Stable indirect adaptive switching control for fuzzy dynamical systems based on T-S multiple models

Nikolaos A. Sofianos and Yiannis S. Boutalis

Department of Electrical and Computer Engineering, Democritus University of Thrace, University Campus of Kimmeria, 67100 Xanthi, Greece

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Stable indirect adaptive switching control for fuzzy dynamical systems based on T–S multiple models

Nikolaos A. Sofianos* and Yiannis S. Boutalis*

Department of Electrical and Computer Engineering, Democritus University of Thrace, University Campus of Kimmeria, 67100 Xanthi, Greece

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A new indirect adaptive switching fuzzy control method for fuzzy dynamical systems, based on Takagi–Sugeno (T–S) multiple models is proposed in this article. Motivated by the fact that indirect adaptive control techniques suffer from poor transient response, especially when the initialisation of the estimation model is highly inaccurate and the region of uncertainty for the plant parameters is very large, we present a fuzzy control method that utilises the advantages of multiple models strategy. The dynamical system is expressed using the T–S method in order to cope with the nonlinearities. T–S adaptive multiple models of the system to be controlled are constructed using different initial estimations for the parameters while one feedback linearisation controller corresponds to each model according to a specified reference model. The controller to be applied is determined at every time instant by the model which best approximates the plant using a switching rule with a suitable performance index. Lyapunov stability theory is used in order to obtain the adaptive law for the multiple models parameters, ensuring the asymptotic stability of the system while a modification in this law keeps the control input away from singularities. Also, by introducing the next best controller logic, we avoid possible infeasibilities in the control signal. Simulation results are presented, indicating the effectiveness and the advantages of the proposed method.

Keywords: adaptive control; feedback linearisation; fuzzy systems; multiple models; switching control; T–S models

1. Introduction

Control of nonlinear and unknown or highly uncertain systems is a very challenging and difficult task for the engineers due to the fact that nonlinear control techniques are not systematically developed and the unknown parameters impose a negative impact on the performance of these systems. Concerning the nonlinearity problem, fuzzy logic techniques have been shown to be an adequate and very useful tool. During the last three decades, there have been remarkable efforts from researchers in the field of fuzzy control systems. Takagi and Sugeno (1985) suggested the Takagi–Sugeno (T–S) fuzzy model which uses a set of fuzzy rules to describe a nonlinear system in terms of some local linear subsystems. These linear models – which are found in the consequent part of the rules – are fuzzy blended and the model of the nonlinear system is obtained. The main advantage of T–S models is that they offer to the engineer the possibility to utilise linear control techniques in order to control the global nonlinear system. Another advantage of T–S models is that they have the ‘universal function approximation’ property, in the sense that they are able to approximate every smooth nonlinear function to any degree of accuracy in a convex compact region (Ying 1998; Zeng, Zhang, and Xu 2000). The stability conditions of these systems were first proposed in Tanaka and Sano (1994) and Wang, Tanaka, and Griffin (1996) and recently in Sala and Arino (2008), Gassara, Hajjaji, and Chaabane (2010), Lam and Narimani (2010) and Lee, Park, and Joo (2010).

Apart from the nonlinear nature of the systems, there is another factor – as already mentioned – that imposes difficulties to the control problem solution; that is the partial knowledge of the system. This means that the system must be controlled and remain stable although some of its parameters are unknown. Adaptive control techniques – direct and indirect – are used to provide the answer to this problem. Several conventional adaptive control methods have been used in the past years (see Ioannou and Sun 1996; Tao 2003; Ioannou and Fidan 2006 and the references therein for details) but the adaptive fuzzy techniques have shown to ensure better performance under certain circumstances (Ordonez, Zumberge, Spooner, and Passino 1997). Stable adaptive fuzzy controllers were first

*Corresponding authors. Email: nsofian@ee.duth.gr; ybout@ee.duth.gr
proposed by Su and Stepanenko (1994), Wang (1994), Chen, Lee, and Chang (1996), Chen and Chen (1996) and since then much research effort has been devoted in this field concerning direct and indirect adaptive fuzzy control (Chao and Tong 1999; Feng, Cao, and Rees 2002; Golea, Golea, and Benmahammed 2003; Park and Cho 2004; Khaber, Hamzaoui, Zehar, and Essoubouli 2006; Labiod and Guerra 2007; Wang, Chen, and Dai 2007; Chen 2008; Huang et al. 2012) in the framework of various techniques such as sliding mode control (Chan, Rad, and Wang 2001; Wang, Rad, and Chan 2001), observer based control (Tong, Li, and Wang 2004; Bouklroune, Tadjine, M’Saad, and Farza 2009; Tong and Li 2009; Hyun, Park, and Kim 2010), hybrid adaptive control (Li and Tong 2003), backstepping design (Hsu and Fu 2001; Yang, Feng, and Ren 2004; Zhou, Feng, and Feng 2005; Li and Yang 2011) mixed feedforward/feedback based adaptive fuzzy control (Chiu 2006), self-structuring fuzzy control design (Chen, Wang, and Lee 2011) etc. This article focuses on the indirect adaptive control, which means that the T–S fuzzy model of a system is estimated on-line. Although the aforementioned fuzzy indirect adaptive control approaches are very efficient, their main drawbacks are that the use of only a single adaptive identification model, may lead to instability or poor performance problems, especially when the parameter uncertainty is very high, the parameters are changing rapidly and discontinuously and the initial estimations for the original parameters of the plant are highly inaccurate (Narendra and George 2002).

In this article, motivated by the weakness of some methods which use a single fuzzy identification model, we overcome the aforementioned drawbacks by introducing a T–S multiple model-based adaptive switching control (TSMMASC) architecture that is suitable for control problems which incorporate the above characteristics. Multiple models have been used in the past with great success (Middleton, Goodwin, Hill, and Mayne 1988; Narendra and Balakrishnan 1994, 1997; Narendra and George 2002; Narendra, Driollet, Feiler, and George 2003). This article contains three main contributions. First, to our knowledge, this is the first time where T–S fuzzy models are used in the framework of multiple models in order to cope with plant nonlinearities. By this way, all the advantages of fuzzy modelling enhance the advantages of multiple models in adaptive control. A stability analysis is provided, offering at the same time the context for future variations in the adaptive control of nonlinear systems using fuzzy multiple models. More specifically, instead of using a single identification model, we propose an architecture which contains more than one T–S identification adaptive models. One fuzzy adaptive feedback linearisation controller corresponds to each identification model and a switch defines the controller that will provide the input to the system. By using a switching rule, the control signal to be applied is determined at every time instant by the model which best approximates the plant. At every step the parameters of identification models and controllers are adapted simultaneously so that the asymptotic stability and a satisfactory performance for the system are ensured. In addition, the parameters adaptation is restricted in a way that there are no singularity problems in the control input. Second, the proposed approach improves the results of Park and Cho (2004) and Park and Park (2004), concerning the transient response and the stability of the adaptive controller when the nonlinear control problem incorporates characteristics such as highly unknown and time-varying parameters. Also, the proposed scheme does not require the control gain to be well known as in Narendra and George (2002). Third, the infeasibility problem of the control signal that could arise under certain circumstances when a single feedback linearisation controller is used (Park and Cho 2004; Park and Park 2004; etc.), is overcome by introducing the Next Best Controller Logic (NBCL) in the switching rule criteria.

The rest of this article is organised as follows. In Section 2 the dynamical system under consideration is described using the T–S method. In Section 3 the TSMMASC architecture is described in detail. Moreover, the switching rule and the cost criterion are given. Section 4 describes the identification models and controller design procedure, while the stability analysis along with a discussion on the convergence and the computational cost of the switching algorithm are given in Section 5. Simulation studies showing the superiorities of the proposed method over the single model approach are provided in Section 6. The conclusions are finally drawn in Section 7.

