ABSTRACT†

In this work, we present the Hypercycles, a class of multidimensional graphs, which are generalizations of the n-cube. These graphs are obtained by allowing each dimension to incorporate more than two elements and a cyclic interconnection strategy. Hypercycles, offer simple routing, and the ability, given a fixed degree, to choose among a number of alternative size graphs. These graphs can be used in the design of interconnection networks for distributed systems tailored specifically to the topology of a particular application. We are also presenting a back-track-to-the-origin-and-retry routing, whereupon paths that block at intermediate nodes are abandoned, and a new attempt is made. Intermediate nodes are chosen at random at each point from among the ones that form the shortest paths from a source to a destination. Preliminary simulation results are also presented.

1.0 Introduction

Message passing concurrent computers such as the Hypercube[11, 16], Cosmic Cube[15], MAX[12, 13], consist of several processing nodes that interact via messages exchanged over communication channels linking these nodes into one functional entity. There are many ways of interconnecting the computational nodes, the Hypercube, Cosmic Cube, and the Connection Machine[17] having adopted a regular interconnection pattern corresponding to a binary n-dimensional cube, while MAX adopts a less structured, yet unspecified topology.

Several recent studies attempt extensions and generalizations of the basic tenets of the n-cube. Broder et. al. [4] have proposed product graphs[14] of small "basic" graphs. Their prime concern is to synthesize fault tolerant networks with a given degree of coverage. In these multidimensional graphs, they define a single route from a source to a destination, as the product of routes in each of the constituent dimensions. Routing is exhausted in each dimension before another dimension is considered. Bhunyan and Agrawal [3] have introduced the generalized hypercubes (GHC) which are also graph products of fully connected "basic" graphs. The mixed radix system [2] is used to express the properties of these graphs and their routing. Wittle [18] gives a good overview and comparison of several interconnection networks including the spanning bus and dual bus hypercubes. These are essentially binary n-cubes with broadcast busses connecting the processors in each dimension.

The advantages of having a regularly structured interconnection are many-fold, and they have been proven time and again in their being incorporated in many recent designs[6,11,12,13,15,16,17]. In these structures, easy deadlock-free routing [7] can be accomplished by locally computing each successive intermediate node - for a path that originates at a source node and terminates at a destination node- as a function of the current position and the desired destination. Many regular problems (such as the ones found in image processing, physics etc.) have been mapped on such regular structures, and run on the corresponding machines exhibiting significant speedups. In contrast, embedded real-time applications, such as those addressed by the MAX project [12, 13], tend to exhibit variable structures that do not necessarily map optimally to an n-cube. In addition, since the size of a binary n-cube is given as 2^n (n being the degree of the graph), it means that a particular configuration cannot be expanded but in predefined quantum steps. For example, if a given embedded application requires a system comprised of 9 nodes, the next larger n-cube with 16 nodes must be chosen. This constitutes a significant increase in resource allocation, especially in the light of the power-mass limited environment of a spacecraft.

Hypercycles[9] can be considered as products of "basic" graphs that allow, as compared to the Generalized Hypercubes (GHC) [3], a richer set of component "basic" graphs ranging in complexity from the simple rings to the fully connected ones used in the GHC. Also, contrary to Broder et. al.[4], we define the component graphs and provide analytical expressions for routing, our aim being twofold:

(a) To provide computer interconnection networks that match the node requirements of a given embedded system. Since our primary target is spacecraft applications which are weight and power limited, (the nodes in a spacecraft computer network are the primary weight contributor rather than the communication media) the exact matching of the node requirements is of paramount importance.

(b) To increase throughput of a given network by providing routing expressions that can be computed analytically (and hence are candidates for VLSI implementation) and which provide a maximum number of alternate paths from a source to a destination. The existence of alternate paths guarantees that a message will not be blocked waiting for its single route to be freed, but it would in turn search for the availability of alternate paths. This strategy also provides for fault protection, since a faulty path can be marked permanently busy, and thus messages can be routed around it. (Such an approach of adaptive routing is applied in the hypercube through the Hyperswitch [6]).

