of \( n \) points \( x_1, x_2, \ldots, x_n \). Now the family \( \{ B_{\epsilon/2}(P_M(z)) : z \in \mathcal{U} \} \), whose elements are weakly compact sets, has the finite intersection property and then it has a non-empty intersection. Therefore \( P_M \) is a.l.s.c. and according to the theorem of Deutsch and Kenderov \( P_M \) admits a continuous \( \epsilon \)-approximate selection for each \( \epsilon > 0 \).

Sufficiency. Suppose \( X \) is non-reflexive. It follows from Theorem 1 that there exist an equivalent norm \( \| \cdot \| \) and a convex proximinal set \( M \subset X \) such that the metric projection \( P_M: X \to 2^M \) is f.l.s.c. with respect to \( \| \cdot \| \), but it lacks a.l.s.c. Apply the theorem of Deutsch and Kenderov again, the sufficiency part, to prove that for some \( \epsilon > 0 \) \( P_M \) fails to admit a continuous \( \epsilon \)-approximate selection.
3. Example of an a.l.s.c. Metric Projection, Generated by a Three-Dimensional Subspace of a Five-Dimensional Space, which Does Not Have a Continuous Selection

In the sequel $S^n$ and $B^n$ will stand for the unit sphere and the closed unit ball of the $n$-dimensional Euclidean space $\mathbb{R}^n$, respectively. The Euclidean norm is denoted by $|\cdot|$.

The following simple example of an a.l.s.c. u.s.c. mapping $\phi: \mathbb{R} \to 2^{\mathbb{R} \times \mathbb{R}}$ which fails to admit a continuous selection has motivated our further considerations. The example is a modification of an analogous example due to Ch. Dangalchev [6]. Earlier examples of the same nature, but without upper semi-continuity involved, have been constructed by Pelant, c.f. [7], and Beer [1].

Suppose $(\theta_n)_{n=1}^\infty$ is a strictly decreasing sequence of positive reals so that $\lim_{n \to \infty} \theta_n = 0$. Define another sequence $(\omega_n)_{n=1}^\infty$ by $\omega_n = 2^{-1}(\theta_n + \theta_{n+1})$. The points $P_n$ and $Q_n$ have coordinates $(\omega_n, 1)$ and $(\theta_n, -1)$, respectively. Denote by $A_n$ the triangle $[Q_n, P_n, Q_{n+1}]$ for $n = 1, 2, \ldots$. The mapping $\phi$ is defined as follows: For $x > 0$ ($x < 0$), $\phi(x) = A_n$ if $x = 0$,

$$
\phi(x) = \begin{cases} 
A_n & \text{if } x = \omega_n \\
[Q_n, (x, 1)] & \text{if } \omega_n < x < \omega_{n-1} \\
[(0, -1), (0, 1)] & \text{if } x = 0,
\end{cases}
$$

and for $x < 0$ $\phi(x) = -\phi(-x)$.

It is easily checked that $\phi$ is a.l.s.c. Suppose $f: \mathbb{R} \to \mathbb{R}^2$ is a selection for $\phi$ which is continuous at both $\omega_n$ and $-\omega_n$. Then $f(\omega_n) = P_n$ and $f(-\omega_n) = -P_n$. On the other hand $\lim_{n \to \infty} P_n \neq (0, 0)$ and $f$ cannot be continuous at 0.

We consider next a multivalued mapping $\bar{\phi}$, in a certain sense similar to $\phi$, which admits a mechanical interpretation: The images of $\bar{\phi}$ might be viewed as sets of contact when a cylinder-like solid is rolling over a plane.

Denote $D_1 = \{(\cos \phi, \sin \phi, 1) \in \mathbb{R}^3: \phi \in [0, 2\pi]\}$ and $D_2 = \{(\cos \phi, \sin \phi, -1) \in \mathbb{R}^3: \phi \in [0, 2\pi]\}$. Let $\gamma_1$ and $\gamma_2$ be the planes carried by $D_1$ and $D_2$, respectively. Suppose $(\theta_n)$ and $(\omega_n)$ are two strictly decreasing sequences both defined for every integer $n \in \mathbb{Z}$ and satisfying $2\omega_n = \theta_n + \theta_{n+1}$. Moreover, suppose $\theta_0 = \pi/2$, $\theta_{-n} = \pi - \theta_n$, $\lim_{n \to \infty} \theta_n = \pi$, $\lim_{n \to -\infty} \theta_n = 0$. Then $\omega_{-n} = \pi - \omega_n$, $\lim_{n \to -\infty} \omega_n = \pi$, $\lim_{n \to +\infty} \omega_n = 0$. Put $P_n = (\cos \omega_n, \sin \omega_n, 1)$ and $Q_n = (\cos \theta_n, \sin \theta_n, -1)$. We now define a three-dimensional convex body $W$ by description of its surrounding surface $\Xi$ (see Fig. 1). The segments $[(1, 0, -1), (1, 0, 1)]$ and $[(-1, 0, -1), (-1, 0, 1)]$ are part of $\Xi$. So are the triangles $A_n = [Q_n, P_n, Q_{n+1}]$. Let $C_n$ be the cones, with vertices $Q_n$, generated by $D_1$. The conical sectors $K_n$ which also
belong to $\mathcal{E}$ are cut off from $C_n$ by $\gamma_1$, $\gamma_2$ and the planes through $\Delta_{n-1}$ and $\Delta_n$. Observe that for each $n$ the plane supported by $\Delta_n$ meets $\gamma_1$ at a line $l_n$ which is a tangent to $D_1$ at $P_n$. Hence the planes through two adjacent triangles $\Delta_{n-1}$ and $\Delta_n$ are tangent to the surface of $C_n$, the segments $[P_{n-1}, Q_n]$ and $[Q_n, P_n]$ being generatrices for $C_n$. To complete the definition of $\mathcal{E}$ note that $X \in \mathcal{E}$ implies $-X \in \mathcal{E}$. Finally, define $W = \overline{\mathcal{E}}$.

