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Solving fuzzy (stochastic) linear programming problems using superiority and inferiority measures

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Abstract

In this paper, the author presents a model to measure the superiority and inferiority of fuzzy numbers/fuzzy stochastic variables. Then, the new measures are used to convert the fuzzy (stochastic) linear program into the corresponding deterministic linear program. Numerical examples are provided to illustrate the effectiveness of the proposed method.

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1. Introduction

The main objective of this paper is to solve fuzzy (stochastic) linear programming problems more efficiently; the author proposes a new method to measure the superiority and inferiority of the triangular fuzzy numbers/fuzzy stochastic variables. Here, the triangular fuzzy numbers/fuzzy random variables are considered because the triangular form is the simplest type of fuzzy numbers/fuzzy random variables. Moreover, we can express and estimate many other types of fuzzy numbers with this simple form of fuzzy number. A triangular fuzzy number gives us the most important information about a fuzzy number: lower and upper bounds of the number and its most possible value. The other types of fuzzy numbers/fuzzy random variables are out of scope of this paper and will be considered in future research works. The new measures are used to evaluate any violation of the constraints to convert the fuzzy (stochastic) linear program into the standard deterministic linear program (LP). In literature, the general direction to handle the challenge of solving fuzzy (stochastic) linear programming problems is to defuzzify and/or derandomize fuzzy numbers/fuzzy stochastic variables [11,6,4,2,10,3,9,7,17,18,16,13,14,12,15,8]. The main difference between the developed methods is the way to defuzzify and/or derandomize fuzzy numbers/fuzzy stochastic variables. The most popular approach to convert the fuzzy linear program (FLP) into the conventional deterministic linear program (LP) is the method of ordering fuzzy numbers such as the area compensation method [5], the expected mid-point of fuzzy numbers [9], the grade of possibility and necessity [1], the signed distance method [3]. Similarly, the fuzzy stochastic
linear program (FSLP) has also been handled by defuzzifying and derandomizing fuzzy random variables in the sequential manner [17,18,16,8] or in the simultaneous manner [13,14,21,12,20,15,19]. In the sequential approaches, the defuzzifying process is performed first. Then the derandomizing process is done later. The defuzzifying process often utilizes ranking operations or the discretizing process of fuzzy sets via α-levels to defuzzify partly/completely fuzzy stochastic variables. The derandomizing process implements traditional stochastic programming techniques, for example, the chance constrained programming approach or the two-stage programming approach (referred to [18]). The main disadvantage of the sequential method is to create a large number of additional constraints and variables. In the simultaneous approaches, both defuzzifying and derandomizing processes have been performed at the same time by calculating the expected value of fuzzy random variables. Although the obtained deterministic linear program is quite simple, the computation process of the expected value is quite complicated and time consuming. In this paper, the proposed method still follows the sequential method but the new measures could reduce significantly number of additional constraints and variables in the obtained LP. In addition, the main point of the proposed approach is completely different with traditional defuzzifying approach using ranking method; we propose two new measures of inferiority and superiority without ranking fuzzy (stochastic) variables/numbers. These characteristics could be the main reason to simplify our solution process and make the conversion execute more efficiently. One more difference between the proposed approach and traditional ones is that the traditional methods defuzzify fuzzy numbers/fuzzy random variables by using the absolute values between the converted points of fuzzy numbers while the proposed approach uses the relative relationship between fuzzy numbers/fuzzy stochastic variables via the superiority/inferiority degrees.

In fuzzy stochastic linear programming problem, fuzziness and randomness could happen at the same time because the nature of two uncertainties is different. Fuzzy number represents the incomplete, imprecise information. The stochastic variable represents randomness or chance of events. Fuzzy numbers can vary randomly in real life. For example, the estimation of tolerance of machining products can be estimated as a fuzzy number. These values can vary from time to time, cycle-by-cycle in the production lot. Thus, tolerance values could be modeled as fuzzy random variables. In real-life, fuzzy stochastic linear programming arises in several situations. The parameters of linear programs such as the right-hand-sides (RHSs) and coefficients of the objective and constraints could be fuzzy random variables due to the fact that they depend on many factors. Thus, it is difficult to determine exactly the values of these parameters. Moreover, the factors, which are fluctuating due to uncertain environment, could make these parameters vary. These circumstances often happen in long-term planning, development strategies [17] engineering design [25], and financial modeling [26], in which the described conditions (objectives, constraints, coefficients) cannot be determined precisely and certainly. An illustrated example of fuzzy stochastic linear programming could be the case of the production planning problem. Consider the objective of minimizing total cost. This objective can be expressed as a fuzzy stochastic variable because total cost includes cost of inventory holding, materials, and operation. Production output depends on process parameters (for example, cutting speeds, feed rates) and machine running time. However, machine running time is fluctuating and hard to estimate precisely. In addition, available resources, demand, and constraints’ coefficients can also be modeled as fuzzy random variables because the vague perceptions with hard statistical data in several environmental conditions such as seasonal factors, market prices, and suppliers which contribute to constraint parameters. Another example is the case of preventive maintenance. Equipment breaks down from time to time, causing losses in production output. To reduce the number of breakdowns, preventive maintenance can be made. Preventive works include inspection, repair, and/or replacement components if necessary. These works cost money in terms of materials, wages, and loss of production due to downtime for preventive works. The length of the downtime is also uncertain due to the complexity of inspection, repair and/or replacement jobs and the skills of maintenance staff. The problem is to determine the preventive frequency such that total downtime, which includes downtime due to breakdowns and downtime due to preventive maintenance, is minimized subject to their associated costs would not exceed the available budget. Here, the running time of the machine is also uncertain. Therefore, we would rather consider these times and their associated costs as fuzzy random variables. These examples motivate the author to propose a new model for solving fuzzy stochastic linear programming problems. One more example of such applications will be discussed in Section 5. The next section presents some important concepts of fuzzy random variables, which will be used as a foundation of the proposed method. Then, the superiority and
inferiority measures of the triangular fuzzy numbers/fuzzy stochastic variables are developed in Section 3. In Section 4, the fuzzy linear programming problem will be transformed into the corresponding deterministic LP. Section 5 will present the solution method for fuzzy stochastic linear programming problem with the illustrated example by utilizing the superiority and inferiority model. Finally, Section 6 is the conclusion.