2. Problem statement

In this section we briefly describe the control problem of this article utilising the T–S formulation and some model expressions borrowed from the linear systems (Ioannou and Fidan 2006).

Let a continuous-time nonlinear unknown system be described by using the T–S method. The fuzzy rules for a nth-order plant are of the following form:

Rule i: IF \( x_1(t) \) is \( M^i_1 \) and \( x_2(t) \) is \( M^i_2 \) and \( \cdots \) and \( x_n(t) \) is \( M^i_n \)

THEN \( \dot{x}(t) = A_i x(t) + B_i u(t) \)

where \( i = 1, \ldots, I \) is the number of fuzzy rules, \( M^i_p, p = 1, \ldots, n \) are the fuzzy sets,
where \( \mathbf{I}_{(n-1)} \) is an \((n-1) \times (n-1)\) identity matrix.

Given a pair of \( x(t), u(t) \), the final form of the fuzzy system is given by the following equation:

\[
\dot{x}(t) = \sum_{i=1}^{l} h_i(x(t))(A_i x(t) + B_i u(t)) \sum_{i=1}^{l} h_i(x(t)) \tag{1}
\]

where \( h_i(x(t)) = \prod_{p=1}^{n} M_p^i(x_p(t)) \geq 0 \) and \( M_p^i(x_p(t)) \) is the grade of membership of \( x_p(t) \) in \( M_p^i \) for all \( i = 1, \ldots, l \) and \( p = 1, \ldots, n \).

The state space parametric model (SSPM) (Ioannou and Fidan 2006) expression of (1) is given by the following equation:

\[
\dot{x}(t) = A_d x(t) + \sum_{i=1}^{l} h_i(x(t))(A_i - A_d)x(t) + B_i u(t) \sum_{i=1}^{l} h_i(x(t)) \tag{2}
\]

The matrix \( A_d \) is stable and corresponds to the state matrix of an arbitrary reference model which is described by the following equation:

\[
\dot{x}_m = A_d x_m \tag{3}
\]

where \( x_m \in \mathbb{R}^n \) is the state vector of the desired reference model. According to the problem statement, the parameters matrices \( A_i, B_i \) are unknown and an estimation model must be used in order to design the control signal based on the certainty equivalence approach. Utilizing the series parallel model (SPM) (Ioannou and Fidan 2006), the estimation model can be formed as

\[
\dot{x}(t) = A_d \hat{x}(t) + \sum_{i=1}^{l} h_i(x(t))(\hat{A}_i - A_d)x(t) + \dot{B}_i u(t) \sum_{i=1}^{l} h_i(x(t)) \tag{4}
\]

where the symbol ‘\(^\wedge\)’ denotes the estimated values for the parameters of the real plant. The SPM expression will be used in the sequel.

3. T–S multiple models based adaptive switching control

The usual methodologies encountered in the indirect adaptive fuzzy control techniques, use a single identification adaptive model which changes its parameter values trying to approximate the dynamic behaviour of the plant model (Park and Cho 2004; Park and Park 2004; Qi and Brdys 2008). In this section, we introduce the new TSMASC architecture along with a switching rule and a cost criterion in order to achieve better performance and stability.

3.1. Architecture

The proposed fuzzy control architecture is depicted in Figure 1. The unknown T–S plant model receives an input from one of the available fuzzy adaptive controllers and produces an output which is identical to its state vector. The control objective is to make the plant’s state track the reference model’s state, i.e. \( e_m = x - x_m \to 0 \). In order to achieve this objective, we use \( N \) T–S identification models \( \{M_k\}_{k=1}^{N} \) of the plant, which are operating in parallel. Every linear submodel of each T–S model rule is expressed by using the SPM formulation (4). These \( N \) T–S models are of the same architecture concerning the number of rules, the membership functions and the premise variables, but they differ in their initial estimations for the plant’s parameters. More precisely, the uncertain parameters of \( A_i, B_i \) are denoted as \( E_{AB} \in \Xi \subseteq \mathbb{R}^n \), where \( \Xi \) is a compact space indicating the region of all the possible parameters values combinations and \( s \) is equal to the number of the unknown parameters (i.e. in case there are two unknown parameters, the space of \( \Xi \) is two dimensional). The initial parameters estimates are distributed uniformly in space \( \Xi \). Every T–S model produces an output \( \hat{x}_k \), which is equal to its state vector and the identification error is given by \( e_k = x - \hat{x}_k \). For every identification model \( M_k \), there is a corresponding feedback linearisation adaptive controller \( C_k \) with an output \( u_k \), and all the controllers are updated in an indirect way, using the certainty equivalence approach. By applying the controller \( C_k \) to the T–S model \( M_k \), the output is given by a dynamical equation identical to that of the reference model (3) as we will prove in the upcoming section. The objective of the TSMMASC architecture is to improve the performance of the system, and ensure its stability. This is performed by choosing at every time instant the appropriate controller according to a switching rule, which is based on a cost criterion \( J_k \).
3.2. Switching rule and cost criterion

In this subsection the description of the switching scheme which orchestrates the suitable controller selection is given. A switching rule selects at every time instant the most appropriate controller for the plant. This rule is based on some indices $J_k$ which are relevant to the model identification errors $e_k = x - x_k$, while the performance of the identification T–S models is evaluated in parallel at every time instant. An additive hysteresis constant $h$ (Middleton et al. 1988; Liberzon 2003, p. 135) and a $T_{\text{min}}$ (Narendra and Balakrishnan 1994) are also used. The switching rule description is as follows. If

$$J_f(t) = \min_{k \in \Lambda} \{J_k(t)\},$$

$\Lambda = \{1, \ldots, N\}$, and $J_f(t) + h < J_0(t)$ is valid in the time interval $[t, t + T_{\text{min}}]$, where $J_0(t)$ is the index of the current active T–S model $M_{cr}$, then the controller $C_{f}$ is chosen and is tuned utilising the certainty equivalence approach from the estimations of the corresponding T–S model $M_f$. If $J_f(t) + h > J_0(t)$ is valid at any time instant of the time interval $[t, t + T_{\text{min}}]$, then the controller $C_{cr}$ remains active, meaning that it is the ideal controller for that time interval. The above procedure is repeated at every step. It has to be noted that the signal of the dominant controller $C_f$ or $C_{cr}$ is used to control the plant as well as all the other T–S identification models. If $J_f(t)$ is not unique (i.e. there are two or more models with equal minimums), the choice of the dominant controller is made arbitrarily among these models. The terminology $C_j$ and $M_j$ will be used for the dominant controller and the dominant T–S model, respectively, in the following sections.

There are many different mathematical expressions for the cost criteria depending on the problem specifications. In this article we use a cost criterion of the following form:

$$J_k(t) = a_c e_k^2(t) + b_c \int_0^t e_k^2(r) dr$$  \hspace{1cm} (5)

where $a_c, b_c$ are design parameters. If $J_k(t) = e_k^2(t)$, possible parameters changes are detected very quickly but the switching between the controllers may be very rapid and the chattering effect may lead to poor control. On the other hand, if $J_k(t) = \int_0^t e_k^2(r) dr$, the parameters changes are not detected on time and the switching between the controllers may be infrequent. Consequently, the criterion (5), which embodies both instantaneous and past measures, may lead to a smooth and satisfactory control signal in case the parameters of the plant are time-varying.

Remark 1: In order to avoid switching at very high frequencies, a time interval $T_{\text{min}}$ is allowed to elapse between the switchings so that $T_{i+1} - T_i \geq T_{\text{min}}, \forall i$ where $\{T_i\}_{i=0}^{\infty}$ is a switching sequence with $T_0 = 0$ and $T_i < T_{i+1}, \forall i$. The choice of an arbitrarily small $T_{\text{min}}$ leads in a globally stable system (Narendra and Balakrishnan 1994).
4. T–S identification models, controller design and NBCL

In this section, the appropriate mathematical expression for the T–S identification models is given, a fuzzy controller inspired from Kang, Kwon, Lee, and Park (1998) is constructed and finally the new NBCL is presented.