Thus we have a closed convex bounded and symmetric set with non-empty interior. Let $\| \cdot \|$ be the norm generated by $W$ via the Minkowski functional. Denote by $M$ the Minkowskian space $(\mathbb{R}^3, \| \cdot \|)$.

For each point $x \in S^3$, with coordinates $(x_1, x_2, x_3)$, $|x_3| \neq 1$, define $\pi(x) = (x_1/\sqrt{x_1^2 + x_2^2}, x_2/\sqrt{x_1^2 + x_2^2}, 0)$, i.e., $\pi$ projects $x$ along the "meridian" on the "equator" $E = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 = 1\}$.

It is clear from the definition of $W$ that for every $y = (y_1, y_2, y_3) \in bdW$ with $|y_3| \neq 1$ there exists a uniquely determined normal vector $v(y) \in S^3$.

Consider the set

$$r = \{v(y) \in S^3 : y = (y_1, y_2, y_3), |y_3| \neq 1, \|y\| = 1\},$$

which is symmetric since $W$ is symmetric itself. There is no difficulty in verifying that $E$ is a homeomorphic image of $r$ via $\pi$. Then $r$ might be viewed as a parametric curve with a parameter $\varphi$, where $\varphi$ is the oriented angle between the axis $Ox_3$ and $\pi(v(y))$.

Denote by $h_w(\cdot)$ the support function generated by $W$, i.e., $h_w(x) = \max \{ \langle x, z \rangle : z \in W \}$. For each $x \in r$ let

$$F_x = \{ y \in W : \langle x, y \rangle = h_w(x) \}.$$
Define \( \phi \) as a composed map \( E \rightarrow \pi^{-1} r \rightarrow -F W \), where \(-F(x) = -(F_x)\) whenever \( x \in r \). The images of \( \phi \) are the “contact sets” of \( W \) and a plane “rolling” around \( W \). Evidently, \( \phi \) is a.l.s.c. The absence of a continuous selection for \( \phi \) is shown in the same way as this was done for the mapping \( \phi \).

At the final stage of our construction we introduce a new norm in \( \mathbb{R}^2 \times M \) such that the metric projection \( P_M: \mathbb{R}^2 \times M \rightarrow 2^M \) restricted on the circumference \( C = \{ (\xi, 0) \in \mathbb{R}^2 \times M: |\xi| = 1 \} \) is identical with \( \phi \).

For arbitrary \( x \in r \) let

\[
G_x = \{ (\xi, \eta) \in \mathbb{R}^2 \times M: \xi = \pi(x), \eta \in F_x \},
\]

and define the new unit ball \( V \) by the formula

\[
V = \overline{co} \left( \bigcup_{x \in r} G_x \cup \frac{1}{2} B^* \right).
\]

Obviously, \( V \) is a closed bounded set with non-empty interior. It will suffice for symmetry to show that \( \bigcup_{x \in r} G_x \) is symmetric. Indeed, if \((\xi, \eta) \in \bigcup_{x \in r} G_x \), there is \( x \in r \) such that \( \xi = \pi(x) \) and \( \eta \in F_x \). Since \( W \) is symmetric, then \(-\eta \in F_{-x} \). On the other hand \(-\xi = \pi(-x) \) since \( r \) is symmetric. Hence \((-\xi, -\eta) \in G_{-x} \) and \(-x \in r \). Thus \( V \) defines a norm in \( \mathbb{R}^2 \times M \) which we also denote by \( \| \cdot \| \).

Identifying in notation \( \{0\} \times M \) with \( M \), let \( P_M \) be the metric projection generated by the three-dimensional subspace \( M \). We claim that

\[
P_M(\xi, 0) = \{0\} \times F_x
\]

whenever \( \xi \in \mathbb{R}^2, |\xi| = 1 \) and \( x = \pi^{-1}(\xi) \). The orthogonal projection along \( M \) maps \( V \) on \( B^2 \). For an arbitrary \( \xi \in S^2 \), denote \( m_\xi = \{(\xi, \eta): \eta \in M \} \). It is clear that \( d((0, 0), m_\xi) \geq 1 \). Suppose \( \xi \) is a fixed point on \( S^2 \) and \( \xi = \pi(x) \).