2. Fuzzy random variable

This section will summarize some important concepts of the fuzzy random variable, which will be used as a basis for the development of the new model. There are several types of definition of fuzzy random variable. Here, the attention is restricted to the concepts of Luhandjula [18] for the definition of the fuzzy random variable and its characteristics.

Definition 1 [18]. Consider a probability space \((\Omega, \mathcal{F}, P)\), a fuzzy random variable on this space is a fuzzy set-valued mapping \( \tilde{X} : \Omega \rightarrow F_0(\mathbb{R}) \)

\[ w \mapsto \tilde{X}_w \]

such that for any Borel set \( B \) [22] and for every \( z \in (0, 1) \)

\[ \tilde{X}_z^{-1}(B) = \{ w \in \Omega : \tilde{X}_w^z \subset B \} \in \mathcal{F} \]

where \( F_0(\mathbb{R}) \) and \( \tilde{X}_w^z \) stand for the set of fuzzy numbers with compact supports and the \( z \)-level set of the fuzzy set \( \tilde{X}_w \), respectively.

Theorem 1 [18]. \( \tilde{X} \) is a fuzzy random variable if and only if given \( w \in \Omega \), \( \tilde{X}_w^z \) is a random interval \( \forall z \in (0, 1] \).

3. Superiority and inferiority between triangular fuzzy numbers or fuzzy random variables

The development of the new model will be started with the definition of a set of triangle fuzzy numbers (T-numbers).

Definition 2 [23]. Let \( \tilde{T} \) be a family of triangular fuzzy numbers. It is defined as follows:

\[ \tilde{T} = \{ \tilde{\delta} = (\delta, a, b), a, b \geq 0 \} \quad \text{and} \quad \mu_\delta(x) = \begin{cases} \max \left( 0, 1 - \frac{\delta - x}{a} \right) & \text{if } x \leq \delta, \ a > 0 \\ 1 & \text{if } a = 0 \text{ and/or } b = 0 \\ \max \left( 0, 1 - \frac{\delta - x}{b} \right) & \text{if } x > \delta, \ b > 0 \\ 0 & \text{otherwise} \end{cases} \quad (1) \]

where the scalars \( a, b \geq 0 \ (a, b \in \mathbb{R}) \) are called the left and right spreads, respectively. This type of fuzzy numbers is quite popular and allows quantification of quite different types of information. A crisp number \( \delta \in \mathbb{R} \) can be illustrated as a triangular fuzzy number \( \tilde{\delta} = (\delta, 0, 0) \).

In real life, objects are often compared by their attributes. The comparison of each attribute is done based on a certain measurement to indicate the difference between objects. In common sense, one object is better than the others if the best value of its attribute is better than the best value of the same attribute of the other objects. In another way, the worst value of the attribute may also be used to compare two objects. For example, to compare the ability to complete a job of two operators \( A \) and \( B \), the shortest time is used. The performance of operator \( A \) is better than the operator \( B \) if the shortest time of \( A \) is shorter than \( B \)'s shortest time. Using this idea, two new comparison measures of superiority and inferiority between fuzzy numbers are defined as follows. Starting from the \( \alpha \)-cut of two fuzzy numbers \( \tilde{P} \) and \( \tilde{Q} \) (see Fig. 1), we have
If \( \bar{P}_x \geq \bar{Q}_x \) then \( \sup \{ s : \mu_{\bar{P}}(s) \geq \alpha \} - \sup \{ t : \mu_{\bar{Q}}(t) \geq \alpha \} \geq 0 \).

This amount is the superiority of \( \bar{P} \) over \( \bar{Q} \). Therefore, the total superiority of \( \bar{P} \) over \( \bar{Q} \) is the area of \( \bar{P} \) larger than \( \bar{Q} \). Mathematically, this area is

\[
S = \int_0^1 \{ \sup \{ s : \mu_{\bar{P}}(s) \geq \alpha \} - \sup \{ t : \mu_{\bar{Q}}(t) \geq \alpha \} \} \, dx \geq 0 \quad \text{if } \bar{P} \geq \bar{Q}
\]

or

\[
S = 0; \quad \text{otherwise}
\]

Similar result can also be obtained for the inferior degree of \( \bar{P} \) to \( \bar{Q} \)

\[
I = \int_0^1 \{ \inf \{ s : \mu_{\bar{P}}(s) \geq \alpha \} - \inf \{ t : \mu_{\bar{Q}}(t) \geq \alpha \} \} \, dx \geq 0 \quad \text{if } \bar{P} \geq \bar{Q}
\]

or

\[
I = 0; \quad \text{otherwise}
\]

Generally, we have the following definition of the superiority and inferiority between general fuzzy numbers.