4.1. T–S identification models

The TSMMASC approach makes use of N architecturally identical T–S models with different initialisations concerning the parameters estimations. Every linear submodel in the consequent part of every T–S model rule is expressed by using the SPM formulation. The fuzzy rules that describe every T–S model $M_k$ are of the following form:

T–S Identification Model $M_k$

Rule i: IF $x_i(t)$ is $M_{k_1}^i$ and $x_j(t)$ is $M_{k_2}^i$ and ... and $x_n(t)$ is $M_{k_n}^i$

THEN $\hat{x}_k(t) = A_k \hat{x}_k(t) + (A_k - A_d)x(t) + B_k u(t)$

where $k \in \Lambda = \{1, ..., N\}$ and $i = 1, ..., l$. It should be noted that the number N of T–S identification models is independent of the number n of the state variables. The final form of every T–S model is inferred by a fuzzy blending of the linear subsystems and is given by the following equation:

$$\dot{\hat{x}}_k(t) = A_k \hat{x}_k(t) + \sum_{i=1}^{l} h_k(x)(A_k - A_d)x(t) + \hat{B}_k u(t)$$

where $h_k(x) = \prod_{i=1}^{n} M_{k_i}^i(x_p(t)) \geq 0$ and $M_{k_i}^i(x_p(t))$ is the grade of membership of $x_p(t)$ in $M_{k_i}^i$, $k \in \Lambda$, $i = 1, ..., l$ and $p = 1, ..., n$. Also, $M_{k_i}^i(x_p(t)) = M_{k_i}(x_p(t))$ and $h_k(x) = h_k(x)$ for all $k, i, p$. The matrices of all the T–S models are of the following form:

$$\hat{A}_k = \begin{bmatrix} 0 & I_{(n-1)} \\ 0 & \vdots \\ \hat{\alpha}_n & \hat{\alpha}_{n-1} & \cdots & \hat{\alpha}_1 \end{bmatrix}, \quad \hat{B}_k = \begin{bmatrix} 0 \\ \vdots \\ \hat{b}_m \end{bmatrix}$$

4.2. Controller design

Using the feedback linearisation technique (Kang et al. 1998), and denoting the dominant controller as $C_p$, the control signal at every time instant can be described by the following equation:

$$u(t) = u(t) = \sum_{i=1}^{l} h_j(x)(a^d - \hat{a}^d)^T x(t)$$

where $(a^d)^T = [\hat{a}_n \hat{a}_{n-1} \cdots \hat{a}_2 \hat{a}_1]$ and $(a^d)^T = [a_n a_{n-1} \cdots a_2 a_1]$ are the nth rows of the estimated state and reference matrices, respectively.

For the sake of compactness, the following notation will be adopted in the sequence:

$$h_j(x) = h_k(x) = h_i$$

Applying the control input $u(t)$ given by (7) to the dominant T–S model $M_j$, we obtain

$$\dot{\hat{x}}_j(t) = A_d \hat{x}_j(t) + \frac{1}{\sum_{i=1}^{l} h_i} \left( \sum_{i=1}^{l} h_i A_j - A_d \right) x(t) + \hat{B}_j \sum_{i=1}^{l} h_j (a^d - \hat{a}^d)^T x(t)$$

where $\theta_{(n-1)}$ is a $(n-1) \times (n-1)$ zero matrix.
It is obvious that:

\[ \dot{x}_j(t) = A_d \dot{x}_j(t) \]  

(8)

It can be noticed that when the input \( u_j(t) \) is applied to the corresponding estimated plant \( M_j \), this plant is linearised and it is asymptotically stable, with a behaviour identical to that of the desired reference model (3).

4.3. The NBCL

The NBCL is derived from the necessity to provide a feasible control signal during the control procedure of the plant. When a single adaptive feedback linearisation controller of type (7) is used, there is a possibility to produce a zero control signal although \( x(t) \neq 0 \). This is the case when at any time instant the following equality holds for \( j = 1 \) (i.e. a single T–S identification model):

\[ \sum_{i=1}^{l} h_{ij}(x)(a^d - \hat{a}^j)^T x(t) = 0 \]  

(9)

Then, from (7) it holds that \( u(t) = u_j(t) = 0 \). Although the estimated parameters \( \hat{a}^j \) will change their values at the next step, as we will prove in the next section, the zero control signal may lead to instabilities or poor performance of the plant. In our case, where multiple models are used for the control of the plant, there is also a possibility for a zero control signal when (9) holds for the winning controller \( C_j \). In order to avoid this zero control signal, we impose the following criterion in the switching rule:

If \( C_j \) is the dominant controller and

\[ \sum_{i=1}^{l} h_{ij}(x)(a^d - \hat{a}^j)^T x(t) = 0 \quad \text{and} \quad x(t) \neq 0 \]

Then \( u(t) = u_{i(nbc)}(t) \), where \( j(nbc) = \arg \min_{k \in \Lambda, k \neq j} \{J_k(t)\} \).

This criterion forces the switching rule to choose the best alternative controller which will not provide a zero signal for that time instant.

5. Stability analysis, convergence and computational cost

The stability analysis of the proposed architecture along with a discussion on the convergence and the computational cost of the switching algorithm are given in this section.

5.1. Stability analysis

The identification error for every T–S model is defined as

\[ e_k = x - \hat{x}_k \]  

(10)

The error \( e_k \) corresponds to the difference between the state of the plant and the state of the T–S model \( M_k \).

The derivative of the identification error is given by the following equation:

\[ \dot{e}_k = \hat{x} - \dot{x}_k \]

\[ = A_d e_k - \sum_{i=1}^{l} h_{i} \hat{A}_{ki} \]

\[ x - \sum_{i=1}^{l} h_{i} \hat{B}_{ki} u \]

\[ = A_d e_k - \sum_{i=1}^{l} h_{i} \left[ \begin{array}{cccc}
0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 \\
\end{array} \right] x \]

\[ + \sum_{i=1}^{l} h_{i} \left[ \begin{array}{cccc}
0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 \\
\end{array} \right] u \]

\[ = A_d e_k - \sum_{i=1}^{l} h_{i} \left[ \begin{array}{cccc}
0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 \\
\end{array} \right] u \]  

(11)

where \( \hat{A}_{ki} = \hat{A}_{ki} - A_i \), \( \hat{B}_{ki} = \hat{B}_{ki} - B_i \), \( \hat{a}_{ki} = \hat{a}_{ki} - a_i \), \( \hat{b}_{ki} = \hat{b}_{ki} - b_i \), \( \hat{a}_{ki} = [\hat{a}_{ki}, \hat{a}_{ki}, \ldots, \hat{a}_{ki}]^T \) is an \( n \times 1 \) vector, \( \theta_{(n-1)} \) is an zero matrix of dimension \( n \times (n-1) \) and \( p = 1, \ldots, n \).

