Neumann Boundary Value Problems with Discontinuities

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Abstract—In this letter, we study Neumann problems with nonlinear boundary conditions. We do not assume that the right-hand side or the nonlinearity of the boundary condition are Carathéodory functions. We use the subdifferential for locally Lipschitz functionals due to Clarke [1]. © 2003 Elsevier Science Ltd. All rights reserved.

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1. INTRODUCTION

Our goal here is to have an existence result for the following Neumann problem:

\[-\Delta_p u = f(u), \quad \text{a.e. on } \Omega,\]

\[\frac{\partial u}{\partial n_p} = g(u), \quad \text{a.e. on } \partial \Omega, \quad 2 \leq p < \infty.\]

We suppose that \(\Omega \subseteq \mathbb{R}^N\) is a bounded domain with a \(C^1\)-boundary \(\partial \Omega\). By \(\Delta_p u\), we denote the \(p\)-Laplacian operator, namely, \(\Delta_p = \text{div}(\|Du(z)\|^{p-2}Du(z))\).

In our problem, the functions \(f, g\) are not continuous. So we cannot use the usual critical point theory, because the energy functional is not \(C^1\). But, for \(p = 2\), Ambrosetti and Badiale [2] had used an interesting technique, so that the energy functional becomes \(C^1\). Therefore, they used the classical critical point theory to obtain their existence result.

We do not use the method of upper and lower solutions. Heikkila and Laksmikantham [3] had used this method to obtain several existence theorems for all types of differential equations. But in the literature, there is not any result, without the \textit{a priori} existence of upper and lower solutions, for Neumann problems. As far as we know, this is the first result that concerns Neumann problems without the existence of upper and lower solutions.

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2. PRELIMINARIES

Let us recall some basic results about the subdifferential of Clarke [1]. A function \( f: Y \to \mathbb{R} \) is said to satisfy a Lipschitz condition (on \( Y \)) provided that, for some nonnegative scalar \( K \), one has
\[
|f(y) - f(x)| \leq K\|y - x\|
\]
for all points \( x, y \in Y \). Let \( f \) be Lipschitz near a given point \( x \), and let \( v \) be any other vector in \( X \). The generalized directional derivative of \( f \) at \( x \) in the direction \( v \), denoted by \( f^\circ(x; v) \), is defined as follows:
\[
f^\circ(x; v) = \limsup_{t \to 0} \frac{f(y + tv) - f(y)}{t},
\]
where \( y \) is a vector in \( X \) and \( t \) a positive scalar. If \( f \) is Lipschitz of rank \( K \) near \( x \), then the function \( \phi: X \to f^\circ(\mathbf{v}; x) \) is finite, positively homogeneous, subadditive, and satisfies \( |f^\circ(x; v)| \leq K\|v\| \). In addition, \( f^\circ \) satisfies \( f^\circ(x; -v) = (-f)^\circ(x; v) \). Now we are ready to introduce the generalized gradient which is denoted by \( \partial_C f(x) \) as follows:
\[
\partial_C f(x) = \{ \mathbf{v} \in X^*: f^\circ(x; v) \geq \langle \mathbf{w}, v \rangle, \text{ for all } \mathbf{w} \in X \}.
\]

3. EXISTENCE THEORY

Let us denote by \( X = W^{1,p}(\Omega) \). We do not assume that \( f, g \) is continuous, so we can introduce the following functions:
\[
f_1(u) = \liminf_{u_\ast \to u} f(u'), \quad f_2(u) = \limsup_{u_\ast \to u} f(u').
\]
The same holds for \( g \).

Our hypotheses for \( f, g \) are the following.

**H(f):** \( f: \mathbb{R} \to \mathbb{R} \) is a function such that
\[
\begin{align*}
(i) & \text{ is } N\text{-measurable (i.e., for every } u: \Omega \to \mathbb{R} \text{ measurable, } x \to f_{1,2}(u(x)) \text{ is measurable too);} \\
(ii) & \text{ there exists } h: \mathbb{R} \to \mathbb{R} \text{ with } |h(u)| \leq c_1 + c_2|u|^p \text{ for all } u \in \mathbb{R}, \text{ almost all } x \in \Omega \text{ and for some } c_1, c_2 > 0, \text{ such that } h(u) \to \infty \text{ as } u \to \pm \infty \text{ and there exists } M > 0 \text{ such that } -F(u) \geq h(|u|) \text{ for } |u| \geq M \text{ with } F(u) = \int_0^u f(r) dr; \\
(iii) & \text{ for all } u \in \mathbb{R}, |f(u)| \leq c_1 + c_2|u|^{p-1}, \mu < p, c_1, c_2 > 0, (1/\mu + 1/\mu' = 1), \text{ and } u \to f(u) \text{ is nondecreasing.}
\end{align*}
\]