The Hypercycles, being regular graphs, retain the advantages of easy routing and regularity. Yet, since we are dealing with a class, rather than isolated graphs, we have the flexibility of adopting any particular graph (from the class) that closely matches the requirements of a given application. Since the graphs belong to the same class, routing is accomplished via the same methods and thus the same hardware can conceivably be used to configure structures with different sizes and topologies.

This work describes such a class of generalized interconnection networks, with routing strategies that are similar to that of the n-cube [4]. Yet, these networks offer richer topologies, and contain both the n-cube and the ring as special cases. While the n-cube is based on representation of nodes in base 2, we generalize using the mixed radix system representation. Such a representation, includes the binary (and hence the n-cube) as well as the arbitrary base b representation as special cases.

This work is divided into three parts. Section 2.0 introduces the Mixed Radix System. Section 3.0 presents some basic graph terminology and notation, while Section 4.0 introduces the Hypercycles and discusses their properties.

2.0 Mixed Radix Number System

The mixed radix representation [2], is a positional number representation, and it is a generalization of the standard b-base representation, in that it allows each position to follow its own base independently of the other.

Thus, given a decimal number M factored into r factors m_1,m_2,m_3,...,m_r as M = m_1 \times m_2 \times m_3 \times \cdots \times m_r, then any number 0 \leq X \leq M-1 can be represented as the following r-tuple

\[(X)_{m_1,m_2,...,m_r} = x_1x_2...x_r|_{m_1,m_2,...,m_r},\]

where 0 \leq x_i \leq (m_i - 1) ; i = 1,2,...,r and the x_i's are chosen in such a way so as \[X = \sum_{i=1}^{r} w_i x_i\] where \[w_i = \frac{M}{m_1 m_2 \cdots m_i} \]

We use the notation \[(X)_{m_1,m_2,...,m_r} = x_1x_2...x_r|_{m_1,m_2,...,m_r},\] to indicate the radices involved. Since for most cases we shall be dealing with a single set of radices, \(m_1,m_2,...,m_r\), we shall omit, when obvious, the
As an example, if we chose \( M = 10 = 2 \times 5 \), then any number between 0 and 9 can be represented with two digits, the first one ranging from 0 to 9. Specifically, \( N = \{ a_1, a_2, \ldots, a_9 \} \) is the set of edges defined as \( E = \{ e_{i,j} = (a_i, a_j) \mid i = 1,2,\ldots,N \} \).

\[
\begin{align*}
(\alpha_1, \alpha_2, \ldots, \alpha_r) &\equiv (\alpha_1 + \alpha_2, \ldots, \alpha_r + \alpha_r) \pmod{m_1, m_2, \ldots, m_r} \\
&\equiv (\alpha_1 + \alpha_2, \ldots, \alpha_r + \alpha_r) \\
&\equiv (\alpha_1, \alpha_2, \ldots, \alpha_r)
\end{align*}
\]

which is exactly the representation in base-b.

### 3.0 Graph Notation

An undirected graph \( G \) is defined as the following tuple: \( G = (N, E) \), where \( N \) is the set of nodes (vertices), and \( E \) is the set of edges. An example is the graph where \( N = \{ a_i \mid i = 1,2,\ldots,N \} \) and \( E = \{ e_{i,j} = (a_i, a_j) \mid i = 1,2,\ldots,N \} \) where \( a_i, a_j \in N \) and \( d_i \) is the degree of node \( a_i \) (i.e., the number of edges incident to a particular node). The degree of a graph, denoted \( d(G) \), is defined as the maximum of the node degrees. A walk in \( G \) is a sequence of edges \( e_1 e_2 \ldots e_l \), such that if \( e_i = (a_i, a_{i+1}) \) then \( e_{i+1} = (a_{i+1}, a_{i+2}) \) and \( e_i \in E \). The length \( l \) of such a walk is defined as the number of edges occurring in it.