**Definition 3.** Let \( \bar{P} \) and \( \bar{Q} \) be two fuzzy numbers. Then, the superiority of \( \bar{P} \) over \( \bar{Q} \) is defined as

\[
S(\bar{P}, \bar{Q}) = \int_0^1 \{ \sup \{ s : \mu_{\bar{P}}(s) \geq \alpha \} - \sup \{ t : \mu_{\bar{Q}}(t) \geq \alpha \} \} \, dx
\]  

(2)

Analogously, we define the inferiority of \( \bar{P} \) to \( \bar{Q} \) as

\[
I(\bar{P}, \bar{Q}) = \int_0^1 \{ \inf \{ s : \mu_{\bar{P}}(s) \geq \alpha \} - \inf \{ t : \mu_{\bar{Q}}(t) \geq \alpha \} \} \, dx
\]  

(3)

If we consider to measure the superiority and inferiority between two triangular fuzzy numbers \( \bar{P} = (u, a, b) \), \( \bar{Q} = (v, c, d) \in \mathbb{T} \), then we have

**Theorem 2.** Consider two fuzzy numbers \( \bar{P} = (u, a, b) \), \( \bar{Q} = (v, c, d) \in \mathbb{T} \), and if \( \bar{P} \leq \bar{Q} \) then the superiority of \( \bar{Q} \) over \( \bar{P} \) is

\[
S(\bar{Q}, \bar{P}) = v - u + \frac{d - b}{2}
\]  

(4)

and inferiority of \( \bar{P} \) to \( \bar{Q} \) is

\[
I(\bar{P}, \bar{Q}) = v - u - \frac{e - a}{2}
\]  

(5)
Proof. The \( \alpha \)-cut of \( \widetilde{P} \) and \( \widetilde{Q} \) are (see Fig. 2)
\[
\widetilde{P}_\alpha = [P^L_\alpha(x), P^U_\alpha(x)] = [u - a(1 - \alpha), u + b(1 - \alpha)] \\
\widetilde{Q}_\alpha = [Q^L_\alpha(x), Q^U_\alpha(x)] = [v - c(1 - \alpha), v + d(1 - \alpha)]
\]
The superiority of fuzzy number \( \widetilde{Q} \) over \( \widetilde{P} \) at \( \alpha \)-level is
\[
S(\widetilde{Q}, \widetilde{P})_\alpha = Q^U_\alpha - P^U_\alpha = v - u + (c - a)(1 - \alpha)
\]
Therefore,
\[
S(\widetilde{Q}, \widetilde{P}) = \int_0^1 (v - u) + (c - a)(1 - \alpha) \, d\alpha = \left[ (v - u)\alpha + (c - a) - \frac{\alpha^2}{2} \right]_0^1 = (v - u) + \frac{(c - a)}{2}.
\]
Similarly, the inferiority of fuzzy number \( \widetilde{P} \) to \( \widetilde{Q} \) at \( \alpha \)-level is
\[
I(\widetilde{P}, \widetilde{Q})_\alpha = Q^L_\alpha - P^L_\alpha = v - u - a(1 - \alpha)
\]
Thus,
\[
I(\widetilde{P}, \widetilde{Q}) = \int_0^1 (v - u) - a(1 - \alpha) \, d\alpha = \left[ (v - u)\alpha - a - \frac{\alpha^2}{2} \right]_0^1 = (v - u) - \frac{a}{2}.
\]

From these results, we have the following property:

Property 1
\[
S(\tilde{f}_j(a_1), \tilde{f}_j(a_2)) \neq I(\tilde{f}_j(a_1), \tilde{f}_j(a_2))
\] (6)

Similar to the fuzzy numbers, we define the superiority and inferiority between fuzzy stochastic variables \( \widetilde{P} \) and \( \widetilde{Q} \) from their \( \alpha \)-cut intervals. For any \( \alpha \in (0, 1] \), consider the \( \alpha \)-cut of the fuzzy sets \( \widetilde{P}_w, \widetilde{Q}_w \)
\[
\widetilde{P}_w^\alpha = [P^L_w(x), P^U_w(x)], \quad \widetilde{Q}_w^\alpha = [Q^L_w(x), Q^U_w(x)]
\]
By Theorem 1, these intervals are random intervals. If \( \widetilde{P}_w, \widetilde{Q}_w \) are \( T \)-numbers, the \( \alpha \)-cut of \( \widetilde{P}_w, \widetilde{Q}_w \) are
\[
\widetilde{P}^\alpha_w = [P^L_w(x), P^U_w(x)] = [u(w) - a(w)(1 - \alpha), u(w) + b(w)(1 - \alpha)] \\
\widetilde{Q}^\alpha_w = [Q^L_w(x), Q^U_w(x)] = [v(w) - c(w)(1 - \alpha), v(w) + d(w)(1 - \alpha)]
\]
Thus, the superiority of \( \widetilde{Q} \) over \( \widetilde{P} \) at \( \alpha \)-level is
\[
S(\widetilde{Q}, \widetilde{P})_\alpha = Q^U_w - P^U_w \Rightarrow S(\widetilde{P}, \widetilde{Q})_\alpha = P^U_w - Q^U_w
\]
and the inferiority of \( \widetilde{P} \) to \( \widetilde{Q} \) at \( \alpha \)-level is \( I(\widetilde{P}, \widetilde{Q})_\alpha = Q^L_w - P^L_w \).

