Independence number of iterated line digraphs

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Abstract

In this paper, we deal with the independence number of iterated line digraphs of a regular digraph $G$. We give pertinent lower bounds and give an asymptotic estimation of the ratio of the number of vertices of a largest independent set of the $n$th iterated line digraph of $G$.

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1. Introduction, notation

Initially, we were interested in the independence number of the De Bruijn digraphs. In [4], we have proved that asymptotically, for a given even integer $d \geq 2$, the number of the vertices of a largest independent set of the de Bruijn digraph $B(d, D)$ is one half of the number of vertices of $B(d, D)$. It is known that $B(d, D)$ is the $(D - 1)$th iterated line digraph of $B(d, 1)$.

In our paper, we generalize this result, more precisely, we prove that asymptotically, for a regular digraph $G$ of degree $d \geq 2$ (even or not), the ratio of the number of vertices of a largest independent set of the $n$th iterated line digraph $L^n(G)$ of $G$ is $\frac{1}{2}$.

Some notation and basic definitions are necessary.

We consider digraphs $G$ without multiple arcs. We denote by $V(G)$ the vertex set of $G$ and by $v(G)$ its cardinality. We denote by $\mathcal{A}(G)$ the set of arcs of $G$. For an arc $(x, y)$, $x$ is the starting point and $y$ is the ending point.

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For $x \in V(G)$, a vertex $y$ such that $(x, y) \in \mathcal{A}(G)$ is a successor of $x$. $N^+_G(x)$ is the set of successors of $x$. The outdegree $d^+_G(x)$ of $x$ is the number of successors of $x$. A vertex $y$ such that $(y, x) \in \mathcal{A}(G)$ is a predecessor of $x$. $N^-_G(x)$ is the set of predecessors of $x$. The indegree $d^-_G(x)$ of $x$ is the number of predecessors of $x$.

An element $x$ of $V(G)$ such that $(x, x) \in \mathcal{A}(G)$ is called a vertex with loop and the arc $(x, x)$ is a loop. We denote by $V_L(G)$ the set of vertices with loop and by $v_L(G)$ its cardinality.

A regular digraph of degree $d$ is a digraph $G$ such that $d^+_G(x) = d^-_G(x) = d$ for any $x \in V(G)$.

For a digraph $G$, the line digraph $L(G)$ is the digraph whose vertex set is $\mathcal{A}(G)$ and whose arcs are the pairs $((x, y), (y, z))$ where $(x, y)$ and $(y, z)$ are arcs of $G$. Clearly, for each arc $(x, y) \in \mathcal{A}(G)$, we have $d^+_L((x, y)) = d^+_G(y)$ and $d^-_L((x, y)) = d^-_G(x)$. It is also clear that $V_L(L(G)) = \{(x, x); x \in V_L(G)\}$.

For an integer $n \geq 1$, the $n$th iterated line digraph is the digraph defined recursively by $L_1(G) = L(G)$ and $L^n(G) = L(L^{n-1}(G))$.

$L^n(G)$ is also the digraph whose vertices are the walks of $G$ of length $n$ and whose arcs are the pairs $(x_1 \ldots x_{n+1}, y_1 \ldots y_{n+1})$ of walks of length $n$, with $x_2 \ldots x_{n+1} = y_1 \ldots y_n$.

For a vertex $x = x_1 \ldots x_i \ldots x_{n+1}$ of $L^n(G)$, $p_i(x) = x_i$ is the $i$th coordinate of $x$.

It is easy to prove that for any $n \geq 1$, we have $v_L(L^n(G)) = v_L(G)$.

It is clear that for a regular digraph of degree $d$, $L^n(G)$ is a regular digraph of degree $d$ with $v(L^n(G)) = v(G)d^n$.

It is known that a line digraph $G$ has the following property (P):

- for vertices $x$ and $y$ of $G$, we have either $N^+_G(x) = N^+_G(y)$ or $N^+_G(x) \cap N^+_G(y) = \emptyset$.

It was proved that a digraph is a line digraph if and only if it has property (P) (see [3]).

