Cycles in a tournament with pairwise zero, one or two given vertices in common
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Received 16 September 2006; accepted 11 July 2007
Available online 22 August 2007

Abstract

Chen et al. [Partitioning vertices of a tournament into independent cycles, J. Combin. Theory Ser. B 83 (2001) 213–220] proved that every $k$-connected tournament with at least $8k$ vertices admits $k$ vertex-disjoint cycles spanning the vertex set, which answered a question posed by Bollobas.

In this paper, we prove, as a consequence of a more general result, that every $k$-connected tournament of diameter at least 4 contains $k$ vertex-disjoint cycles spanning the vertex set.

Then, for a connected tournament of diameter at most 3, we determine a relation between the maximum number of vertex-disjoint cycles and the maximum number of vertex-disjoint cycles spanning the vertex set of $T$. Also, by using a lemma of Chen et al. [Partitioning vertices of a tournament into independent cycles, J. Combin. Theory Ser. B 83 (2001) 213–220], we prove that a $k$-connected tournament of order at least $5k - 3$, of diameter distinct from 3 (resp. 3) admits $k$ (resp. $k - 1$) vertex-disjoint cycles spanning the vertex set of $T$, with only one exception. Finally, we give results on cycles with pairwise one or two vertices in common. A few open problems are raised.

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Keywords: Connected tournament; Disjoint cycles; Diameter

1. Introduction, basic notions

Bollobas (see [7]) raised the following question:

If $k$ is a positive integer, what is the least integer $g(k)$ so that all but a finite number of $g(k)$-connected tournaments contain a spanning subgraph consisting of $k$ vertex-disjoint cycles?

Chen et al. answered in [4]:

Every $k$-connected tournament $T$ of order $n \geq 8k$ contains $k$ vertex-disjoint cycles that span $V(T)$.

As it was proved that $k \leq g(k)$ (see [7]), it follows that $g(k) = k$.

In this paper, we prove that every $k$-connected tournament $T$ of diameter at least 4, contains $k$ vertex-disjoint cycles that span $V(T)$.

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doi:10.1016/j.disc.2007.07.018
Then we determine a relation between the maximum number of vertex-disjoint cycles of a connected tournament \( T \) of diameter at most 3 and the maximum number of vertex-disjoint cycles that span \( T \).

Via a result of Reid [6], we prove that for \( 1 \leq m \leq v'(T) \), there exist \( m \) vertex-disjoint cycles spanning \( V(T) \), where \( v'(T) \) is the maximum number of vertex-disjoint cycles spanning the vertex set \( V(T) \).

Via a lemma of Chen et al. [4], we prove that a \( k \)-connected tournament of order at least \( 5k - 3 \) is vertex-partitionable into \( k \) cycles if its diameter is distinct from 3 (with only one exception), and into at least \( k - 1 \) cycles if the diameter is 3.

We finish by giving results on the number of cycles spanning the connected tournament \( T \) with pairwise one or two given vertices in common.

The notation and definitions are those of [1]:

A tournament is a digraph \( T \) such that for any distinct vertices \( x, y \), exactly one of the couples \( (x, y) \) and \( (y, x) \) is an arc of \( T \).

By paths or cycles of a tournament \( T \), we mean directed paths or directed cycles of \( T \). If \( C = x_1x_2...x_{m-1}x_m \) is a path or a cycle, we write \( C = x_1Px_m \), where \( P = x_2...x_{m-1} \), etc. By disjoint cycles, we mean vertex-disjoint cycles.

A cycle of length 3 is a 3-cycle or a triangle.

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A vertex \( x \) dominates a vertex \( y \), if \( (x, y) \) is an arc of \( T \). We say also that \( y \) is dominated by \( x \). For disjoint subsets \( A \) and \( B \) of \( V(T) \), \( A \) dominates \( B \) if every \( x \in A \) dominates every \( y \in B \).

In our figures a thick arrow shows dominance.

A regular tournament of order \( n \) is a tournament such that \( d^+(x) = d^-(x) = (n - 1)/2 \) for any vertex \( x \), where \( d^+(x) \) is the out-degree of \( x \) (the number of successors of \( x \)) and \( d^-(x) \) the in-degree of \( x \) (the number of predecessors of \( x \)).