Theorem 3. Consider two triangular fuzzy random variables \( \widetilde{P} \leq \widetilde{Q} \), the superiority of fuzzy random variable \( \widetilde{P} \) over \( \widetilde{Q} \) is
\[
S(\widetilde{Q}, \widetilde{P}) = v(w) - u(w) + \frac{d(w) - b(w)}{2}
\] (7)

![Fig. 2. Superiority and inferiority between \( \widetilde{P} \) and \( \widetilde{Q} \) at \( \alpha \)-level.](image-url)
and the inferiority of fuzzy random variable $P$ to $Q$ is

$$I(P, Q) = v(w) - u(w) + \frac{a(w) - c(w)}{2}$$

(8)

Proof

$$S(Q, P) = Q(w) - P(w) = (v(w) - u(w)) + (a(w) - b(w))(1 - z)$$

$$\Rightarrow S(Q, P) = \int_0^1 (v(w) - u(w)) + (a(w) - b(w))(1 - z) \, dz = v(w) - u(w) + \frac{d(w) - b(w)}{2}$$

Similarly, we also have

$$I(P, Q) = P(w) - Q(w) = (v(w) - u(w)) + (a(w) - c(w))(1 - z)$$

$$\Rightarrow I(P, Q) = \int_0^1 (v(w) - u(w)) + (a(w) - c(w))(1 - z) \, dz = v(w) - u(w) + \frac{a(w) - c(w)}{2}$$

The main difference of the proposed approach with traditional ones in solving the conventional Fuzzy Linear Programming problems (for example, the signed distance method [24]) or Fuzzy Stochastic Programming problems (for example, Liu’s approaches and Luhandjula’s methods), is the defuzzifying method by using the superiority and/or inferiority degree between two fuzzy numbers/fuzzy stochastic variables. The traditional methods defuzzify fuzzy numbers/fuzzy random variables by using the absolute values between the converted points of fuzzy numbers while the proposed approach uses the relative relationship between fuzzy numbers/fuzzy stochastic variables via the superiority/inferiority degrees. In addition, the inferior and superior measures could help our conversion FLP and/or FSLP into LP to avoid ranking fuzzy (stochastic) numbers/variables. This fact could help to reduce significantly number of constraints in the obtained LP model. □

4. Fuzzy linear programming

4.1. Fuzzy linear programming with fuzzy constraints

First, we consider the following fuzzy linear program:

(P1)  \[
\begin{align*}
\text{Max} & \quad cx \\
\text{s.t.} & \quad \sum_{j=1}^n (\tilde{a}_{ij})x_j \leq \tilde{b}_i, \quad i = 1, 2, \ldots, m \\
& \quad x_j \geq 0, \quad j = 1, 2, \ldots, n
\end{align*}
\]

This problem requires maximizing the objective subject to the superiority of right-hand sides (RHS) over the left-hand sides (LHS) of constraints and the inferiority of LHS to the RHS. This requirement corresponds to maximize the objective function with paying a penalty for any violation of superiority of RHS over LHS and inferiority of the LHS to the RHS. Here, we include both superiority and inferiority to the equivalent model because of Property 1. It means that the problem (P1) is reformulated as follows:

(P2)  \[
\begin{align*}
\text{Max} & \quad cx - \left[ p_i S_i \left( \sum_{j=1}^n (\tilde{a}_{ij})x_j, \tilde{b}_i \right) + q_i I_i \left( \tilde{b}_i, \sum_{j=1}^n (\tilde{a}_{ij})x_j \right) \right]; \quad i = 1, 2, \ldots, m \\
\text{s.t.} & \quad x_j \geq 0; \quad j = 1, 2, \ldots, n
\end{align*}
\]

where $p_i > 0$, $q_i > 0$ are penalty coefficients.

It is noticed that the penalty costs $p_i$ and $q_i$ are basically determined without any rule. The larger value of penalty costs will be the strict allowance for violation is required. Therefore, depending of the application situation, the decision maker can select the suitable value for penalty costs. In our case, we use a rule of thumb to select the suitable penalty values. In the most optimistic case, the decision maker can select penalty costs $p_i$ and $q_i$ as the minimum value of $c_j$. In the most pessimistic case, the penalty costs $p_i$ and $q_i$ are selected as the maximum value of $c_j$.  

\[1982 \quad N. \text{Van Hop} \quad \text{Information Sciences 177 (2007) 1977–1991}\]
Example 1. In a competitive business environment, a company produces two products I and II. Product I is manufactured approximately 1 h by both machines A and B. Product II is manufactured approximately 2 h and 1 h by machines A and B, respectively. Subject to many factors such as machine breakdown, waiting for material, bottleneck, the available time of machine A is approximately 4 h and 2 h for machine B. In addition, product I is needed to be mixed after processing on both machines A and B. The estimated mixing time for product I is 2 h. The available time for mixing is approximately 3 h. The prices for product I and II are 2 and 1 Dollar(s) per kilogram, respectively. The management of the company wants to determine how much to produce for each product to maximize the total revenue.