### 4.0 Hypercycles

We define now an \( r \)-dimensional Hypercycle, as the following regular undirected graph:

\[
G_m^{r} = \{ N, E \}^{r}
\]

where \( m = m_1, m_2, m_3, \ldots, m_r \) and \( E \) is the set of edges defined as:

\[
E = \{ e_{i,j} = (a_i, a_j) \mid i = 1,2,\ldots,N \} \quad \text{where} \quad a_1, a_2, \ldots, a_r \in N
\]

are at most distance \( \left\lfloor \frac{\log_2 l_{max}}{\rho_i} \right\rfloor \) away. A walk, from node \( \alpha \) to node \( \alpha^* \), of length \( l_{max} \) is defined as the maximum distance between any pair of nodes. A graph is regular, if all nodes have the same degree.

**Example:** Suppose that \( M = 12 = 4 \times 3 \) and \( \rho = 1,1,2 \). Therefore, for the graph generated by the above mixed radix and connectivity vector, node \( (\alpha_4, \alpha_3) = (0,1) \) is distance one away from node \( (10,4,3) = 31 = \beta_1 \beta_2 \). Indeed, according to the definition, \( \beta_1 = 3 = (0-1) \mod 4 = (\alpha_1 - \xi_1) \mod m_1 \), and \( \beta_2 = 2 = (0-1) \mod 4 = (\alpha_2 - \xi_2) \mod m_2 \).

The previously defined Hypercycles are regular graphs of degree.\(^6\)

\[
d = \sum_{i=1}^{r} (\alpha_i - \xi_i) \mod m_i
\]

where \( d_{i,j} \) is the connectivity vector. The distance, \( d \), between any two points is given as \( l_{max} = \left\lfloor \frac{\log_2 m_i}{\rho_i} \right\rfloor \). Thus, because each point is connected to points that are distance \( \xi_i \) away, all the points can be reached in a maximum of \( \left\lfloor \frac{l_{max}}{\rho_i} \right\rfloor \) steps.

\^[6^]{Observe that \( \left\lfloor \frac{\log_2 m_i}{\rho_i} \right\rfloor \) and \( \left\lfloor \frac{\log_2 m_i}{\rho_i} \right\rfloor \) imply that \( \left( \alpha + \xi \right) \mod d \) and \( \left( \alpha + \xi \right) \mod d \) are two distinct nodes in the graph.}

\^[1]{This number is calculated as follows: Given that there are \( r \) digits in a node address, there are the maximum \( r \) differences between a source and a destination address. Also, for each digit (since it is expressed in base \( m_i \) ) the maximum distance between any two points is given as \( l_{max} = \left\lfloor \frac{\log_2 m_i}{\rho_i} \right\rfloor \). Thus, because each point is connected to points that are distance \( \xi_i \) away, all the points can be reached in a maximum of \( \frac{l_{max}}{\rho_i} \) steps.
Given an origin \((\alpha)_{m_1, m_2, \ldots, m_r} = \alpha_1, \alpha_2, \ldots, \alpha_r\)
and a destination \((\beta)_{m_1, m_2, \ldots, m_r} = \beta_1, \beta_2, \ldots, \beta_r\),
and if \(q_i\) denotes their distance along dimension \(i\), their total distance,
denoted as \(\text{dis}(\alpha, \beta)\) and defined as the sum of the individual distances
along all the dimensions, is given as

\[
\text{dis}(\alpha, \beta) = q = \sum_{i=1}^{r} q_i
\]

For these nodes, there are a total of \(L = \binom{q}{q_1, q_2, \ldots, q_r} = \frac{q!}{q_1! q_2! \cdots q_r!}\)
distinct walks of length \(q\) that connect them. These paths can be
constructed by sequentially modifying the source address, each time
substituting a source digit by an intermediate walk digit, as specified in
equation (2) above, until the destination is reached. As an example, the
following walk connects the source to the destination:

\[
\begin{align*}
\alpha & = \alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_r, \\
\beta & = \beta_1, \beta_2, \beta_3, \ldots, \beta_r,
\end{align*}
\]

Figure 1, gives an example of two distinct walks of equal length that
connect a source to a destination, for a hypercycle.