The rotational tournament \( R(q_1, ..., q_m) \) is the tournament whose vertex set is the additive group \( \mathbb{Z}_{2m+1} \) of the integers modulo \( 2m + 1 \) and whose arcs are the couples \((i, j)\) with \( j - i \in \{q_1, ..., q_m\} \), where \( \{q_1, ..., q_m\} \) is a subset of \( \mathbb{Z}_{2m+1} \) such that \( q_i \neq 0 \) and \( q_i + q_j \neq 0 \) for \( 1 \leq i, j \leq m \). It is easy to see that \( R(q_1, ..., q_m) \) is a regular tournament.

By a connected tournament, we mean a strongly connected tournament (i.e., a tournament such that for any distinct vertices \( x \) and \( y \), there exists a path from \( x \) to \( y \)). By a \( k \)-connected tournament, we mean a tournament whose vertex strong-connectivity is at least \( k \). Recall that the vertex strong connectivity \( k(G) \) of a digraph \( G \), is the greatest number \( k \) such that the removal of any \( k - 1 \) vertices of \( G \) does not disconnect \( G \). It is known that if \( T \) is connected, every vertex \( x \) is contained in at least a 3-cycle (it is a consequence of Moon’s theorem which states that in a connected tournament \( T \) of order \( n \), for every \( x \in V(T) \) and every integer \( k \in \{3, ..., n\} \), there exists a \( k \)-cycle through \( x \) in \( T \)). In a strongly connected digraph \( G \), the distance \( d(x, y) \) from a vertex \( x \) to a vertex \( y \), is the minimum length of a directed path from \( x \) to \( y \). The diameter of \( G \), denoted \( D(G) \), is the maximum of these distances.

Camion’s theorem states that a tournament \( T \) is Hamiltonian if and only if \( T \) is connected (see [3]). Redei’s theorem states that any tournament contains a Hamiltonian path (see [5]). By Menger’s theorem, in a \( k \)-connected tournament, for distinct vertices \( x \) and \( y \), there exist \( k \) internally disjoint paths going from \( x \) to \( y \).

An acyclic tournament is a tournament without cycles. It is known that in a nonacyclic tournament \( T \), there exists at least a 3-cycle.

2. Disjoint cycles

We start by:

**Theorem 2.1.** Let \( T \) be a connected tournament of diameter \( D(T) \geq 4 \). Let \( a \) and \( b \) be vertices of \( T \) with \( d(a, b) \geq 4 \). Suppose that there exist \( m \) internal-disjoint paths \( P_i = aQ_ih_i, 1 \leq i \leq m \), with \( m \geq 1 \). Then there exists a family \( (C_i)_{1 \leq i \leq m} \) of \( m \) disjoint cycles spanning \( V(T) \) and satisfying:

\[
V(P_i) \subset V(C_i) \quad \text{and} \quad V(Q_i) \subset V(C_i) \quad \text{for} \ 2 \leq i \leq m. \tag{6}
\]

**Proof.** The case \( m = 1 \) is true since \( T \) is connected and therefore Hamiltonian. So, we can consider \( m \geq 2 \). First, we verify that there exist a family of \( m \) disjoint cycles satisfying condition (6).

Let \( Q_i = a_iQ_i'b_i \). The family \( (C_i')_{1 \leq i \leq m} \) with \( C_i' = aQ_i'b_i \) and \( C_i' = a_iQ_i'b_ia_i \) for \( 2 \leq i \leq m \), suffices. Among these families, we take a family \( (C_i)_{1 \leq i \leq m} \) to be a collection of cycles having a maximum number of vertices.
Let $A = \bigcup_{1 \leq i \leq m} V(C_i)$ and $B = V(T) \setminus A$. Suppose that $B$ is nonempty. For a vertex $x$ of $B$ and for a cycle $C_i$, either $x$ dominates $C_i$ or $x$ is dominated by $C_i$. Indeed, otherwise the vertices of $C_i$ and $x$, would form a cycle (see Fig. 1) and within the other cycles, we would have $m$ cycles having $|A| + 1$ vertices, a contradiction.

Furthermore, we claim that either $x$ dominates $A$, or $x$ is dominated by $A$.

Indeed, two cases are possible:

Case 1: $x$ dominates $V(C_1)$ (Fig. 2).

Then $x$ dominates $b$ and for $2 \leq i \leq m$, $x$ dominates $a_i$ (as otherwise $aa_ixb$ would be a path of length 3, impossible as $d(a, b) \geq 4$). So, $x$ dominates $A$.

Case 2: $x$ is dominated by $V(C_1)$ (Fig. 3).