For $d \geq 2$ and $D \geq 2$, the De Bruijn digraph $B(d, D)$ is the digraph whose vertex set is $\mathbb{Z}_d^D$ and whose arcs are the couples $(x_1, x_2 \ldots x_D, x_2 \ldots x_D i)$ with $i \in \mathbb{Z}_d$. The De Bruijn digraph $B(d, 1)$ is the complete digraph $K_d^+$ (with a loop in each vertex). $B(d, D)$ is a regular digraph of degree $d$. The undirected De Bruijn graph $UB(d, D)$ is the underlying graph of $B(d, D)$.

An independent set $S$ of a digraph $G$ is a set of vertices such that there are no arcs between any two distinct elements of $S$. The independence number of $G$ is the maximum cardinality of the independent sets of $G$ and is denoted by $\alpha(G)$.

2. Results on regular line digraphs

Our first result is:

**Proposition 2.1.** For any regular line digraph $H$ of degree $d \geq 2$, we have

$$\alpha(H) \leq \frac{v(H)}{2}.$$  

**Proof.** Let $S$ be an independent set of $H$ with $|S| = s = \alpha(H)$. If $S$ does not contain vertices with loop, every vertex $x$ of $S$ has its neighbours in $V(H) \setminus S$. Consequently, there are $2sd$
Theorem 3.1. For $0 < x < 1$, we have

$$\lim_{m \to +\infty} \sum_{0 \leq k \leq m} \left( \frac{m}{k} \right)^2 x^{2k} (1-x)^{2m-2k} = 0.$$
The proof which follows, shorter than our initial proof, was suggested by one of the referees. We begin by:

**Lemma 3.2.** For $0 < x < 1$, we have $\lim_{m \to +\infty} \left( \frac{m}{w_m} \right)^x (1 - x)^{m - w_m} = 0$, where $w_m = \lfloor (m + 1)x \rfloor$ for $m \in \mathbb{N}^*$.

**Proof.** We can write $(m + 1)x = w_m + r_m$ with $0 \leq r_m < 1$, and then $x = (w_m + r_m)/(m + 1)$. We denote $b_m = \left( \frac{m}{w_m} \right)^x (1 - x)^{m - w_m}$. We have

$$b_m = \left( \frac{m}{w_m} \right)^x (\frac{w_m + r_m}{m + 1})^{w_m} \left( \frac{m - w_m + 1 - r_m}{m + 1} \right)^{m - w_m},$$

that is

$$b_m = \frac{m!}{w_m!} \frac{(w_m + r_m)^{w_m}(m - w_m + 1 - r_m)^{m - w_m}}{(m + 1)^m}.$$

By Stirling formula (see [2] for proofs), we know that

$$\lim_{n \to +\infty} \frac{n!}{(n/e)^n \sqrt{2\pi n}} = 1.$$

Let us denote

$$U_m = \frac{n!}{(n/e)^n \sqrt{2\pi n}}.$$

It is easy to prove that $\lim_{m \to +\infty} w_m = +\infty$ and $\lim_{m \to +\infty} m - w_m = +\infty$.

Consequently, we have $\lim_{m \to +\infty} U_{w_m} = 1$, that is

$$\lim_{m \to +\infty} \frac{w_m!}{(w_m/e)^w_m \sqrt{2\pi w_m}} = 1$$

and $\lim_{m \to +\infty} U_{m - w_m} = 1$, that is

$$\lim_{m \to +\infty} \frac{(m - w_m)!}{((m - w_m)/e)^{m - w_m} \sqrt{2\pi (m - w_m)}} = 1.$$