**H(g):** \( g: \mathbb{R} \to \mathbb{R} \) is a function such that
\[
\begin{align*}
(i) & \text{ is } N\text{-measurable (i.e., for every } u: \Omega \to \mathbb{R} \text{ measurable, } x \to g_{1,2}(u(x)) \text{ is measurable too);} \\
(ii) & \text{ there exists some } k_1, k_2, k_3, k_4 \in \mathbb{R} \text{ such that } -k_4 \leq g(u) \leq k_1 + k_2|u|^p \text{ with } p^* = Np/(N - p) \text{ and } 1 < p. \text{ Finally, } u \to g(u) \text{ is nonincreasing.}
\end{align*}
\]

**Theorem 1.** If Hypotheses **H(f), H(g)** hold, then problem (1) has a solution in \( u \in X \).

**Proof.** Let \( \Phi(u) = -\int_\Omega F(u(x)) dx, \psi_1 = (1/p) \|Du\|_p^p, \psi_2 = \int_{\partial\Omega} G(u(x)) d\sigma, \text{ with } F(u) = \int_0^u f(r) dr, G(u) = \int_0^u g(r) dr.\)

Our energy functional is \( R = \Phi + \psi_1 + \psi_2 \).

First, we are going to prove that \( R \) is weakly lower semicontinuous. Note that \( \psi_1 \) is convex and lower semicontinuous, so is weakly lower semicontinuous. Now let \( u_n \to u \text{ weakly in } X. \text{ Then it is well known that } u_n \to u \text{ strongly in } L^p(\Omega) \text{ and in } L^p(\partial\Omega). \text{ Now it is clear that } \Phi, \psi_2 \text{ is weakly lower semicontinuous too. So } R \text{ is weakly lower semicontinuous.}

Now, we claim that \( R \) is coercive. Indeed, suppose not. Then there is \( M > 0 \) and a sequence \( u_n \in X \) with \( \|u_n\| \to \infty \) such that \( |R(u_n)| \leq M \) for all \( n \in \mathbb{N}. \text{ That means,}
\[
\Psi(u_n) + \psi_1(u_n) + \psi_2(u_n) \leq M.
\]
Set $y_n = u_n/\|u\|^p$.

Now divide inequality (2) with $\|u_n\|^p$ and we have

$$
- \int_{\Omega} \frac{F(u_n)}{\|u_n\|^p} \, dx + \frac{1}{p} \|Dy_n\|^p_p - k_4 \int_{\partial \Omega} \frac{|u_n|^l}{\|u_n\|^p} \, d\sigma \leq \frac{M}{\|u_n\|^p}.
$$

Here we have used $H(g)(ii)$. Note that $\|u_n\|^l_{L^p(\partial \Omega)} \leq K \|u_n\|^l_{L^p(\partial \Omega)} \leq K \|u\|^l_{L^p(\partial \Omega)}$ (see [4, p. 217]). Now using Hypothesis $H(f)(iii)$, the fact that $X$ embeds continuously in $L^p(\Omega)$ and that $l < p$, we have from (2) that $\lim \inf \|Dy_n\|^p_p \leq \lim \sup \|Dy_n\|^p_p \to 0$. But, $y_n \to y$ weakly in $X$, so from the weak lower semicontinuity of the norm we have $\|Dy\|^p_p \leq \lim \inf \|Dy_n\|^p_p \to 0$. From that we obtain that $y = c \in \mathbb{R}$. Recall that, $y_n \to y$ weakly in $X$ and $\|Dy_n\|^p_p \to \|Dy\|^p_p = 0$, so $y_n \to c$ strongly in $X$. Since $\|\|y\|| = 1$ we have that $c \neq 0$. So, we have that $\|y_n\| \to \infty$.

Now using $H(f)(ii)$ we have that $\Phi(u_n) \to \infty$. Thus, we have a contradiction, since we suppose inequality (2). Therefore, there exists some $u \in X$ such that $0 \leq R(y) - R(u)$ for all $y \in X$ (see [5, p. 154, Prop. 38.12(d)]).

Therefore, we can say that

$$(\Phi)(y) - (\Phi)(u) \leq \psi_1(y) - \psi_1(u) + \psi_2(y) - \psi_2(u), \quad \text{for all } y \in X.$$

Choose now $y = u + tv$, divide with $t > 0$ and take the limit as $t \to 0$. Recall that $(-\Phi)$ is convex (since $f$ is nondecreasing) and that $\psi_1 + \psi_2$ are locally Lipschitz. So, we obtain that

$$(-\Phi)'(u, v) \leq \psi_1'(u, v) + \psi_2'(u, v).$$

Thus, all $w \in \partial (-\Phi)(x)$ belongs to $\partial C(\psi_1 + \psi_2)$ (by $\partial C$ we denote the subdifferential for locally Lipschitz functionals due to [1]).