Then $a$ dominates $x$ and for $2 \leq i \leq m$, $x$ is dominated by $b_i$ (as otherwise $axb_ib$ would have length 3). Therefore, $x$ is dominated by $A$.

Let $E$ be the set of vertices of $B$ dominating $A$ and $F$ be the set of vertices of $B$ dominated by $A$. As $T$ is connected, $E \neq \emptyset$, $F \neq \emptyset$ and $E$ cannot dominate $F$.

Consequently, there exists $x \in F$ which dominates $y \in E$ (see Fig. 4).

If $C_1 = aPa$, then $C_1'' = xyaPxa$ is a cycle (see Fig. 5).

The cycles $C_1'', C_2, \ldots, C_m$ satisfy (6) and they have $|A| + 2$ vertices, a contradiction.

Conclusion: $B = \emptyset$ and $A = V(T)$.

\textbf{Theorem 2.2.} A $k$-connected tournament of diameter $D(T) \geq 4$ contains $k$ disjoint cycles spanning $V(T)$.

\textbf{Proof.} Let $a$ and $b$ be vertices of $T$ with $d(a, b) \geq 4$. By Menger’s theorem there exist $k$ internal-disjoint paths $P_i = aQ_ib$, $1 \leq i \leq k$. The rest follows from Theorem 2.1.

□
Remark. There exist $k$-connected tournaments of diameter at least 4 having less than $8k$ vertices, the minimum order being $3k + 2$ (see Fig. 6 with $R$, $S$, $T$ tournaments of order $k$). So, this result expands that of Chen et al.

We denote by $v(T)$ the maximum number of disjoint cycles of a connected tournament $T$ and by $v'(T)$ the maximum number of disjoint cycles spanning $T$. Since every connected tournament has a Hamiltonian cycle, clearly we have $1 \leq v'(T) \leq v(T)$. We state:

**Theorem 2.3.** Let $T$ be a connected tournament of diameter at most 3. Then either $v'(T) = v(T)$ or $v'(T) = v(T) - 1$.

**Proof.** Let $C_1, \ldots, C_v$ be a family of disjoint cycles, where $v = v(T)$, such that $A = \bigcup_{1 \leq i \leq v} V(C_i)$ has maximum cardinality.

If $|A| = n$, we have $v'(T) = v(T)$ and we are done.

So, assume $|A| < n$. Thus, there is not a family of $v$ disjoint cycles spanning $V(T)$. Let us consider the tournament $T' = T - A$. Since $A$ has maximum cardinality, $T'$ is acyclic and there exists a Hamiltonian path $x_1 \ldots x_m$ of $T'$ where $x_i$ dominates $x_j$ if and only if $i < j$. 
Each vertex of $T'$ either dominates or is dominated by a cycle $C_i$.

Two cases are possible:

Case 1: $m = 1$. Then $T'$ is a single vertex $x_1$. Since $T$ is connected, there exists a 3-cycle $x_1y_1y_2x_1$ of $T$ and $y_1, y_2$ are in $A$.

If $y_1, y_2$ were in the same cycle $C_i$, the vertices of this cycle and $x_1$ would induce a connected tournament and therefore we would have a new cycle. Then we would have $v$ disjoint cycles spanning $T$, a contradiction.

Consequently $y_1, y_2$ are in two distinct cycles, say $C_i$ and $C_j$ (Fig. 7).

It is easy to see that the vertices of these cycles and $x_1$ can be assembled in a single cycle and with the remaining $v - 2$ cycles, we get $v - 1$ disjoint cycles spanning $T$. So, $v'(T) = v(T) - 1$.

Case 2: $m > 1$.

There exists a path $P$ from $x_m$ to $x_1$ of length 2 or 3 such that the internal vertices are in $A$.

Suppose $P$ is of length 2, let $P = x_m y_1 x_1$, $y_1$ would be in a cycle $C_i$ (Fig. 8). It is easy to see that the vertices of this cycle and the vertices of $T'$, would induce a connected tournament. Therefore, we would have a new cycle, and then we would have $v$ disjoint cycles spanning $T$, contrary to $|A| < n$.

Therefore, $P$ has length 3. Let $P = x_m y_1 y_2 x_1$.

By the same reasoning, $y_1$ and $y_2$ are in two distinct cycles $C_i$ and $C_j$ (Fig. 9).