Consider the following functions as Lyapunov function candidates:

\[ V_k(e_k, \hat{a}_{ki}, \hat{b}_{ki}) = e_k^T P_k e_k + \sum_{i=1}^{l} \frac{(\hat{a}_{ki})^T \hat{a}_{ki}}{r_{ki}^1} + \sum_{i=1}^{l} \frac{(\hat{b}_{ki})^2}{r_{ki}^2} \]  

(12)

where \( r_{ki}^1, r_{ki}^2 > 0 \) are the learning rate constants, \( V_k \geq 0 \), and \( P_k = P_k^T \) is the solution of the following Lyapunov equation:

\[ A_k^T P_k + P_k A_k = -Q_k \]  

(13)

for known \( Q_k > 0 \). The time derivative of \( V_k \) is given as follows:

\[ \dot{V}_k = \dot{e}_k^T P_k e_k + e_k^T P_k \dot{e}_k + \sum_{i=1}^{l} \frac{2 (\hat{a}_{ki})^T \hat{a}_{ki}}{r_{ki}^1} + \sum_{i=1}^{l} \frac{2 (\hat{b}_{ki})^2}{r_{ki}^2} \]

\[ = \dot{e}_k^T A_k^T P_k e_k + e_k^T P_k A_k \dot{e}_k + \sum_{i=1}^{l} \frac{2 (\hat{a}_{ki})^T \hat{a}_{ki}}{r_{ki}^1} \]
\[ + \sum_{i=1}^{l} 2 \frac{\tilde{b}^{ki} \hat{b}^{ki}}{r_{2}^{ki}} - \sum_{i=1}^{l} h_{i} x^{T} \left[ 0_{n \times (n-1)} \quad \tilde{a}^{ki} \right] P_{k} e_{k} \]
\[ - \hat{e}_{k}^{T} P_{k} \sum_{i=1}^{l} h_{i} \left[ 0_{n \times (n-1)} \quad \tilde{a}^{ki} \right]^{T} x \]
\[ - \sum_{i=1}^{l} h_{i} x^{T} \left[ 0 \quad 0 \ldots \tilde{b}^{ki} \right] P_{k} e_{k} \]
\[ - \hat{e}_{k}^{T} P_{k} \sum_{i=1}^{l} h_{i} \left[ 0 \quad 0 \ldots \tilde{b}^{ki} \right]^{T} u \]
\[ = \hat{e}_{k}^{T} \left( A_{d}^{x} P_{k} + P_{k} A_{d} \right) e_{k} + \sum_{i=1}^{l} 2 \frac{\tilde{b}^{ki} \hat{b}^{ki}}{r_{1}^{ki}} + \sum_{i=1}^{l} 2 \frac{\tilde{b}^{ki} \hat{b}^{ki}}{r_{2}^{ki}} \]
\[ - \sum_{i=1}^{l} h_{i} x^{T} \tilde{a}^{ki} P_{k} e_{k} - \hat{e}_{k}^{T} P_{k} \sum_{i=1}^{l} h_{i} \left( \tilde{a}^{ki} \right)^{T} x \]
\[ - \sum_{i=1}^{l} h_{i} u^{T} \hat{b}^{ki} P_{k} e_{k} - \hat{e}_{k}^{T} P_{k} \sum_{i=1}^{l} h_{i} \hat{b}^{ki} u \]

\[ = \hat{e}_{k}^{T} \left( A_{d}^{x} P_{k} + P_{k} A_{d} \right) e_{k} + \sum_{i=1}^{l} 2 \frac{\tilde{b}^{ki} \hat{b}^{ki}}{r_{1}^{ki}} + \sum_{i=1}^{l} 2 \frac{\tilde{b}^{ki} \hat{b}^{ki}}{r_{2}^{ki}} \]
\[ - \sum_{i=1}^{l} h_{i} x^{T} \tilde{a}^{ki} P_{k} e_{k} - \hat{e}_{k}^{T} P_{k} \sum_{i=1}^{l} h_{i} \left( \tilde{a}^{ki} \right)^{T} x \]
\[ - \sum_{i=1}^{l} h_{i} u^{T} \hat{b}^{ki} P_{k} e_{k} - \hat{e}_{k}^{T} P_{k} \sum_{i=1}^{l} h_{i} \hat{b}^{ki} u \]

Therefore using the above values for \((\hat{a}^{ki})^{T}, \hat{b}^{ki}\) we obtain

\[ \dot{V}_{k} = -\hat{e}_{k}^{T} Q_{k} e_{k} \leq 0 \] (15)

Consequently, \(e_{k}, \hat{a}^{ki}, \hat{b}^{ki} \in L_{\infty}\) and thus \(a^{ki}, \hat{b}^{ki} \in L_{\infty}\) too.

The next goal is to show that the adaptation mechanism ensures the boundedness of the control signal. Although, \(a^{ki}, \hat{b}^{ki}\) are bounded, the term \(1/\hat{b}^{ki}\) in the control signal expression may become unbounded whenever the adaptive algorithm generates values approximately zero or equal to zero for \(\hat{b}^{ki}\). In particular, \(\hat{b}^{ki}\) should not go to zero. More precisely the following assumption is necessary; the sign of \(b^{ki}\) and a lower bound \(b^{ki}_{0} > 0\) for \(|b^{ki}|\) are known for all \(i = 1, \ldots, l\). Therefore, a modification in the adaptive law for \(\hat{b}^{ki}\) is required. The restricted adaptive law takes the following form:

\[ (\dot{a}^{ki})^{T} = r_{1}^{ki} \sum_{i=1}^{l} h_{i} P_{k}^{T} e_{k} x^{T}, \]
\[ \dot{b}^{ki} = \begin{cases} \frac{1}{r_{2}^{ki}} \sum_{i=1}^{l} h_{i} P_{k}^{T} e_{k} u, & \text{if } |\hat{b}^{ki}| > b^{ki}_{0} \text{ or} \\
0, & \text{if } |\hat{b}^{ki}| = b^{ki}_{0} \text{ and } P_{k}^{T} e_{k} \text{sgn}(h_{i}) \geq 0 \end{cases} \quad \text{for all } i, k. \] (16)

where \(P_{k} \in R^{n \times n}\) is the \(n\)th column of \(P_{k}\).

After some straightforward mathematical manipulations, the time derivative of \(V_{k}\) is given by the following equation:

\[ \dot{V}_{k} = -\hat{e}_{k}^{T} Q_{k} e_{k} + \sum_{i=1}^{l} 2 \frac{\tilde{b}^{ki} \hat{b}^{ki}}{r_{1}^{ki}} + \sum_{i=1}^{l} 2 \frac{\tilde{b}^{ki} \hat{b}^{ki}}{r_{2}^{ki}} \]
\[ - 2 \sum_{i=1}^{l} h_{i} P_{k}^{T} e_{k} x^{T} \tilde{a}^{ki} - 2 \sum_{i=1}^{l} h_{i} P_{k}^{T} e_{k} u \]
\[ \text{In order to make the time derivative of } V_{k} \text{ negative}, \ (\dot{a}^{ki})^{T}, \hat{b}^{ki} \text{ are chosen as follows:} \]
\[ (\dot{a}^{ki})^{T} = r_{1}^{ki} \sum_{i=1}^{l} h_{i} P_{k}^{T} e_{k} x^{T} \]
\[ \dot{b}^{ki} = \begin{cases} \frac{1}{r_{2}^{ki}} \sum_{i=1}^{l} h_{i} P_{k}^{T} e_{k} u, & \text{if } |\hat{b}^{ki}| > b^{ki}_{0} \text{ or} \\
0, & \text{if } |\hat{b}^{ki}| = b^{ki}_{0} \text{ and } P_{k}^{T} e_{k} \text{sgn}(h_{i}) < 0 \end{cases} \quad \text{for all } i, k. \] (14)

The initialisation of the \(\hat{b}^{ki}\) for every one of the \(N\) \(T\)–\(S\) models is chosen so that the following inequality holds:

\[ \hat{b}^{ki}(0) \text{sgn}(b^{ki}) \geq b^{ki}_{0} \] (17)

The negativity of \(\dot{V}_{k}\) must be checked again under the new framework of the adaptive law (16). The following theorem ensures that:

(i) The plant is asymptotically stable
(ii) The plant follows the state of the reference model (3).