### 4.2 Average Distance Calculation

Given an \(r\)-dimensional generalized graph \(G^D_m\), and restricting
ourselves in any single dimension \(i\), we can calculate the number of nodes that are distance \(l\) away from a source node \((\alpha)_{m_1, m_2, \ldots, m_r} = \alpha_1, \alpha_2, \ldots, \alpha_r\)
equal to the number of symbols \(\xi_i\) that satisfy
the relation

\[
(l-1) \rho_i < |\alpha_i, \xi_i| \leq l \rho_i.
\]

This number is given as

\[
n^*_l = \begin{cases}
2 \rho_i & \text{if } l \rho_i < \lfloor m_i / 2 \rfloor, \\
(m_i - l) - 2 (l - 1) \rho_i & \text{if } \lfloor m_i / 2 \rfloor \leq l \rho_i \leq \frac{m_i - 1}{2} \rho_i, \\
0 & \text{otherwise}
\end{cases}
\]

Given a node \((\alpha)_{m_1, m_2, \ldots, m_r} = \alpha_1, \alpha_2, \ldots, \alpha_r\) there exist
\(n_j = \sum_{i=1}^{r} n^*_l\) nodes at distance one, and in general

\[
n_i = \sum_{i=1}^{r} n^*_l = \prod_{i=1}^{r} n^*_l
\]

Nodes at distance \(l\). This equation is simplified for the case where \(\rho_i = \lfloor m_i / 2 \rfloor\) to the following:

\[
n_i = \sum_{i=1}^{r} \sum_{j=1}^{r} \sum_{k=1}^{r} (m_i - 1)(m_i - 1) \cdots (m_i - 1)
\]

Therefore, the average distance between any two nodes in a Hypercycle

\[
G^D_m \text{ can be calculated as } \overline{d} = \frac{\sum_{i=1}^{r} n_i}{m_1 m_2 \ldots m_r}
\]

Some typical distances are given in TABLE 1.

---

\(\text{TABLE 1}\)

\[
\begin{array}{c|c}
\hline
m_1 & m_2 & m_3 \\
\hline
1 & 2 & 3 \\
\hline
2 & 3 & 4 \\
\hline
3 & 4 & 5 \\
\hline
\end{array}
\]

\(\text{For the definition of a multinomial number, see [1] pp 32.}\)

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\(\text{Extend} \) is a trademark of Imagine That inc.
TABLE I.

<table>
<thead>
<tr>
<th>GRAPH</th>
<th>DEGREE</th>
<th>DIAMETER</th>
<th>NODES</th>
<th>AVERAGE DISTANCE PER NODE</th>
</tr>
</thead>
<tbody>
<tr>
<td>$m = 62$</td>
<td>6</td>
<td>2</td>
<td>12</td>
<td>1.454</td>
</tr>
<tr>
<td>$\rho = 31$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$m = 53$</td>
<td>6</td>
<td>2</td>
<td>15</td>
<td>1.571</td>
</tr>
<tr>
<td>$\rho = 21$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$m = 522$</td>
<td>6</td>
<td>3</td>
<td>20</td>
<td>1.894</td>
</tr>
<tr>
<td>$\rho = 211$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$m = 333$</td>
<td>6</td>
<td>3</td>
<td>27</td>
<td>2.077</td>
</tr>
<tr>
<td>$\rho = 111$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$m = 3322$</td>
<td>6</td>
<td>4</td>
<td>36</td>
<td>2.4</td>
</tr>
<tr>
<td>$\rho = 11111$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$m = 322222$</td>
<td>6</td>
<td>5</td>
<td>48</td>
<td>2.723</td>
</tr>
<tr>
<td>$\rho = 1111111$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$m = 2222222$</td>
<td>6</td>
<td>6</td>
<td>64</td>
<td>3.047</td>
</tr>
<tr>
<td>$\rho = 111111111$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$m = 2217$</td>
<td>6</td>
<td>6</td>
<td>68</td>
<td>3.403</td>
</tr>
<tr>
<td>$\rho = 112$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$m = 2237$</td>
<td>6</td>
<td>6</td>
<td>84</td>
<td>3.434</td>
</tr>
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<td>$\rho = 11111$</td>
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<td></td>
</tr>
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<td>$m = 357$</td>
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<td>6</td>
<td>105</td>
<td>3.615</td>
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<tr>
<td>$\rho = 111$</td>
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<td></td>
</tr>
<tr>
<td>$m = 555$</td>
<td>6</td>
<td>6</td>
<td>125</td>
<td>3.629</td>
</tr>
<tr>
<td>$\rho = 111$</td>
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<td></td>
<td></td>
</tr>
<tr>
<td>$m = 2222222$</td>
<td>7</td>
<td>7</td>
<td>128</td>
<td>3.528</td>
</tr>
<tr>
<td>$\rho = 1111111111$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Figure 1. Examples of Hypercycles.