It is easy to see that the vertices of these cycles and the vertices of $T'$ yield a new cycle. Combine this with the remaining cycles and we get $v - 1$ disjoint cycles spanning $T$. So, $v' = v - 1$. □
When the diameter is 2 we have a more precise result:

**Theorem 2.4.** Let $T$ be a $k$-connected tournament of order $n$ and diameter 2.

If $n \equiv 0 \pmod{3}$ or $n \equiv 2 \pmod{3}$, then $v'(T) = v(T)$.

If $n \equiv 1 \pmod{3}$ and if $v'(T) = v(T) - 1$, then $v(T) = (n - 1)/3$ and $v'(T) = (n - 4)/3$.

**Proof.** A collection of $m$ disjoint cycles gives rise to $m$ disjoint triangles since the subtournament induced by the vertices of a $p$-cycle, $p \geq 3$, is a connected tournament that contains cycles of all lengths from 3 to $p$. So, $v(T) \leq (T)$ (the maximum number of disjoint triangles). Clearly, (the maximum number of disjoint triangles) $\leq v(T)$. Thus, the maximum number $v(T)$ of disjoint cycles is also the maximum number of disjoint triangles.

So, we can consider $v = v(T)$ disjoint triangles $T_1, \ldots, T_v$.

Let $B = V(T) \setminus A$, where $A$ is the set of the vertices of the $v$ triangles.

For $n \equiv 0$ or $2 \pmod{3}$, if we had $v'(T) < v(T)$, $B$ would contain a number $m \geq 2$ of vertices inducing an acyclic tournament $T'$ (as otherwise we would have additional triangles), and we would have a Hamiltonian path $x_1 \ldots x_m$ of $T'$.

As $d(x_m, x_1) \leq 2$ and $d(x_m, x_1) > 1$, there would exist a path $P = x_my_1x_1$. The vertex $y_1$ would be in a triangle $T_i$ that with the vertices of $T'$ would yield a new cycle. This cycle and the other triangles would be a family of $v$ disjoint cycles spanning $T$, a contradiction. Therefore, we have $v'(T) = v(T)$.

For $n \equiv 1 \pmod{3}$ and $v'(T) = v(T) - 1$, the same reasoning shows that necessarily, we have $|B| = 1$, hence $v(T) = (n - 1)/3$ and $v'(T) = (n - 4)/3$. □

Reid proved in [6]:

**Theorem 2.5.** A 2-connected tournament $T$ of order $n \geq 6$, has two vertex-disjoint cycles of lengths 3 and $n - 3$, unless $T$ is isomorphic to the rotational tournament $R(1, 2, 4)$.

This implies:

**Theorem 2.6.** Let $T$ be a connected tournament with $v'(T) \geq 2$. For any $m$ such that $1 \leq m \leq v'(T)$, there exist $m$ disjoint cycles spanning $T$.

**Proof.** It suffices to prove that if $T$ is spanned by $m$ disjoint cycles, then $T$ is spanned by $m - 1$ disjoint cycles.

Since $T$ is connected, $T$ is Hamiltonian. So, the case $m = 2$ holds. So, for $m \geq 3$, let $C_1, \ldots, C_m$ be $m$ cycles spanning $T$.

If there exist two cycles $C_i, C_j$ such that one does not dominate the other (Fig. 10), then the tournament induced by the vertices of both cycles is connected, and thus, Hamiltonian. By combining this new cycle with the remaining $m - 2$ cycles, we get $m - 1$ disjoint cycles spanning $T$.

On the other hand, suppose that one dominates the other, we consider the tournament $T'$ whose vertices are the cycles $C_1, \ldots, C_m$ and whose arcs are the couples $(C_i, C_j)$ with $C_i$ dominating $C_j$.

As $T$ is connected, $T'$ is connected, so, there exists in $T'$ a 3-cycle $C_i, C_j, C_r$ (Fig. 11).

It is easy to see that the vertices of this 3-cycle induce a 2-connected tournament with at least nine vertices. By Theorem 2.5, $T'$ can be spanned by two disjoint cycles.

With the remaining $m - 3$ cycles, again we get $m - 1$ disjoint cycles spanning $T$. □

![Fig. 10.](image-url)
Chen et al. [4] proved the following:

**Lemma 2.7.** Every k-connected tournament of order at least 5k − 3, contains k disjoint cycles.

With our results we get:

**Theorem 2.8.** Let T be a k-connected tournament with k ≥ 2 and of order n.

(a) If the diameter is at least 4, T can be spanned by k disjoint cycles.
(b) If the diameter is 2 and if n ≥ 5k − 3, T can be spanned by k disjoint cycles, the only exception being for k = 2, when T is the rotational tournament R(1, 2, 4).
(c) If the diameter is 3, and if n ≥ 5k − 3, T can be spanned by at least k − 1 cycles.