**Theorem 5.1:** Given a plant model (1) with the control law (7), the adaptive law (16) and a reference model with state matrix \(A_{\mu}\) the TSMMASC approach guarantees that for all \(i = 1, \ldots, l\) and \(j, k \in \Lambda:\)

(1) \(a^{ki}, \hat{b}^{ki}, 1/\hat{b}^{ki}, e_{k}(t)\) are bounded,
(2) \([e_{k}(t), a^{ki}(t), b^{ki}(t), e_{\text{nl}}(t)] \to 0 \text{ as } t \to \infty.\)

**Proof:** First, we must ensure that \(\hat{b}^{ki}(t)\) do not take values equal to zero or approximately zero. The initial
estimation for $b_i$ fulfils the condition $b_i^{(0)} \text{sgn}(b_i) = |b_i^{(0)}| \geq b_0^i$ for all $i, k$. In case during the estimation process, $|b_i^{(0)}(t)| = b_0^i$ and $P_k^T e_k \text{sgn}(b_i) \geq 0$, then from (16) it holds that

$$\hat b_i(t) = t_1^i \frac{h_i}{\sum_{l=1}^{i} h_l} P_k^T e_k u \hat b_i(t)$$

$$= t_2^i \frac{h_i}{\sum_{l=1}^{i} h_l} P_k^T e_k \text{sgn}(b_i)$$

$$= t_2^i \frac{h_i}{\sum_{l=1}^{i} h_l} P_k^T e_k \text{sgn}(b_i) \hat b_i(t) \geq 0$$

If $|\hat b_i^{(0)}(t)| = b_0^i$ and $P_k^T e_k \text{sgn}(b_i) < 0$ then $\hat b_i(t) = 0$. Consequently, $\hat b_i(t) \hat b_i^{(0)}(t) \geq 0$. This inequality implies that $|\hat b_i^{(0)}(t)| \geq b_0^i$ for all $i, k$. In other words, the algorithm prevents the estimated parameters $b_i^{(0)}(t)$ to get through the $b_0^i$ value and thus $1/\hat b_i^{(0)}$ is finite. Considering the adaptive law (16), the time derivative of $V_k$ takes the following form:

$$\dot V_k = \begin{cases} -e_k^T Q_k e_k, & \text{if } |\hat b_i^{(0)}| > b_0^i \text{ or } P_k^T e_k \text{sgn}(b_i) < 0, \\ -e_k^T Q_k e_k - 2 \sum_{l=1}^{i} \frac{h_l b_l^i P_k^T e_k u}{\sum_{l=1}^{i} h_l}, & \text{if } |\hat b_i^{(0)}| = b_0^i \text{ and } P_k^T e_k \text{sgn}(b_i) \geq 0, \quad \text{for all } i, k \\ \end{cases}$$

The case when $|\hat b_i^{(0)}| = b_0^i$ and $P_k^T e_k \text{sgn}(b_i) < 0$ should be investigated concerning the demand for the negativity of $V_k$. Equality $|\hat b_i^{(0)}| = b_0^i$ implies that $\hat b_i^{(0)} \text{sgn}(b_i) = (\hat b_i^{(0)} - b_0^i) \text{sgn}(b_i) < 0$. Utilizing the fact that $P_k^T e_k \text{sgn}(b_i) < 0$, the following inequality holds:

$$\hat b_i^{(0)} P_k^T e_k u = ((\hat b_i^{(0)} - b_0^i) \text{sgn}(b_i)) (P_k^T e_k \text{sgn}(b_i)) > 0$$

This result means that $V_k \leq -e_k^T Q_k e_k \leq 0$, for all $k$ and $t \geq 0$. Consequently, the function $V_k$ is a Lyapunov function for the error system (11) when the parameters are updated according to (16). This implies that $e_k, \hat a_k^{(i)}, \hat b_k^{(i)}, \hat b_k^{(i)} \in L_\infty$. The terms $1/\hat b_i^{(0)}$ are also uniformly bounded. From Equations (12) and (15) it is obvious that due to the fact that $V_k$ is bounded from below and non-increasing with time, the following equation stands:

$$\lim_{t \to \infty} V_k(e_k(t), \hat a_k^{(i)}(t), \hat b_k^{(i)}(t)) = V(\infty) < \infty$$

Integrating the quantity $e_k^T Q_k e_k$ from zero to infinity, one obtains

$$\int_0^\infty e_k^T Q_k e_k dr \leq -\int_0^\infty \dot V_k dr = (V_k(0) - V_k(\infty))$$

where $V_k(0) = V_k(e_k(0), \hat a_k^{(0)}(0), \hat b_k^{(0)}(0))$. Consequently, $e_k \in L_2 \cap L_\infty$. As mentioned above, at any time instant only one controller $C_j$ is chosen in order to control the plant. The control signal is equal to $u_j(t)$, which is given by (7). Applying $u_j(t)$ in $M_j$ we obtain $\hat x_j(t) = A_\hat x_j(t)$, as it was shown in (8), and taking into account the expression (2), the time derivative of the identification error for $M_j$ is given as follows:

$$\dot e_j(t) = \dot x(t) - \dot \hat x(t)$$

$$= A_d e_j(t) + \sum_{l=1}^{i} h_l(x(t))((A_i - A_d)x(t) + B_d u(t))$$

(22)

In the same way, the time derivative of the reference model error $e_m = x - x_m$ is given as follows:

$$\dot e_m(t) = \dot x(t) - \dot \hat x_m(t)$$

$$= A_d e_m(t) + \sum_{l=1}^{i} h_l(x(t))((A_i - A_d)x(t) + B_d u(t))$$

(23)

From (22) and (23) one has

$$\dot e_j(t) - \dot e_m(t) = A_d (e_j(t) - e_m(t))$$

(24)

Due to the fact that $e_j \in L_2 \cap L_\infty$ for all $j \in k$, and Equation (24) is satisfied at any time instant, it follows that $e_m(t)$ is bounded and consequently $x, u \in L_\infty$. Then, Equation (11) implies that $e_k \in L_\infty$, combined with $e_k \in L_2 \cap L_\infty$, we obtain that $e_k \to 0$ asymptotically. Using (16), we conclude that $\hat a^{(i)}, \hat b^{(i)} \to 0$. Finally from (24) it follows that $\lim_{t \to \infty} e_m(t) \to 0$, which is the objective of the controller.

The overall procedure of the proposed approach can be delineated as follows:

**Step 1:** Represent the nonlinear plant using a T–S fuzzy model (1).

**Step 2:** Based on the representation of Step 1 and utilising the SPM expression, construct $N$ T–S identification models $\{M_k\}_{k=1}^N$.

**Step 3:** For every identification model $M_k$, construct a controller $C_k^i$ based on a feedback linearisation technique so that when the control signal $u_k$ is applied in model $M_k$, the dynamic equation of the specified reference model (3) is obtained.
Step 4: Define a cost criterion $J_k(t)$ of the form (5), a hysteresis $h$, a $T_{\text{min}}$ and a switching rule for the controllers.

Step 5: For $t=0$, choose arbitrarily the signal from one of the available controllers. For $t>0$, repeat Steps 6–8 at each time instant.

Step 6: Calculate all the identification errors $e_k$ and use the adaptive law (16) to update the parameters estimations for every T–S model.

Step 7: Utilising the updated parameters form Step 6, update the controllers $C_k$.

Step 8: Apply the appropriate controller $C_k$ based on criterion (5) and NBCL, to the plant and to all the T–S identification models.

Remark 2: The TSMMASC architecture that has been presented in this article can be applied efficiently on fuzzy dynamical systems of the form (1), as it was shown in Theorem 5.1. These fuzzy systems are assumed to be free of approximation errors, that is, the nonlinear systems are assumed to be modelled exactly by these T–S dynamical models. Notice that this is the case in many papers (Park and Cho 2004; Park and Park 2004; Qi and Brdys 2008; etc.). However, in most cases the approximation errors are inevitable. It has been shown for the linear systems case that these approximation errors may lead to closed-loop instabilities (Ioannou and Sun 1996). See also the relevant discussion in Kosmatopoulos (2010). Some of these methods have been adopted in fuzzy control schemes: Feng et al. (2002) used a robustifying term in the control signal to cope with unmodelled dynamics and bounded disturbances, Zhou et al. (2005) used $\sigma$-modification along with backstepping design and Labiod and Guerra (2007) also used $\sigma$-modification in a direct adaptive fuzzy control scheme. Due to the use of these techniques, the tracking errors converge to a small bounded region around zero.