Figure 2. Throughput vs. offered load using backtrack to the origin and e-cube routing. The offered load and throughput are normalized to the capacity of the interconnection network.

Figure 3. Delay vs. offered load using backtrack-to-the-origin and e-cube routing. The offered load is normalized to the capacity of the interconnection network.
network failed, it could be marked as permanently busy, and packets would be routed around it. This obviously is not the case for the e-cube routing.

6.0 Conclusions and Discussion

In this work, we presented the Hypercycle, a class of multidimensional graphs, which are essentially generalizations of the n-cube. These graphs are obtained by allowing each dimension to incorporate more than two elements and a cyclic interconnection strategy. Although these graphs are not the densest possible, they are attractive, because of their simple routing. Similarly to the n-cube, the destination address is used to sequentially route a message through intermediate nodes as outlined in section 4.1. Also, since the node addresses are represented in a mixed radix as a sequence of r-digits, each one of these digits is processed independently and in parallel with the remaining digits. Thus the hardware involved in the routing can be made fast (because of the parallelism) and simple (since each module need only handle arithmetic mod \( m_j \), as compared to arithmetic mod \( m_1 m_2 \cdots m_r \) needed when all the address digits are necessary as is the case with such networks as the chordal rings \([10]\), or the cube connected cycles \([5]\).

For these graphs, we have also calculated the average distance between any two nodes. Assuming that the graph is the representation of a multiprocessor, the average distance provides a metric of the communication complexity involved in such a structure. Compare for example the average distances of the 7-cube and the \(( m = 5 \times 5, \rho = 1111 )\) graph. The average distances, as they can be determined from TABLE I, are 3.528 and 3.629 respectively, both have similar number of nodes, while their diameters are equal to six and seven respectively.

The graphs presented in this study, are generalizations of some well known graphs such as the binary n-cube, 2- and 3-dimensional meshes, and rings, which are included as special cases. Examples of some special cases are depicted in Figure 1.

The Hypercycles, can be further extended by employing a different connectivity strategy in each dimension. Thus, in its most general form, eqn. 1 can be modified into

\[
b_j = (\alpha_j + \epsilon_j) \text{mod} m_j; \quad \epsilon_j \in \{\epsilon_j^1, \epsilon_j^2, \ldots, \epsilon_j^k\}
\]

provided that the subgraph (projected in dimension \( j \)) is connected. Example of such extensions include the chordal rings \([10]\), cube connected cycles \([5]\) etc. The disadvantage is the increased complexity involved in the routing calculations for that particular dimension.

In conclusion, Hypercycles are generalizations of the basic n-cube and they offer simple routing, and the ability, given a fixed degree, to chose among a number of alternative size graphs. The use of the backtrack-to-origin-and-retry routing strategy offers increased throughput and fault tolerance. These properties are important for the design of concurrent systems (e.g. MAX \([12]\), Hypercube \([11]\)) of varying size and connectivity, and tailored specifically to the topology of a particular application.

REFERENCES

15. Seitz, C. L. "The cosmic cube" *CACM* vol. 28, pp. 22-33, Jan. 1985