**Proof.** (a) We have already proved this result.
(b) The case k = 2 is Reid’s Theorem, the only exception being R(1, 2, 4). Consider now k ≥ 3. By Lemma 2.7, T admits k disjoint cycles and therefore v(T) ≥ k. If v'(T) = v(T), then v'(T) ≥ k. So, by Theorem 2.6 T can be spanned by $k$ disjoint cycles. If v'(T) = v(T) − 1, then Theorem 2.4 implies:

$$n \equiv 1 \pmod{3} \quad \text{and} \quad v'(T) = \frac{n - 4}{3}.$$ 

In this case, for k ≥ 4, we have v'(T) ≥ (5k − 7)/3 ≥ k. For k = 3, n ≥ 12. As n ≡ 1 (mod 3), we see that n ≥ 13. So, v'(T) = (n − 4)/3 ≥ 3 = k. Again, Theorem 2.6 is employed to obtain the result.
(c) By Lemma 2.7, v(T) ≥ k. Then by Theorem 2.3, we conclude v'(T) ≥ k − 1. □

We finish this section by raising three questions:

**Open problem 1:** For k ≥ 2 what is the least integer h(k) so that any k-connected tournament of order n ≥ h(k), can be spanned by k disjoint cycles?

What is the least integer r(k) so that any k-connected tournament of order n ≥ r(k), can be spanned by at least k − 1 disjoint cycles?

**Comments:** According to Chen et al. [4] we have h(k) ≤ 8k.

We conjecture that h(k) ≤ 5k + 1.

With our results, we have r(k) ≤ 5k − 3.

**Open problem 2:** For integers r and s with 1 ≤ r ≤ s, is there a connected tournament T satisfying v'(T) = r and v(T) = s?

**Comments:** It is easy to see that the answer is yes when r = s.

The answer is also yes when r = s − 1 (personal communication of F. Havet).

The Bermond, Thomassen conjecture (see [2]) states that for r > 1, a digraph of minimum out-degree at least 2r − 1, contains r disjoint cycles. We ask:

**Open problem 3:** Is the Bermond, Thomassen conjecture true for tournaments?

**Comments:** We proved only that a tournament of minimum out-degree at least 3r − 2, admits r disjoint cycles. The proof uses the fact that the removal of a maximum number of disjoint triangles gives rise to an acyclic tournament. We
obtain a better result for regular tournaments. So, for fixed \( r \geq 1 \) let \( T \) be a regular tournament of degree \( \geq (5r - 4)/2 \). Then the order \( n \) of \( T \) satisfies \( n \geq 2 \times (5r - 4)/2 + 1 \). That is, \( n \geq 5r - 3 \).

It is known that the vertex strong connectivity \( k(T) \) of \( T \) satisfies \( k(T) \geq n/3 \) (Thomassen’s result, see [8]), hence \( k(T) \geq r \). Consequently, \( T \) is a \( r \)-connected tournament of order \( n \geq 5r - 3 \), and by Chen et al.’s Lemma 2.7, \( T \) admits \( r \) disjoint cycles.

3. Cycles with pairwise one or two given common vertices

Our first result is:

**Theorem 3.1.** Let \( T \) be a \( k \)-connected tournament and \( x \) be a vertex of \( T \). There exist \( k \) cycles \( C_i \), \( 1 \leq i \leq k \), spanning \( V(T) \) and such that

\[
V(C_i) \cap V(C_j) = \{ x \} \quad \text{for } 1 \leq i < j \leq k.
\]

**Proof.** Let \( A \) be the set of the successors of \( x \) and let \( B \) be the set of predecessors of \( x \). \( A \cup \{ x \} \) and \( B \) are complementary in \( V(T) \) with \( |A \cup \{ x \}| \geq k \) and \( |B| \geq k \). It is well known that in this case, there exist \( k \) independent arcs \((x_i, y_i)\) with \( x_i \in A \cup \{ a \}, y_i \in B \), \( 1 \leq i \leq k \) (consequence of Menger’s theorem). Since \( x \) has no successors in \( B \), the vertices \( x_i \) are distinct from \( a \). The triangles \( T_i = x_i x y_i, 1 \leq i \leq k \), form a family satisfying \( V(T_i) \cap V(T_j) = \{ x \} \) for distinct \( i \) and \( j \).