Let \( x_1 = \) the amount of product I to be produced.
\( x_2 = \) the amount of product II to be produced.

As results, we have the following FLP problem:

Maximize \( 2x_1 + x_2 \)

Subject to

\[
\begin{align*}
\tilde{1}x_1 + \tilde{2}x_2 & \leq \tilde{4} \\
\tilde{1}x_1 + \tilde{1}x_2 & \leq \tilde{2} \\
\tilde{2}x_1 & \leq \tilde{3} \\
x_1, x_2 & \geq 0
\end{align*}
\]

Assume all fuzzy numbers are in the form of \( \tilde{m} = (m, 0.5, 0.5) \in \tilde{T} \). Convert this example to the form of linear program (P2), we have

Maximize \( 2x_1 + x_2 - 10 \ast (\lambda_1 + \lambda_2 + \lambda_3) - 10 \ast (\lambda_4 + \lambda_5 + \lambda_6) \)

Subject to

\[
\begin{align*}
(x_1 + 2x_2 - 4) + \frac{(0.5x_1 + 0.5x_2 - 0.5)}{2} &= \lambda_1 \\
(x_1 + x_2 - 2) + \frac{(0.5x_1 + 0.5x_2 - 0.5)}{2} &= \lambda_2 \\
(2x_1 - 3) + \frac{(0.5x_1 - 0.5)}{2} &= \lambda_3 \\
(x_1 + 2x_2 - 4) + \frac{(0.5 - 0.5x_1 - 0.5x_2)}{2} &= \lambda_4 \\
(x_1 + x_2 - 2) + \frac{(0.5 - 0.5x_1 - 0.5x_2)}{2} &= \lambda_5 \\
(2x_1 - 3) + \frac{(0.5 - 0.5x_1)}{2} &= \lambda_6 \\
\lambda_k & \geq 0 \quad \text{for} \quad k = 1, 2, \ldots, 6
\end{align*}
\]

The solution of this linear program is \( x_1 = 1.57, x_2 = 1.47 \). Then, \( X^* = (1.57, 1.47) \) may be considered as the final solution of the original problem.

4.2. Fuzzy linear programming with fuzzy constraints and objective function

Next, we extend our consideration to the more general case of fuzzy objective function as follows:

\[
\begin{cases}
\text{Max} \quad \sum_{j=1}^{n} \tilde{c}_j x_j \\
\text{s.t.} \quad \sum_{j=1}^{n} (\tilde{a}_{ij}) x_j \leq \tilde{b}_i; \quad i = 1, 2, \ldots, m \\
x_j \geq 0; \quad j = 1, 2, \ldots, n
\end{cases}
\]

(P3)
The equivalent form of (P3) will be (similar to the conversion of [7])

\[
\begin{align*}
\text{(P4)} & \quad \begin{cases}
\text{Max} & \quad \theta \\
\text{s.t.} & \quad \sum_{j=1}^{n} \tilde{c}_j x_j \geq \theta \\
& \quad \sum_{j=1}^{n} (\tilde{a}_{ij}) x_j \leq \tilde{b}_i; \quad i = 1, 2, \ldots, m \\
& \quad x_j \geq 0; \quad j = 1, 2, \ldots, n
\end{cases} \\
\iff & \quad \begin{cases}
\text{Max} & \quad \theta - p_0 S \left( \theta, \sum_{j=1}^{n} \tilde{c}_j x_j \right) - q_0 I \left( \sum_{j=1}^{n} \tilde{c}_j x_j, \theta \right) - p_1 S \left( \sum_{j=1}^{n} (\tilde{a}_{ij}) x_j, \tilde{b}_i \right) - q_1 I \left( \tilde{b}_i, \sum_{j=1}^{n} (\tilde{a}_{ij}) x_j \right) \\
& \quad \text{s.t.} \quad x_j \geq 0; \quad j = 1, 2, \ldots, n
\end{cases}
\end{align*}
\]

It is noticed that crisp variable \( \theta \) could be fuzzified as \( (\theta, 0, 0) \) and (P5) is the standard LP.

