Bernstein Polynomial Collocation Method for Elliptic Boundary Value Problems

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We present a summary of recent developments in application of Bernstein polynomials to solution of elliptic boundary value problems with a pseudospectral method. Solution is approximated using Bernstein polynomial interpolant defined at points of the extrema of Chebyshev polynomials i.e. the Chebyshev-Gauss-Lobatto (CGL) nodes. This approach brings improvement comparing to the Bernstein interpolation at equidistant nodes we used previously [1]. We show that this approach leads to spectral convergence and accuracy comparable to that of pseudospectral methods with orthogonal polynomials (Chebyshev, Legendre). The algorithm is implemented in open source library bernstein-poly, which is available online.

1 Introduction

Pseudospectral methods employ global polynomial interpolants for approximation of elliptic partial differential equations. Consider Bernstein polynomials defined over arbitrary interval $[a, b]$, 

$$B_{i,n}(x) = \binom{n}{i} \frac{(x-a)^i (b-x)^{n-i}}{(b-a)^n}, \quad i = 0, 1, \ldots, n.$$ 

(1)

For two-dimensional tensor product domains Bernstein interpolant takes the following form

$$u(x, y) = \sum_{i=0}^{n} \sum_{j=0}^{m} \beta_{i,j} B_{i,n}(x) B_{j,m}(y).$$ 

(2)

The expansion coefficients $\beta_{i,j}$ are obtained by collocation conditions at sufficient number of CGL nodes. The idea consists of replacing exact partial derivatives by derivatives of Bernstein interpolating polynomial. General order derivatives for Bernstein polynomials defined over arbitrary interval are defined using

$$D^{(p)} B_{i,n}(x) = \frac{n!}{(n-p)! (b-a)^p} \sum_{k=\max(0, i+p-n)}^{\min(i+p, n)} \binom{p}{k} B_{i-k,n-p}(x).$$ 

(3)

Numerical efficiency of the Python implementation is achieved by using vectorisation whenever possible. Therefore, similar to Chebyshev pseudospectral algorithms presented in [2, 3] we employ the concept of a pseudospectral differentiation matrix $D^{(p)}$. The elements $d_{i,j}$ of the differentiation matrix are generated by evaluating $j$th basis function derivative at $i$th collocation point.

2 Numerical Results

We present the example solutions to Poisson and Helmholtz equations on a square domain, with homogenous Dirichlet boundary conditions defined on the edges. Pseudospectral discretization of the Laplace operator using Bernstein polynomials is defined in matrix form as

$$L = B \otimes D^{(2)} + D^{(2)} \otimes B.$$ 

(4)

Fig.1 shows approximate solution (polynomial order, $N=20$) and the absolute error distribution for the Poisson problem with forcing function $f = 2\pi^2 \sin(\pi x) \sin(\pi y)$. Spectral convergence of the present global approximation method, and the elapsed time needed to assemble and solve linear system (Intel Core2Duo 2.33GHz, 4Gb RAM) is shown in Fig. 2a. However, the resulting system matrices have high condition numbers, shown in Fig. 2b (cf. for $N = 20$, cond$_{Cheb} = 3192.52$).

Helmholtz operator is cast in following matrix form

$$L = B \otimes D^{(2)} + D^{(2)} \otimes B + k^2 (B \otimes B).$$ 

(5)

Fig. 3 shows spectrally accurate solution of this problem for $f = \exp(-10((y-1)^2 + (x-1/2)^2))$ and $k = 6$. The values of $u(0, 0)$ for Chebyshev and Bernstein polynomial collocation methods match to ten digits, in the worst case (Table 1).

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Fig. 1: Poisson equation on a square. Numerical solution a and absolute error b.

Fig. 2: Poisson equation on a square. Convergence and elapsed time to assemble and solve linear system a and condition numbers for resulting matrices b.

Fig. 3: Numerical solution to Helmholtz BVP, with k=6.

Table 1: Approximate solution value u(0,0) for Chebyshev and Bernstein collocation.

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References