Remark 3: This article deals with time-invariant systems and provides a stability analysis for that class of systems. Notice that this is the case in most papers which are dealing with adaptive control systems (Narendra and Balakrishnan 1997; Feng et al. 2002; Park and Cho 2004; Park and Park 2004; Qi and Brdys 2008; etc.). However, in all these papers (including this one), some of the simulation examples include plants which parameters are changing linearly or discontinuously with time. The reason for this fact is that adaptive control techniques are developed and are applied mainly to unknown systems that are subjected to unpredictable parameters variations due to faults, etc. It is known (Ioannou and Sun 1996) that if the unknown parameters are not constant all the time (even if they vary sufficiently slowly with time), the adaptive laws that have been developed especially for the time-invariant systems may become unstable when applied to time-variant systems. Consequently, the stability in the above methods for the case of time-variant unknown parameters has been shown via simulations rather than via analysis. The time-varying parameters case is a complicated problem due to the nonlinear nature of the controlled plants.

Remark 4: Equation (13) could also be formulated as

$$A_k^TP + PA_d = -Q, \quad Q > 0$$

where $Q$ is common to all subsystems. In this case, the stability results of this section remain valid. However, Equation (13) offers the flexibility to the algorithm scheme to choose a unique $Q_k$ matrix and thus a unique $P_k$ matrix for every identification model. $P_k$ matrices are used as design parameters in Equation (16) in order to provide stability to the system. They also serve as adaptation gains in the weight adaption laws (16). The value of $Q_k$ could change according to the value of the cost criterion $J_k$. For example, the systems with the higher values for $J_k$ could use a more positive definite matrix $Q_k$ in (13) than the other systems. This could lead to a more positive definite $P_k$, and finally a more negative $V_k$ (see (18)) along with a faster convergence rate.

5.2. Convergence and computational cost of the switching algorithm

There are three main tools which are used in this article in order to formulate the switching algorithm. These are the cost criterion (5) which embodies past measures of the squared identification errors, the constant hysteresis $h$ and a $T_{\text{min}}$ between the successive switchings. Under the stability results of Theorem 5.1 for the adaptive controllers and identification models, it has been proved (Ye 2008) that by using these three tools, only a finite number of switchings will take place during the control process and finally the plant will be controlled by a unique dominant controller.

The computational cost of the proposed algorithm is another crucial factor for the success of control process. It is obvious that the computational burden
depends on the number of T–S adaptive identification models. Due to the fact that in this article we distribute uniformly the identification models in a compact space \( \mathcal{S} \), the larger is this space, the larger will be the number of the models. Theoretically, this number can be arbitrarily large, but in real problems there must be a compromise between the performance and the computational requirements of the controller. Thus, the number of models must be kept as small as possible.

The unknown parameters \( \mathcal{E}_{AB} \) lie in the three-dimensional compact space \( \mathcal{S} \):

\[
\mathcal{S} = \{0.2 \leq l \leq 1, \ 0.5 \leq m \leq 4, \ 4 \leq M \leq 12\}
\]

The control objective is to force the system (26) and its T–S model to follow the reference model (3) where

\[
A_d = \begin{bmatrix} 0 & 1 \\ -4 & -4 \end{bmatrix}
\]

From the Lyapunov Equation (13) and for \( Q_k = I_{2 \times 2} \), one obtains

\[
P_k = \begin{bmatrix} 0.1250 & 0.1250 \\ 0.1250 & 0.1562 \end{bmatrix}, \quad P_{ks} = \begin{bmatrix} 0.1250 & 0.1562 \end{bmatrix}^T.
\]

The plant’s parameters are time-varying and are given in Table 1.

6. Simulation studies

To illustrate the control scheme discussed in the preceding sections and demonstrate the advantages of the TSMMAASC technique over a single adaptive controller, 12 simulations were conducted on two nonlinear, unknown, rapidly time-varying mechanical systems. Simulations took place for both original plants and T–S plant models. The conclusions about the efficiency of the proposed control scheme are the same for both cases since the simulations showed that there is not much difference in the state trajectories of the original plants and the T–S plants.

6.1. Problem 1

Let the inverted pendulum system (26) which is a highly unstable system and is widely used as a benchmark control problem (Wang et al. 1996; Park and Cho 2004) be defined as:

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= \frac{g \sin(x_1) - aml \sin(2x_1)/3 - a \cos(x_1)u}{4l/3 - aml \cos^2(x_1)}
\end{align*}
\] (26)

where \( x_1 \) denotes the angle (in radians) of the pendulum from the vertical, \( x_2 \) is the angular velocity, \( g = 9.8 \text{ m/s}^2 \) is the gravity constant, \( m \) is the mass of the pendulum, \( M \) is the mass of the cart, \( 2l \) is the length of the pole, \( u \) is the force applied to the cart and \( a = 1/(m + M) \).

The nonlinear system (26) can be approximated using the following two fuzzy rules when \( x_1 \in (-\pi/2, \pi/2) \):

Rule 1: IF \( x_1(t) \) is about 0 THEN \( \dot{x}(t) = A_1 x(t) + B_1 u(t) \)

Rule 2: IF \( x_1(t) \) is about \( \pm \pi/2 \) THEN \( \dot{x}(t) = A_2 x(t) + B_2 u(t) \) where

\[
A_1 = \begin{bmatrix} 0 & 1 \\ a_2 & a_1 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 \\ a \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 1 \\ a_2 & a_1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ a \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 \\ -a \end{bmatrix}
\]

and \( \beta = \cos(88^\circ) \). The membership functions are depicted in Figure 2.

Figure 2. The membership functions of the fuzzy model.
The following three simulation experiments for both original and T–S model plant took place:

1. Use of a single identification adaptive T–S model $M_1$ with one adaptive fuzzy controller $C_1$.
2. Use of three identification adaptive T–S models $\{M_k\}_{k=1}^3$ with three adaptive fuzzy controllers $\{C_k\}_{k=1}^3$.
3. Use of 10 identification adaptive T–S models $\{M_k\}_{k=1}^{10}$ with 10 adaptive fuzzy controllers $\{C_k\}_{k=1}^{10}$.

The initialisation for the single adaptive identification model is given as follows:

$$(\hat{a}^{11})^T = \begin{bmatrix} 14 & 0.02 \end{bmatrix}, \quad (\hat{a}^{12})^T = \begin{bmatrix} 7.8 & 0.05 \end{bmatrix},$$
$$\hat{b}^{11} = -0.099, \quad \hat{b}^{12} = -0.003.$$

Table 1. Plant parameters values.

<table>
<thead>
<tr>
<th>Time $t$ (s)</th>
<th>$l$ (met)</th>
<th>$m$ (kg)</th>
<th>$M$ (kg)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t \in [0, 0.2)$, $t \in (1, 1.2)$</td>
<td>0.5</td>
<td>2</td>
<td>12</td>
</tr>
<tr>
<td>$t \in [0.2, 0.4)$, $t \in [1.2, 1.4)$</td>
<td>0.2</td>
<td>3</td>
<td>7</td>
</tr>
<tr>
<td>$t \in [0.4, 0.6)$, $t \in [1.4, 1.6)$</td>
<td>0.8</td>
<td>0.5</td>
<td>9</td>
</tr>
<tr>
<td>$t \in [0.6, 0.8)$, $t \in [1.6, 1.8)$</td>
<td>1</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>$t \in [0.8, 1)$, $t \in [1.8, 2)$</td>
<td>0.3</td>
<td>1.5</td>
<td>8</td>
</tr>
<tr>
<td>$t \geq 2$</td>
<td>0.7</td>
<td>3.2</td>
<td>6</td>
</tr>
</tbody>
</table>

Note that the initialisation of the single identification model of case 1 is included in the three models of case 2, and the initialisations of the three models of case 2 are included in case 3, i.e. $M_1 \subset \{M_k\}_{k=1}^3 \subset \{M_k\}_{k=1}^{10}$. The initial estimates $\hat{a}^{ki}, \hat{b}^{ki}$ of all the unknown parameters are distributed uniformly in the space $\Sigma$, and they are adjusted on line by the adaptive law (16) where $r_1 = 1$ and $r_2 = 0.2 \forall k, i$. The values $a_c = 8, b_c = 1$ are used for the cost criterion (5), $T_{min} = 0.05$ s and $h = 0.01$. Also the lower bounds for the $b^i$s are $b^0_0 = 0.015$ and $b^0_1 = 0.0015$. The initial states for the real plant, the T–S plant model, the estimation models and the reference model are $x(0) = \dot{x}(0) = \ddot{x}(0) = [\pi/3, 0]^T, \forall k$.