As in the proof of Theorem 2.1, we show that there exist \( k \) cycles \( C_i \), \( 1 \leq i \leq k \), spanning \( V(T) \) and such that: \( V(T_i) \subseteq V(C_i) \) for \( 1 \leq i \leq k \) and \( V(C_i) \cap V(C_j) = \{ x \} \) for \( 1 \leq i \leq j \leq k \). So, the result is proved. \( \square \)

By using the reasoning of Theorem 2.1 one can prove that in a connected tournament \( T \), the maximum number of cycles with pairwise exactly \( x \) in common, is also the maximum number of cycles spanning \( V(T) \) with pairwise exactly \( x \) in common.

It is easy also to see that the maximum number of these cycles cannot exceed \( \min(d^+(a), d^-(a)) \).

The second result is:

**Theorem 3.2.** Let \( T \) be a \( k \)-connected tournament and \( x \) and \( y \) two vertices of \( T \). There exist \( k \) cycles \( C_i \), \( 1 \leq i \leq k \), spanning \( V(T) \) and such that:

\[
V(C_i) \cap V(C_j) = \{ x, y \} \quad \text{for } 1 \leq i < j \leq k.
\]

**Proof.** Without loss of generality, we may assume that \( y \) dominates \( x \).

By Menger’s theorem there exist \( k \) internal-disjoint paths \( P_i = x Q_i y \), \( 1 \leq i \leq k \). As in the proof of Theorem 2.1, we show that there exist \( k \) cycles \( C_i \), \( 1 \leq i \leq k \), spanning \( V(T) \) and such that: \( V(P_i) \subseteq V(C_i) \) for \( 1 \leq i \leq k \) and \( V(C_i) \cap V(C_j) = \{ x, y \} \) for \( 1 \leq i < j \leq k \). So, the result is proved. \( \square \)

For regular tournaments, we propose:

**Open problem 1:** Let \( T \) be a regular tournament of order \( n \).

(a) Is it true that for any vertex \( x \), there exist \( (n - 1)/2 \) triangles \( T_i \) spanning \( T \) and such that:

\[
V(T_i) \cap V(T_j) = \{ x \} \quad \text{for } 1 \leq i < j \leq \frac{n - 1}{2}.
\]

(b) Is there a vertex \( x \) such that there exist \( (n - 1)/2 \) triangles \( T_i \) spanning \( T \) and such that:

\[
V(T_i) \cap V(T_j) = \{ x \} \quad \text{for } 1 \leq i < j \leq \frac{n - 1}{2}.
\]

**Comments:** It is easy to prove that condition (a) is satisfied by regular tournaments of order \( n \) whose vertex strong connectivity is \( (n - 1)/2 \). We conjecture that the converse is true.
The Paley tournament (also known as the quadratic residue tournament) of order \( p \equiv 3 \pmod{4} \), with \( p \) prime, is the tournament whose vertices are the elements of \( \mathbb{Z}_p \) and whose arcs are the couples \((x, y)\) such that \( x - y \) is a nonzero quadratic residue.

It is the rotational tournament \( R(-q_1, \ldots, -q_m) \) where \( q_1, \ldots, q_m \) are the \( m = (p - 1)/2 \) nonzero quadratic residues modulo \( p \) (recall that a quadratic residue is an element \( q \) of \( \mathbb{Z}_p \) for which there exists \( x \in \mathbb{Z}_p \) such that \( q = x^2 \)).

We state that condition (a) and consequently condition (b) are true for Paley tournaments. Indeed, a Paley tournament \( T \) of order \( p \) is regular and consequently connected. Then for a vertex \( x \) of a Paley tournament \( T \) of order \( p \), there exists a triangle \( xbcx \).

For \( 1 \leq i \leq (p - 1)/2 \), it is easy to prove that \( T_i = x, x + q_i(b - x), x + q_i(c - x), x \) is a triangle.

It is also easy to prove that the \( (p - 1)/2 \) triangles \( T_i \) span \( V(T) \) and satisfy:

\[
V(T_i) \cap V(T_j) = \{x\} \quad \text{for} \quad 1 \leq i < j \leq \frac{p - 1}{2}.
\]

The proof is based on the fact that if \( q \) and \( r \) are quadratic residues, then \( qr \) is also a quadratic residue and that \( q \neq 0 \) is a quadratic residue if and only if \( -q \) is not a quadratic residue.

We are not certain that condition (a) is satisfied by any rotational tournament.

Acknowledgments

The author thanks the two referees for their substantial help.

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