This implies that the sequence $b_m$ is equivalent when $m \to +\infty$ to the sequence

$$v_m = \frac{(m/e)^m \sqrt{2\pi m}}{2\pi (w_m/e)^{w_m} (m - w_m)/e)^{m - w_m} \sqrt{w_m(m - w_m)}} \times \frac{(w_m + r_m)^{w_m}(m - w_m + 1 - r_m)^{m - w_m}}{(m + 1)^m},$$

that is

$$v_m = \sqrt{\frac{m}{2\pi w_m(m - w_m)} (m + 1)^m \frac{(w_m + r_m)^{w_m}}{w_m^m} \frac{(m - w_m + 1 - r_m)^{m - w_m}}{(m - w_m)^{m - w_m}}}.$$
We have \( \frac{m^m}{(m + 1)^m} < 1 \). We have also
\[
\frac{(w_m + r_m)^{w_m}}{w_m^{w_m}} \leq \left( 1 + \frac{1}{w_m} \right)^{w_m}
\]
and
\[
\frac{(m - w_m + 1 - r_m)^{m - w_m}}{m - w_m^{m - w_m}} \leq \left( 1 + \frac{1}{m - w_m} \right)^{m - w_m}
\]
and as
\[
\lim_{m \to +\infty} \left( 1 + \frac{1}{w_m} \right)^{w_m} = \lim_{m \to +\infty} \left( 1 + \frac{1}{m - w_m} \right)^{m - w_m} = e,
\]
the sequences \( (w_m + r_m)^{w_m} / w_m^{w_m} \) and \( (m - w_m + 1 - r_m)^{m - w_m} / (m - w_m)^{m - w_m} \) are bounded.

As
\[
\frac{m}{w_m(m - w_m)} = \frac{m}{(mx + x - r_m)(m(1 - x) - x + r_m)},
\]
clearly, we have
\[
\lim_{m \to +\infty} \frac{m}{w_m(m - w_m)} = 0.
\]
It follows \( \lim_{m \to +\infty} v_m = 0 \), hence \( \lim_{m \to +\infty} b_m = 0 \). So, the lemma is proved.

Now, we can prove Theorem 3.1:

For fixed \( x, 0 < x < 1 \) and for \( 0 \leq k \leq m \), we denote \( F_k = \binom{m}{k} x^k (1 - x)^{m - k} \). Let us put
\[
\varphi(m) = \max_{0 \leq k \leq m} F_k.
\]
We have
\[
\sum_{0 \leq k \leq m} \binom{m}{k}^2 x^{2k} (1 - x)^{2m - 2k} \leq \varphi(m) \sum_{0 \leq k \leq m} \binom{m}{k} x^k (1 - x)^{m - k}
\]
that is
\[
\sum_{0 \leq k \leq m} \binom{m}{k}^2 x^{2k} (1 - x)^{2m - 2k} \leq \varphi(m).
\]
We have
\[
F_{k+1} / F_k = (m - k)x / ((k + 1)(1 - x)) < 1 \text{ for } k > (m + 1)x - 1 \text{ and } F_{k+1} / F_k > 1 \text{ for } k < (m + 1)x - 1.
\]
This means that the maximum of \( F_k \) is reached for \( k = \lfloor (m + 1)x \rfloor = w_m \).

Consequently, \( \varphi(m) = F_{w_m} \) and by Lemma 4.1, we have \( \lim_{m \to +\infty} \varphi(m) = 0 \), which implies \( \lim_{m \to +\infty} \sum_{0 \leq n \leq m} \binom{m}{k}^2 x^{2k} (1 - x)^{2m - 2k} = 0 \). So, Theorem 3.1 is proved.

Let \( G \) be a regular digraph of degree \( d \geq 2 \) and let us denote \( H = L(G) \).
We fix an integer \( r \) verifying \( 1 \leq r \leq d - 1 \) and an element \( f \) of \( F_r(H) \).
For an integer \( n \geq 1 \), we define
\[
\Psi_{H,f,n} : V(L^n(H)) \to \mathbb{Z} \text{ by } \Psi_{H,f,n}(x_1 \ldots x_{n+1}) = \sum_{1 \leq i \leq n+1} (-1)^{i+1} f(x_i).
\]
It is clear that \( \Psi_{H,f,n}(x) \) is minimum when the coordinates in odd rows of \( x \) have 0 as image by \( f \) and the coordinates in even rows have 1 as image and
that $\Psi_{H,f,n}(x)$ is maximum when the opposite holds. Consequently, $\Psi_{H,f,n}(x)$ goes from $-[(n+1)/2]$ to $[(n+2)/2]$.