The results from the experiments 1–3, are depicted in Figures 3–6. In Figures 3(a–b), 4(a–b) and 5(a–b), the states of the real plant (26) and the reference model (3) are given. In Figure 6(a–f) the states of the T–S model plant for all the experiments are given. The dashed line in all these figures depicts the reference model’s state. In Figures 3(c), 4(c) and 5(c), the respective control signals are depicted. In Figures 3(d), 4(d) and 5(d), the switching sequence of the controllers is given. Finally in Figures 4(e–f) and 5(e–f) zoom outs of crucial parts of the previous figures are given. Due to the fact that the initial $J_k(t)$ is identical for all the models $M_k$, the same controller $C_1$
is used to provide the first control signal to the system in all cases, i.e. the initial control signal is the same for the three cases. In case 1, where a single identification T–S fuzzy adaptive model is used, the controller signal leads the system to instability. The single controller is inadequate to cope with such a difficult problem due to the large region of the parameters uncertainty and the rapidly time-varying parameters. The states in Figure 3(a–b) are displayed for less than 1 s due to the fact that state $x_1$ exceeds its limits and the system becomes unstable. In case 2, where three T–S fuzzy adaptive identification models are used, the system is stabilised and follows the state of the reference model after about 9 s. It can be noticed (Figure 4d) that all the three controllers are used during the simulation and finally the controller $C_2$ is the dominant controller of the system. The control signal is oscillating more intensively for the first second due to the controllers’ switching. Also, Figure 4(a) shows that the inverted pendulum oscillates around the vertical position and finally is stabilised. In case 3, where the number of the multiple models is increased to 10, there is a significant improvement in the system’s performance. More precisely, the states $x_1$, $x_2$ are driven to the equilibrium point at about 2.5 s with lower peaks, and the energy that is required for the control signal is less than that of case 2 (Figure 5c). Also the precision is very high since the tracking error is almost zero (of the order $10^{-7}$).
before the tenth second. It has to be noted that due to the high unstable nature of the inverted pendulum, chattering effect problems may come up. In order to avoid this situation we use the cost criterion (5) along with $T_{\text{min}}$ and hysteresis $h$, tools that provide a more smooth control signal. There is a switch to the controller $C_9$ at about 1.6 s which causes a very small burst in the control signal. The switchings are taking place in the first 1.7 s where the plant is time-varying and not fully identified. Five ($C_1, C_2, C_3, C_6, C_9$) of the 10 candidate controllers are used during the control process and finally the controller $C_3$ is the dominant controller of the system (Figure 5d). The same conclusions can be drawn for the case where the T–S model of the plant is controlled (Figure 6). The simulation results demonstrate the superiority of the proposed method over the single adaptive model case (Park and Cho 2004; Park and Park 2004). In addition, there is a significant improvement in performance when increasing the number of the adaptive models but there is also an increase in the computational cost. More specifically using a standard Laptop and a not optimised code written in Matlab® 2010 software, the computational cost for the case of three and 10 models is 3.10 s and 12.32 s respectively while the real-time is equal to 10 s.
6.2. Problem II

Let the mass-spring mechanical system be described by the following differential equation (Lian and Liou 2006):

\[ m\ddot{x} + F_f + F_s = u \]  
\[(27)\]

where \( F_f = c\dot{x} \) with \( c > 0 \), denotes the viscous damping force, \( F_s = k(1 + a^2 x^2)x \) with constants \( k, a \), is the spring’s force, \( m \) is the mass, \( u \) is the control signal and \( x \) is the mass displacement from a reference point. Equation (27) takes the following form:

\[ \dot{x} = (-c/m)x - (k/m)(1 + a^2 x^2)x + (1/m)u \]  
\[(28)\]

Equation (28) can be expressed as (29) by taking two first-order differential equations, using the definitions \( x_1 = x \) and \( x_2 = \dot{x} \).

\[ \begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= (-c/m)x_2 - (k/m)(1 + a^2 x_1^2)x_1 + (1/m)u
\end{align*} \]  
\[(29)\]

Assuming that \( x_1 \in (-d, d) \), the nonlinear term \( x_1^3 \) can be expressed as

\[ x_1^3 = M_1(x_1)0x_1 + M_2(x_1)d^2 x_1 \]  
\[(30)\]

where \( M_1(x_1) = (d^2 - x_1^2)/d^2 \) and \( M_2(x_1) = 1 - M_1(x_1) \) are the membership functions and \( d = 2 \). The nonlinear system (29) can be approximated using the following two fuzzy rules:

Rule 1: IF \( x_1(t) \) is \( M_1 \) THEN \( \dot{x}(t) = A_1 x(t) + B_1 u(t) \)

Rule 2: IF \( x_1(t) \) is \( M_2 \) THEN \( \dot{x}(t) = A_2 x(t) + B_2 u(t) \)

Figure 6. State response of T–S plant model using (a)–(b) one; (c)–(d) three and (e)–(f) 10 adaptive identification models.
where
\[
\begin{align*}
A_1 &= \begin{bmatrix} 0 & 1 \\ a_1^1 & a_1^2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{k}{m} & -\frac{c}{m} \end{bmatrix} \\
B_1 &= \begin{bmatrix} 0 & 1 \\ b^1 \\ b^2 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} \\
A_2 &= \begin{bmatrix} 0 & 1 \\ a_2^1 & a_2^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ -\frac{k}{m(1 + a_2^1 d_2)} & -\frac{c}{m} \end{bmatrix} \\
B_2 &= \begin{bmatrix} 0 & 1 \\ b^2 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix}
\end{align*}
\]

and \( a_2^1 = 0.95 \). The unknown parameters \( E_{AB} \) lie in the three-dimensional compact space \( \Xi \):
\[
\Xi = \left\{ \frac{0.375 \leq k/m \leq 9.5, \; 0.0555 \leq c/m \leq 3.5, \; 0.555 \leq 1/m \leq 5} \right\}
\]

The control objective is to force the system (29) and its T–S model to follow the reference model (3) where
\[
A_d = \begin{bmatrix} 0 & 1 \\ -5 & -5 \end{bmatrix}
\]

From the Lyapunov Equation (13) and for \( Q_k = I_{2 	imes 2} \), one obtains
\[
\begin{align*}
P_k &= \begin{bmatrix} 0.10 & 0.10 \\ 0.10 & 0.12 \end{bmatrix}, & P_{ks} &= \begin{bmatrix} 0.10 & 0.12 \end{bmatrix}^T
\end{align*}
\]

The plant’s parameters are time-varying and are given in Table 2.