**Example 2**

Maximize \( \tilde{2} x_1 + \tilde{1} x_2 \)

Subject to

\[
\begin{align*}
1 \tilde{x}_1 + 3 \tilde{x}_2 & \leq 4 \\
\tilde{1} x_1 + \tilde{1} x_2 & \leq 2 \\
2 \tilde{x}_1 & \leq 3 \\
x_1, x_2 & \geq 0
\end{align*}
\]

Assume that all fuzzy numbers are in the form of \( \tilde{m} = (m, 0.5, 0.5) \in \tilde{T} \). Penalty values for three constraints are 3, 2, 3 per unit of violation and 10 for the objective violation, respectively. The equivalent model (P5) of this example will be

Maximize \( \theta - 10\theta_1 - 2\theta_2 - 3(\lambda_1 + \lambda_4) - 2(\lambda_2 + \lambda_3) - 3(\lambda_3 + \lambda_6) \)

Subject to

\[
\begin{align*}
\theta - 2x_1 - x_2 + \frac{(0 - 0.5x_1 - 0.5x_2)}{2} &= \theta_1 \\
\theta - 2x_1 - x_2 + \frac{(0.5x_1 + 0.5x_2 - 0)}{2} &= \theta_2 \\
(x_1 + 3x_2 - 4) + \frac{(0.5x_1 + 0.5x_2 - 0.5)}{2} &= \lambda_1 \\
(x_1 + x_2 - 2) + \frac{(0.5x_1 + 0.5x_2 - 0.5)}{2} &= \lambda_2 \\
(2x_1 - 3) + \frac{(0.5x_1 - 0.5)}{2} &= \lambda_3 \\
(x_1 + 3x_2 - 4) + \frac{(0.5 - 0.5x_1 - 0.5x_2)}{2} &= \lambda_4 \\
(x_1 + x_2 - 2) + \frac{(0.5 - 0.5x_1 - 0.5x_2)}{2} &= \lambda_5 \\
(2x_1 - 3) + \frac{(0.5 - 0.5x_1)}{2} &= \lambda_6 \\
\lambda_k & \geq 0; \quad k = 1, 2, \ldots, 6 \\
\theta, \theta_1, \theta_2 & \geq 0
\end{align*}
\]

The optimal solution is \( X^* = (1.57, 0.94) \).
5. Fuzzy stochastic linear programming problem

Consider the following fuzzy stochastic linear program:

\[
\text{(P6)} \quad \begin{align*}
\text{Max} & \quad cx \\
\text{s.t.} & \quad \sum_{j=1}^{n} (\tilde{a}_{ij})_w x_j \leq (\tilde{b}_i)_w; \quad i = 1, 2, \ldots, m \\
x_j & \geq 0, \quad w \in \Omega, \quad k = 1, 2, \ldots, l
\end{align*}
\]

where \( c \) is a \( 1 \times n \) matrix, \( A \), \( b \) are \( m \times n \) and \( m \times 1 \) matrices of fuzzy random variable constraint coefficients defined on a probability space \((\Omega, F, P)\). Using the concept of superiority and inferiority degrees, we reformulate problem (P6) by paying penalty cost for any violation of constraints due to both variations of fuzziness and randomness. The corresponding deterministic program for the problem (P6) is

\[
\text{(P7)} \quad \begin{align*}
\text{Max} & \quad cx - p_E \left[ \sum_{i=1}^{m} \lambda_i(w) \right] - q_E \left[ \sum_{i=m+1}^{2m} \lambda_i(w) \right] \\
\text{s.t.} & \quad S_i \left( \sum_{j=1}^{n} (\tilde{a}_{ij})_w x_j, (\tilde{b}_i)_w \right) = \lambda_i(w); \quad i = 1, 2, \ldots, m \\
& \quad I_i \left( (\tilde{b}_i)_w, \sum_{j=1}^{n} (\tilde{a}_{ij})_w x_j \right) = \lambda_i(w); \quad i = m + 1, m + 2, \ldots, 2m \\
x_j & \geq 0, \quad w \in \Omega
\end{align*}
\]

where \( E \) denotes the expected value. Comparing to the approach of Luhandjula [18], the proposed method has used the superiority and inferiority to defuzzify the problem instead of using discrete interval values of \( \alpha \)-levels of fuzzy numbers. Thus, the number of constraints in the obtained deterministic model (P7) reduces drastically. To illustrate the efficiency of the proposed method, we reconsider Luhandjula’s example [18].

Example 3

Maximize \( 3x_1 + 2x_2 \)
Subject to
\[
(\tilde{A}, \tilde{b}) = \begin{cases}
(\tilde{A}_{w1}, \tilde{b}_{w1}) = \left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 4 \end{pmatrix} \right\} \\
(\tilde{A}_{w2}, \tilde{b}_{w2}) = \left\{ \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 5 \\ 4 \end{pmatrix} \right\}
\end{cases}
\]

\( p(w_1) = 0.25 \), \( p(w_2) = 0.75 \); \( \Omega = (w_1, w_2) \)

and \( \tilde{m} \) denotes a triangular fuzzy number with the following membership function:

\[
\mu_{\tilde{m}}(x) = \begin{cases}
0; & x \leq m - 1 \\
x - (m - 1); & (m - 1) < x \leq m \\
-x + (m + 1); & m < x \leq (m + 1) \\
0; & (m + 1) < x
\end{cases}
\]

or we can express fuzzy number \( \tilde{m} \in \tilde{T} \) as \( \tilde{m} = (m, 1, 1) \).
The solution is $x_1 = 4, x_2 = 1$. The objective value of this solution is $3x_1 + 2x_2 = 3 \times 4 + 2 \times 1 = 14$. The Luhandjula’s solution is $x_1 = 1.34, x_2 = 1$ and its objective value is $3x_1 + 2x_2 = 3 \times 1.34 + 2 \times 1 = 6.02$.