It is clear that if $n$ is odd, for $x \in V(L^n(H))$ we have $\Psi_{H,f,n}(x) = -\Psi_{H,1-f,n}(x)$ and that if $n$ is even, for $x \in V(L^n(H))$ we have $\Psi_{H,f,n}(x) = 1 - \Psi_{H,1-f,n}(x)$.

**Lemma 3.3.** Let $m \geq 2$ be an integer. For $0 \leq i \leq m$, we have

(a) $|\Psi^{-1}_{H,f,2m-1}(i)| = (v(H)/d) \sum_{0 \leq k \leq m-i} \binom{m}{k} \binom{m}{k+i} r^{2k+i}(d-r)^{2m-2k-i}$,

(b) $|\Psi^{-1}_{H,f,2m-1}(-i)| = |\Psi^{-1}_{H,f,2m-1}(i)|$.

**Proof.** (a) For $0 \leq k \leq m-i$, let us denote by $A_{i,k}$ the set of vertices $x = x_1\ldots x_{2m}$ of $L^{2m-1}(H)$ such that exactly $k+i$ coordinates of $x$ in odd rows have 1 as image by $f$ and exactly $k$ coordinates of $x$ in even rows have 1 as image by $f$.

Clearly, the $A_{i,k}$, $0 \leq k \leq m-i$, are disjoint and $\Psi^{-1}_{H,f,2m-1}(i) = \bigcup_{0 \leq k \leq m-i} A_{i,k}$. Let us consider first $1 \leq i \leq m-1$ (which implies $m \geq 2$).

For $0 \leq k \leq m-i-1$, the number of vertices $x_1\ldots x_{2m}$ of $A_{i,k}$ with $f(x_1) = 1$ is $(v(H)/d) r^{m-i-1} (d-r)^{m-k-1} \binom{m}{k} r^k (d-r)^{m-k}$, that is $(v(H)/d) \binom{m}{k+i} r^{k+i} (d-r)^{2m-2k-i}$.

The number of vertices $x_1\ldots x_{2m}$ of $A_{i,k}$ with $f(x_1) = 0$ is $(v(H)/d) (d-r) \binom{m}{k+i} r^{k+i} (d-r)^{2m-2k-i}$.

In a similar way, one can prove that the result holds for $i \in \{0, m\}$.

(b) It is clear that if $\Psi_{H,f,2m-1}(x) = -i$, we have $\Psi_{H,1-f,2m-1}(x) = i$ and so $\Psi^{-1}_{H,f,2m-1}(-i) = \Psi^{-1}_{H,1-f,2m-1}(i)$. By the proof of part (a), we have

$$|\Psi^{-1}_{H,1-f,2m-1}(i)| = \frac{v(H)}{d} \sum_{0 \leq k \leq m-i} \binom{m}{k} \binom{m}{k+i} (d-r)^{2k+i} r^{2m-2k-i}.$$
This implies \(|\Psi^{-1}_{H,1-f,2m-1}(i)|=|\Psi^{-1}_{H,f,2m-1}(i)|\), hence \(|\Psi^{-1}_{H,1-f,2m-1}(-i)|=|\Psi^{-1}_{H,f,2m-1}(i)|\)

The following result is essential:

**Lemma 3.4.** If \(x \in \Psi^{-1}_{H,1-f,2m-1}(i)\) and \(y \in \Psi^{-1}_{H,f,2m-1}(j)\) are adjacent in \(L^{2m-1}(H)\), we have \(i + j \in \{-1, 0, 1\}\).