Similar to Problem I, three simulations were conducted using one \( (M_1) \) three \( \{(M_k)_k=1\} \) and 10 \( \{(M_k)_k=1\} \) identification T–S models with their corresponding feedback linearisation controllers \( C_1, \{C_k\}_{k=1}^3 \) and \( \{C_k\}_{k=1}^{10} \), respectively. The initialisation for the single adaptive identification model is given as follows:
\[
\begin{align*}
(a_1^1)^T &= [-5 \; -2], & (a_1^2)^T &= [-24 \; -2], \\
b_1^1 &= 2.4, & b_1^2 &= 2.4
\end{align*}
\]

Figure 7. (a)–(b) State response; (c) control signal and (d) active controller using a single adaptive identification model.

Table 2. Plant parameters values.

<table>
<thead>
<tr>
<th>Time ( t ) (s)</th>
<th>( k/m )</th>
<th>( c/m )</th>
<th>( 1/m )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( [0, 0.25) )</td>
<td>0.4</td>
<td>1.5</td>
<td>4.95</td>
</tr>
<tr>
<td>( [0.25, 0.5) )</td>
<td>0.9</td>
<td>3</td>
<td>1.2</td>
</tr>
<tr>
<td>( [0.5, 0.75) )</td>
<td>5</td>
<td>0.4</td>
<td>1</td>
</tr>
<tr>
<td>( [0.75, 1) )</td>
<td>1.8</td>
<td>0.15</td>
<td>1.8</td>
</tr>
<tr>
<td>( [1, 1.25) )</td>
<td>2.8</td>
<td>1.7</td>
<td>3.9</td>
</tr>
<tr>
<td>( [1.25, 1.5) )</td>
<td>8.8</td>
<td>1.4</td>
<td>2.25</td>
</tr>
<tr>
<td>( [1.5, 1.75) )</td>
<td>1.1</td>
<td>0.1</td>
<td>3.1</td>
</tr>
<tr>
<td>( [1.75, 2) )</td>
<td>9</td>
<td>1.4</td>
<td>4.2</td>
</tr>
<tr>
<td>( [2, 2.25) )</td>
<td>0.6</td>
<td>2.7</td>
<td>1.3</td>
</tr>
<tr>
<td>( [2.25, 2.5) )</td>
<td>0.7</td>
<td>3.5</td>
<td>0.56</td>
</tr>
<tr>
<td>( t \geq 2.5 )</td>
<td>0.9</td>
<td>0.9</td>
<td>4.7</td>
</tr>
</tbody>
</table>
As in Problem I, it holds that \( M_1 \subset \{ M_k \}_{k=1}^{10} \subset \{ M_k \}_{k=1}^{10} \). Also the simulation parameter values are given as follows: \( r_i = 6 \) and \( r_{2i} = 1 \) \( \forall k, i, a_i = 8, b_i = 1, \ T_{min} = 0.01 \) s, \( h = 0.01, \ b_0 = 0.15 \) and \( b_0 = 0.15. \) The initial states for the real plant, the T–S plant model, the estimation models and the reference model are \( x(0) = x_{m}(0) = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix}^T, \forall k. \)

The results from these three experiments are depicted in Figures 7–9. In Figures 7(a–b), 8(a–b) and 9(a–b), the states of the real plant (29) and the reference model (3) are given. The dashed line in all these figures depicts the reference model’s state.

In Figures 7(c), 8(c) and 9(c), the respective control signals are depicted. Finally, in Figures 7(d), 8(d) and 9(d), the switching sequence of the controllers is given while in Figures 8(e–f) and 9(e–f) zoom outs of crucial parts of the previous figures are given. The initial control signal is the same for the three cases and is provided by \( C_1. \) When a single identification T–S adaptive model is used, the unknown plant needs more than 15 s to follow the reference model. The single controller has a poor performance due to the large region of the parameters uncertainty and the rapidly time-varying parameters. In case where three T–S

Figure 8. (a)–(b) State response; (c) control signal; (d) controllers switching sequence and (e)–(f) zoom out of controllers switching sequence on the time interval [0,1.5] using three adaptive identification models.
adaptive identification models are used, the system follows the state of the reference model after about 8 s and its performance is improved compared with that of the single controller. It can be noticed (Figure 8(d–f)) that all the three controllers are used during the simulation and finally the $C_1$ controller is the dominant controller of the system. When the number of the multiple models is increased to 10, there is an additional improvement in the system’s performance compared with the three controllers scheme. More precisely, the states $x_1$, $x_2$ are close enough to the reference model’s states and approximate them in less than 4 s. The cost criterion (5) provides a smooth control signal and there is no chattering effect. Five ($C_1$, $C_2$, $C_3$, $C_4$, $C_{10}$) of the ten candidate controllers are used during the control process and finally the controller $C_2$ is the dominant controller of the system (Figure 9(d–f)). The results for the T–S model plant case are almost the same since the approximation error for this example is very small. Consequently, there is a significant improvement in the performance when increasing the number of the adaptive models and conclusions of the Problem I are valid for Problem II as well.

7. Conclusions
A new indirect adaptive switching fuzzy control method based on T–S multiple models has been proposed in this article in order to efficiently control a class of unknown nonlinear dynamical fuzzy systems. The unknown T–S plant is estimated by using several adaptive identification models: Figure 9. (a)–(b) State response; (c) control signal; (d) controllers switching sequence and (e)–(f) zoom out of controllers switching sequence on the time interval [0,1] using 10 adaptive identification models.
T–S adaptive models instead of using only a single adaptive fuzzy model. This strategy increases the possibility to obtain more accurate estimations for the plant parameters at every time instant. The control objective is fulfilled by using one feedback linearisation fuzzy adaptive controller for each T–S model. An appropriate switching rule picks the best available controller at every time instant while at the same time the chattering effect is alleviated by allowing a small time interval to elapse after every switching along with a constant hysteresis. The NBCL strategy ensures the feasibility of the control signal at any time instant. Adaptation of the parameters is derived using the Lyapunov theory, ensuring the asymptotic stability of the system and the asymptotic tracking of a stable reference model. Simulation experiments which conducted using two rapidly time-varying and highly uncertain nonlinear systems, demonstrated the superiority of the proposed approach in comparison to the single adaptive fuzzy model case. Also, by increasing the number of the identification models, a significant improvement in the performance has been noticed. As mentioned in Remark 3, the stability of the time-varying plants, which were used in the simulations for comparison reasons, has been shown via simulations rather than via analysis. Future work includes the extension of the proposed method to a more general class of nonlinear systems along with possible robustifications in their adaptive control scheme.

Notes on contributors

Nikolaos A. Sofianos received his Dip-Eng. and MSc degrees in Electrical and Computer Engineering from Democritus University of Thrace (DUTH) in 2004 and 2009, respectively. He is currently a PhD candidate at DUTH. He has served as a Teaching Assistant at DUTH and at Technological Educational Institution of Kavala in the fields of Automatic Control Systems and Intelligent Systems, respectively, from 2005 to 2010. His current research interests include intelligent adaptive control, fuzzy control, multiple models, nonlinear systems and robust control.

Yiannis S. Boutalis received his Diploma of Electrical Engineer in 1983 from Democritus University of Thrace (DUTH), Greece, and PhD degree in Electrical and Computer Engineering (topic Image Processing) in 1988 from the Computer Science Division of National Technical University of Athens, Greece. Since 1996, he serves as a Faculty Member at the Department of Electrical and Computer Engineering, DUTH, Greece, where he is currently an Associate Professor and Director of the Automatic Control Systems lab. Currently, he is also a Visiting Professor for research cooperation at Friedrich-Alexander University of Erlangen-Nuremberg, Germany, Chair of Automatic Control. He served as an Assistant Visiting Professor at University of Thessaly, Greece, and as a Visiting Professor in Air Defence Academy of General Staff of Airforces of Greece. He also served as a Researcher in the Institute of Language and Speech Processing (ILSP), Greece, and as a Managing Director of the R&D SME Ideatech S.A, Greece, specialising in pattern recognition and signal processing applications. His current research interests are focused on the development of computational intelligence techniques with applications in control, pattern recognition, signal and image processing problems.

References