Therefore, we can say that

$$
\int_{\Omega} w(x)y(x) \, dx = \int_{\Omega} \|Du(x)\|^p_{p-2}(Du(x), dy(x)) \, dx + \int_{\partial \Omega} h(x)y(x) \, d\sigma,
$$

with $w(x) \in [f_1(u(x)), f_2(u(x))]$ and for some $h(x) \in \partial C G(u(x))$.

We will show that $\lambda \{x \in \Omega : u(x) \in D(f)\} = 0$ with $D(f) = \{u \in \mathbb{R} : f(u^+) > f(u^-)\}$, that is, the set of upward discontinuities.

So let $w \in \partial (-\Phi)(u)$ and for any $t \in D(f)$, set

$$\rho_+(x) = |1 - \chi_t(u(x))|w(x) + \chi_t(u(x)) \left[ f(u(x)^+) \right],$$

where

$$\chi_t(s) = \begin{cases} 
1, & \text{if } s = t, \\
0, & \text{otherwise.}
\end{cases}
$$

Then $\rho_+ \in L^p(\Omega)$ and $\rho_- \in \partial (-\Phi)(x)$. So

$$
\int_{\Omega} \rho_+(x)y(x) \, dx = \int_{\Omega} \|Du(x)\|^p_{p-2}(Du(x), Du(x))_{\mathbb{R}^n} \, dx + \int_{\partial \Omega} h(x)y(x) \, d\sigma,
$$

for all $y \in X$.

So for $y = \phi \in C_0^\infty(\Omega)$ we have

$$
\int_{\Omega} \rho_+(x)\phi(x) \, dx = \int_{\Omega} \|Du(x)\|^p_{p-2}(Du(x), D\phi(x))_{\mathbb{R}^n} \, dx.
$$

Thus, $\rho_+ = \rho_-$ for almost all $x \in \Omega$. From this it follows that $\chi_t(u(x)) = 0$ for almost all $x \in \Omega$. Since $D(f)$ is countable, and

$$\chi(u(x)) = \sum_{t \in D(f)} \chi_t(u(x)),$$

it follows that $\chi(u(x)) = 0$ almost everywhere (with $\chi(t) = 1$ if $t \in D(f)$ and $\chi(t) = 0$ otherwise).
Therefore, $\Phi(u)$ is strictly differentiable at $u$. Thus,
\[ \int_{\Omega} f(u(x))y(x) \, dx = \int_{\Omega} \|Du(x)\|^{p-2}(Du(x), Du(x))_{R^n} \, dx, \tag{5} \]
for all $y \in C^\infty(\Omega)$.

To finish the proof, we return to the inequality $0 \leq R(y) - R(u)$ for all $y \in X$. Now, we can say that
\[ (-\psi_2)(y) - (-\psi_2)(u) \leq \Phi(y) - \Phi(u) + \psi_1(y) - \psi_1(u), \quad \text{for all } y \in X. \]

Note that $(-\psi_2)$ is convex and that $\Phi, \psi_1$ are locally Lipshitz functionals. So, all $h \in \partial \psi_2$ belongs to $\partial_C(\Phi + \psi_1)$.

As before we will show that $\lambda_0\Omega\{x \in \partial \Omega : u(x) \in D(f)\} = 0$ with $D(y) = \{u \in \mathbb{R} : y(u^+) > g(u^-)\}$, where by $\lambda_0\Omega$ we denote the surface measure.

Setting now $\rho^\pm(x) = [1 - \chi_1(u(x))][h(x) + \chi_1(u(x))[g(u(x)^+)]]$ with $h(x) \in \partial G(u(x))$, we have that $\rho^\pm \in L^p(\partial \Omega)$ and
\[ \int_{\partial \Omega} \rho^\pm(x)y(x) \, d\sigma = \int_{\Omega} \|Du(x)\|^{p-2}(Du(x), Du(x))_{R^n} \, dx - \int_{\Omega} f(u(x))y(x) \, dx, \quad \text{for all } y \in X. \]
That means that $\rho^+ = \rho^- \ a.e., \text{ on } \partial \Omega$. As before we can show that $\chi(u(x)) = 0 \ a.e., \text{ on } \partial \Omega$. Therefore, $\psi_2$ is also strictly differentiable. Thus, we have
\[ \int_{\Omega} f(u(x))y(x) \, dx = \int_{\Omega} \|Du(x)\|^{p-2}(Du(x), Du(x))_{R^n} \, dx + \int_{\partial \Omega} g(u(x))y(x) \, d\sigma, \]
for all $y \in X$.

This completes the proof.

REFERENCES