**Proof.** Let \(x = x_1 \ldots x_{2m}\) and \(y = y_1 \ldots y_{2m}\).

We have \(f(x_1) - f(x_2) + \cdots + f(x_{2m-1}) - f(x_{2m}) = i\) and \(f(y_1) - f(y_2) + \cdots + f(y_{2m-1}) - f(y_{2m}) = j\). If \(y\) is a successor of \(x\), by adding, we obtain \(f(x_1) - f(y_{2m}) = i + j\), which implies \(i + j \in \{-1, 0, 1\}\). If \(y\) is a predecessor of \(x\), by adding, we obtain \(f(y_1) - f(x_{2m}) = i + j\), which again implies \(i + j \in \{-1, 0, 1\}\). □

Now, we can give pertinent lower bounds:

**Theorem 3.5.** Let \(m \geq 2\) be an integer and let us denote

\[
N(m, d, r) = \frac{1}{2} d^{2m} - \frac{1}{2} \sum_{k=0}^{m} \binom{m}{k}^2 r^{2k}(d-r)^{2m-2k} + \sum_{k=0}^{m-2} \left( \frac{m-1}{k} \right) \left( \frac{m-1}{k+1} \right) r^{2k+2}(d-r)^{2m-2k-2}.
\]

Then

(a) \(\alpha(L^{2m}(G)) \geq v(G)N(m, d, r)\),
(b) \(\alpha(L^{2m+1}(G)) \geq d v(G)N(m, d, r)\).

**Proof.** Let us define sets \(A_m\) and \(B_m\), by

\[
A_m = \Psi^{-1}_{H,f,2m-1}(1) \cup \cdots \cup \Psi^{-1}_{H,f,2m-1}(m), \quad \text{and}
\]

\[
B_m = \{ x \in \Psi^{-1}_{H,f,2m-1}(0); \ f(p_1(x)) = 0, \ f(p_{2m}(x)) = 1 \}.
\]

Clearly, \(A_m\) and \(B_m\) are disjoint. Since for \(i\) and \(j\) in \([1, \ldots, m]\), we have \(i + j \notin \{-1, 0, 1\}\), Lemma 3.3 implies that \(A_m\) is an independent set of \(L^{2m-1}(H) = L^{2m}(G)\). \(B_m\) is also an independent set of \(L^{2m}(G)\).

Indeed, the existence of an arc \((x_1 \ldots x_{2m}, y_1 \ldots y_{2m})\) with extremities in \(B_m\) would imply \(f(x_1) - f(x_2) + \cdots + f(x_{2m-1}) - f(x_{2m}) = 0\), \(f(y_1) - f(y_2) + \cdots + f(y_{2m-1}) - f(y_{2m}) = 0\) and by addition, we would have \(f(x_1) - f(y_{2m}) = 0\), that is \(-1 = 0\), which is false.

A vertex of \(A_m\) and a vertex of \(B_m\) are not linked. Indeed, suppose that there exists an arc \((x, y)\) with extremities in \(A_m\) and \(B_m\).

By denoting \(x = x_1 \ldots x_{2m}\) and \(y = y_1 \ldots y_{2m}\), we get

\[
f(x_1) - f(y_{2m}) = \Psi_{H,f,2m-1}(x) + \Psi_{H,f,2m-1}(y).
\]
If \( x \) is in \( A_m \) and \( y \) is in \( B_m \), we have \( f(x_1) - 1 = \Psi_{H,f,2m-1}(x) \), hence \( \Psi_{H,f,2m-1}(x) = 1 \), which is false. And if \( x \) is in \( B_m \) and \( y \) is in \( A_m \), we get \( -f(y_{2m}) = \Psi_{H,f,2m-1}(y) \), again false. We conclude that \( A_m \cup B_m \) is an independent set of \( L^{2m}(G) \).