From the obtained results, it is clear that the proposed method gives better solution than Luhandjula’s method. In addition, the proposed methodology is also superior considering the amount of calculations and complexities embedded in other methodology, especially, the Luhandjula’s method because of a few number of constraints and the simplicity of conversion. Moreover, the proposed method compares fuzzy stochastic variables based on their relative relationships. This comparison will relax the constraints from the deterministic ones. Thus, the larger spreads of fuzzy numbers, the more relaxation amount. On the other hand, the Luhandjula’s method discretizes fuzzy stochastic variables by $z$-levels. This discretizing process creates more constraints, reduces the accuracy of the conversion, and increases the complexity of the model. The number of $z$-levels is depended on the decision maker perspective. This fact makes Luhandjula’s method taking more time to compute and less accuracy to convert. Therefore, the obtained results of the proposed method will be better than the Luhandjula’s results. However, the proposed method converts completely the fuzzy stochastic linear program into the deterministic linear program. The proposed method “destroys”
completely fuzziness characteristics to maximize the objective value. The proposed method has obtained the higher objective value by sacrificing parts of fuzziness embedded in the model. On the other hand, the Luhandjula’s method still maintains partly the fuzziness of the system. The discretizing process via \( \alpha \)-cuts still associates each element of the set with an alpha value (membership). It is a different form of presenting the fuzzy sets (by some discrete intervals). Therefore, the proposed method has to trade-off between obtaining a more maximized objective value and maintaining the fuzziness of the system.

To extend our discussion, we present in detail the product mix problem to illustrate the application of the proposed method. In fact, there are many more cases needed to formulate as fuzzy stochastic linear programming problems since most of information are not ready, imprecise, difficult to measure and/or fluctuating by the time in real-life.

**Example 4 (Product mix problem).** In manufacturing situation, we often have to determine how much of each type of product to produce (the production mix) with the limitations in resources available that satisfies a certain objective, for example, maximizing profit. It is not enough just to produce the largest possible quantity of the most profitable product that current resources permit. When there is a large number of products and resources, the interactive nature of these elements makes the consequences of selecting a particular product mix uncertain. Here, a certain ice-cream manufacturer (IceCo, Inc.) is considered. IceCo produces various types of ice-cream. It has both production and marketing capabilities to produce and sell all or a mixture of the following products: (1) ice-cream cone; (2) ice-cream box, 250 g weight; (3) ice-cream box 500 g weight.

For production of these products, the following materials are used: (1) fresh milk, (2) sugar, (3) flour. Packaging materials are out of consideration. Usage of these raw materials varies for each product is hard to measure exactly due to many factors. The common factors are transportation loss, season, working conditions (temperature, humidity, cleanliness), processing technology and processing parameters. All materials are measured by weight in kilogram and estimated as an approximate number in the form of fuzzy numbers \( \tilde{\mu} = (\mu, 10, 10) \). The estimated raw materials consumptions are fluctuating by time due to processing parameters, supplying risks and quality of imported raw materials, etc. For example, the probability of high productivity process is 0.6 and low productivity process is 0.4. Details of raw material usage are estimated in Table 1.

The following major facilities are used for production: (1) cooking vats (2) mixing machines, (3) frozen machines, (4) packaging machines. All facilities are expressed in ton-hours. These machines are limited by both tonnage weight capacity and hours of operation. Operations of these facilities are independent of the product weight. Labor is also an input in production. Indirect labor, such as administrative, supervisory, and maintenance, is excluded from consideration because the associated costs are fixed in nature. Direct labor required for manufacturing the products, such as labor required in packaging, is considered and details of the requirements are included in the same table. The uncertainty in both aspects of incomplete and fluctuation of information come from the fact that machine downtime due to many reasons and labor absentee and skills are not uniform. The capacity of facilities is expressed in the. Details of the facility usage are presented in form of \( \tilde{\mu} = (\mu, 0.1, 0.1) \) (see Table 2).

IceCo is in the process of preparing a monthly plan for the next manufacturing period. The company undertakes periodic planning, taking into consideration various alternatives available. The fluctuation of raw materials availability comes from the distribution uncertainty as well as inventory control’s risks. Available raw materials and facilities are presented in Table 3 in the form of fuzzy number \( \tilde{\mu} = (\mu, 50, 50) \).

<table>
<thead>
<tr>
<th>Materials required (kg)</th>
<th>Probability = 0.6</th>
<th>Probability = 0.4</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Ice-cream cone (for 1000 pieces)</td>
<td>Ice-cream 250 g</td>
</tr>
<tr>
<td>Milk</td>
<td>35</td>
<td>75</td>
</tr>
<tr>
<td>Sugar</td>
<td>62</td>
<td>25</td>
</tr>
<tr>
<td>Flour</td>
<td>37</td>
<td>40</td>
</tr>
</tbody>
</table>
Prices of end products, raw materials and labor cost are also given in Table 4.
There are two sets of constraints, raw material availability and facility capacity constraints. Based on the material consumption of each product, facility usage, and the resource availability, the constraints can be developed for each material and facility.