We prove next

\[
G_x = m_\xi \cap V
\]

The inclusion \( G_x \subseteq m_\xi \cap V \) follows immediately. Conversely, if \((\xi, \eta) \in m_\xi \cap V \), then \(|\xi| = 1 \) and \((\xi, \eta) = \lim_{n \to \infty} (\xi_n, \eta_n) \) where \((\xi_n, \eta_n) \in co(\bigcup_{x \in r} G_x \cup \frac{1}{2} B^2) \). According to the theorem of Carathéodory \((\xi_n, \eta_n) = \sum_{i=1}^6 \lambda_{ni}(\xi_{ni}, \eta_{ni}) \) where \( 0 \leq \lambda_{ni} \leq 1, \sum_{i=1}^6 \lambda_{ni} = 1, (\xi_{ni}, \eta_{ni}) \in \bigcup_{x \in r} G_x \cup \frac{1}{2} B^2, n = 1, 2, ..., i = 1, ..., 6 \). We may assume, by passing to subsequences, that for every \( i \lim_{n \to \infty} \lambda_{ni} = \lambda_{oi} \) and \( \lim_{n \to \infty} (\xi_{ni}, \eta_{ni}) = (\xi_{oi}, \eta_{oi}) \). So, with abuse of notation, we write

\[
(\xi, \eta) = \sum_{i=1}^{k} \lambda_{oi}(\xi_{oi}, \eta_{oi}), \lambda_{oi} > 0, \quad \sum_{i=1}^{k} \lambda_{oi} = 1, k \leq 6.
\]
Since $\zeta = \sum_{i=1}^{k} \lambda_{oi} \xi_{oi}$ and $|\xi_{oi}| \leq 1$, we have from the strict convexity of $B^2$ that $\xi_{oi} = \xi$ for $i = 1, 2, \ldots, k$. Suppose $i$ is a fixed index. If $(\xi_{ni}, \eta_{ni})$ were in $\frac{1}{2} B^2$ for infinitely many values for $n$, then $\xi_{oi}$ would belong to $\frac{1}{2} B^2$ too. But this is incompatible with our choice of $\xi$. So for large $n$ $(\xi_{ni}, \eta_{ni}) \in \bigcup_{x \in r} G_x$. Therefore there exist uniquely determined points $x_{ni} \in S^2$ such that $\xi_{ni} = \pi(x_{ni}), \eta_{ni} \in F_{x_{ni}}$ whence

$$\langle x_{ni}, \eta_{ni} \rangle = h_w(x_{ni}).$$

(16)

Since $\pi$ is a homeomorphism, then $\lim_{n \to \infty} x_{ni} = \lim_{n \to \infty} \pi^{-1}(\xi_{ni}) = \pi^{-1}(\xi) \in r$. Taking $x = \pi^{-1}(\xi)$ and letting $n$ go to infinity in (16), we obtain $\langle x, \eta_{oi} \rangle = h_w(x)$. Notice that $\eta_{oi} \in W$ since $W$ is a closed set. On the other hand $x \in r$ and then $\eta_{oi} \in F_{x}, \xi = \pi(x)$. So $(\xi, \eta_{oi}) \in G_x$. It follows from the convexity of $G_x$ that $(\xi, \eta) = (\xi, \sum_{i=1}^{k} \lambda_{oi} \eta_{oi}) \in G_x$. Thus (15) is established. In particular, (15) entails $d((0, 0), m_z) = 1$.

We proceed in determining the image of $P_M$ at $(\xi, 0)$ for $\xi \in S^2$. As was shown above $d((\xi, 0), M) = 1$. Suppose $(z, y) \in [(\xi, 0) + V] \cap M$. Then $z = 0$ and $(z, y) = (\xi, 0) + (-\xi, y)$ whence $(-\xi, y) \in V$. So $(\xi, -y) \in m_{z} \cap V = G_x$ whenever $x = \pi^{-1}(\xi)$. We have $-y \in F_{-x}$ which implies $y \in F_{-x}$. Thus $P_M(\xi, 0) \subseteq (0, F_{-x})$ for $\xi = \pi(x)$. Conversely, suppose $(0, y) \in (0, F_{-x})$ where $\xi = \pi(x)$. It follows from the representation $(0, y) = (\xi, 0) + (-\xi, y)$ that $(-\xi, y) \in G_{-x} \subseteq V$ because $-\xi \in \pi(-x)$. On the other hand obviously $(0, y) \in M$. The proof of our claim (14) is completed.

Finally, notice that since the restriction of $P_M$ on the circumference $C$ behaves like the mapping $\varphi$, we need only apply Lemma 2 in order to make sure that $P_M$ satisfies the required properties.

REFERENCES

6. Ch. Dangalchev, personal communication.