Since \( |\Psi_{H,f,2m-1}^{-1}(-i)| = |\Psi_{H,f,2m-1}^{-1}(i)| \) for \( 1 \leq i \leq m \), we deduce

\[ |A_m| = \frac{v(G)d^{2m} - |\Psi_{H,f,2m-1}^{-1}(0)|}{2}. \]

The elements \( x_1 \ldots x_{2m} \) of \( B_m \) are characterized by

\[
\begin{align*}
&f(x_1) = 0, \quad f(x_{2m}) = 1, \\
&f(x_2) - f(x_3) + \cdots + f(x_{2m-2}) - f(x_{2m-1}) = -1.
\end{align*}
\]

By Lemma 3.2b, we deduce that \( |B_m| = r(d - r)|\Psi_{H,f,2m-3}^{-1}(1)| \), let

\[ |B_m| = v(G)r(d - r) \sum_{0 \leq k \leq m-2} \binom{m-1}{k} \binom{m-1}{k+1} r^{2k+1}(d - r)^{2m-3-2k}. \]

Since \( A_m \cup B_m \) is independent, we have \( \alpha(L^{2m}(G)) \geq |A_m \cup B_m| \) and the assertion is proved.

(b) We have \( L^{2m+1}(G) = L^{2m}(H) \). By applying the previous conclusion, the result follows.

This theorem yields in fact \( d - 1 \) lower bounds of \( \alpha(L^n(G)) \) and we conjecture that the best lower bound is provided by \( r = [d/2] \).

Now we can give the main result:

**Theorem 3.6.** For a regular digraph \( G \) of degree \( d \geq 2 \), we have

\[
\lim_{n \to +\infty} \frac{\alpha(L^n(G))}{v(L^n(G))} = \frac{1}{2}.
\]

**Proof.** For \( m \in \mathbb{N} \) and \( 1 \leq r \leq d - 1 \), from Theorem 3.5 and Proposition 2.1, we deduce

\[
v(G) \left( \frac{1}{2}d^{2m} - \frac{1}{2} \sum_{0 \leq k \leq m} \binom{m}{k}^2 r^{2k}(d - r)^{2m-2k} \right) \leq \alpha(L^{2m}(G)) \leq \frac{v(L^{2m}(G))}{2},
\]

hence

\[
\frac{1}{2} - \frac{1}{2} \sum_{0 \leq k \leq m} \binom{m}{k}^2 \left( \frac{r}{d} \right)^{2k} \left( \frac{d - r}{d} \right)^{2m-2k} \leq \frac{\alpha(L^{2m}(G))}{v(L^{2m}(G))} \leq \frac{1}{2}.
\]

By Theorem 3.1, we have

\[
\lim_{m \to +\infty} \sum_{0 \leq k \leq m} \binom{m}{k}^2 \left( \frac{r}{d} \right)^{2k} \left( \frac{d - r}{d} \right)^{2m-2k} = 0.
\]
It follows that
\[
\lim_{m \to +\infty} \frac{\alpha(L^{2m}(G))}{v(L^{2m}(G))} = \frac{1}{2}.
\]
Similarly one can prove that
\[
\lim_{m \to +\infty} \frac{\alpha(L^{2m+1}(G))}{v(L^{2m+1}(G))} = \frac{1}{2}.
\]
Consequently, we have
\[
\lim_{n \to +\infty} \frac{\alpha(L^n(G))}{v(L^n(G))} = \frac{1}{2}. \quad \Box
\]

A cover set of a digraph \( G \) is an independent set \( S \) of \( G \) with no loops and such that any vertex in \( V(G) \setminus S \) has at least one neighbour in \( S \). We denote by \( \alpha'(G) \) the maximum cardinality of the cover sets of \( G \).

It is not difficult to prove that for a regular digraph \( G \) of degree \( d \geq 2 \), \( A_m \cup B_m \) is a cover set of \( L^{2m}(G) \) (\( A_m \) and \( B_m \) are the sets defined in the proof of Theorem 3.5).

As for Theorem 3.6, this implies that
\[
\lim_{n \to +\infty} \frac{\alpha'(L^n(G))}{v(L^n(G))} = \frac{1}{2}.
\]
In particular, this disproves a conjecture of Bryant and Fredricksen (see [1]) stating that, asymptotically, the number of the vertices of a cover set of the binary de Bruijn digraph \( B(2, D) \) cannot exceed \( \frac{4}{9} \) of the total number, \( 2^D \), of vertices of \( B(2, D) \). This conjecture had already been disproved by the author in [4].

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References