The problem is to find the optimal product mix under the technical, raw material, and market considerations as discussed such that the total revenue is maximized. The linear programming model can be formulated with the following variables:

\[ x_1 = \text{Ice-cream cone to be produced (100 pcs)} \]
\[ x_2 = \text{Ice-cream box of 250 g to be produced (pcs)} \]
\[ x_3 = \text{Ice-cream box of 500 g to be produced (pcs)} \]

Maximize \[ Z = 0.375x_1 + 1.5x_2 + 4x_3 \]

Subject to

Material constraints: \[ \tilde{A}_1 x \leq \tilde{b}_1 \]

where

\[ (\tilde{A}_{1,w_1}, \tilde{b}_{1,w_1}) = \left\{ \begin{array}{ccc} 35 & 75 & 87 \\ 62 & 25 & 50 \\ 37 & 40 & 48 \end{array} \right\}, \quad \left( \begin{array}{c} 10,000 \\ 18,000 \\ 22,000 \end{array} \right) \]

\[ (\tilde{A}_1, \tilde{b}_1) = \left\{ \begin{array}{ccc} 48 & 62 & 84 \\ 67 & 33 & 61 \\ 45 & 56 & 48 \end{array} \right\}, \quad \left( \begin{array}{c} 12,000 \\ 20,000 \\ 20,000 \end{array} \right) \]

\[ p(w_1) = 0.6, \quad p(w_2) = 0.4; \quad \Omega = (w_1, w_2) \]
Table 4

<table>
<thead>
<tr>
<th>Item (unit)</th>
<th>Price ($)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ice-cream cone (1000 pcs.)</td>
<td>0.375</td>
</tr>
<tr>
<td>Ice-cream box, 250 g (pcs.)</td>
<td>1.50</td>
</tr>
<tr>
<td>Ice-cream box, 500 g (pcs.)</td>
<td>4.00</td>
</tr>
</tbody>
</table>

Facility constraints: \( \tilde{A}_{2x} \leq \tilde{b}_{2} \)

\[
(\tilde{A}_{2,1}, \tilde{b}_{2,1}) = \left\{ \begin{array}{c}
0.5 \\
0.25 \\
0.24 \\
0.84 \\
0.84 \\
0.25 \\
0.84 \\
0.24 \\
0.84 \\
0.25 \\
end{array} \right\}, \quad (\tilde{A}_{2,2}, \tilde{b}_{2,2}) = \left\{ \begin{array}{c}
0.14 \\
0.14 \\
0.12 \\
0.68 \\
0.14 \\
0.13 \\
0.12 \\
0.68 \\
0.14 \\
0.13 \\
end{array} \right\}
\]

\[
p(w_1) = 0.6, \quad p(w_2) = 0.4; \quad \Omega = (w_1, w_2)
\]

Nonnegative integer variables constraints: \( x \geq 0 \) and integer, \( w \in \Omega \).

Applying the proposed method, we have the following new model:

Maximize \( Z = 0.375x_1 + 1.5x_2 + 4x_3 - 1000 \left( 0.6 \sum_{j=1}^{7} \lambda_{x_j}^{w_1} + 0.4 \sum_{j=1}^{7} \lambda_{x_j}^{w_2} \right) - 1000 \left( 0.6 \sum_{j=1}^{7} \lambda_{x_j}^{w_1} + 0.4 \sum_{j=1}^{7} \lambda_{x_j}^{w_2} \right) \)

Subject to

\[
S(35x_1 + 75x_2 + 87x_3, 10, 000) = \lambda_{s_1}^{w_1} \iff \sum_{j=1}^{7} \lambda_{x_j}^{w_1} = 35x_1 + 75x_2 + 87x_3 - 10, 000 + \frac{10x_1 + 10x_2 + 10x_3 - 50}{2} = \lambda_{s_2}^{w_1}
\]

\[
S(62x_1 + 25x_2 + 50x_3, 18, 000) = \lambda_{s_2}^{w_1} \iff \sum_{j=1}^{7} \lambda_{x_j}^{w_1} = 62x_1 + 25x_2 + 50x_3 - 18, 000 + \frac{10x_1 + 10x_2 + 10x_3 - 50}{2} = \lambda_{s_2}^{w_1}
\]

\[
S(37x_1 + 40x_2 + 48x_3, 22, 000) = \lambda_{s_3}^{w_1} \iff \sum_{j=1}^{7} \lambda_{x_j}^{w_1} = 37x_1 + 40x_2 + 48x_3 - 22, 000 + \frac{10x_1 + 10x_2 + 10x_3 - 50}{2} = \lambda_{s_3}^{w_1}
\]

\[
S(48x_1 + 62x_2 + 84x_3, 12, 000) = \lambda_{s_4}^{w_1} \iff \sum_{j=1}^{7} \lambda_{x_j}^{w_1} = 48x_1 + 62x_2 + 84x_3 - 12, 000 + \frac{10x_1 + 10x_2 + 10x_3 - 50}{2} = \lambda_{s_4}^{w_1}
\]

\[
S(67x_1 + 33x_2 + 61x_3, 20, 000) = \lambda_{s_5}^{w_1} \iff \sum_{j=1}^{7} \lambda_{x_j}^{w_1} = 67x_1 + 33x_2 + 61x_3 - 20, 000 + \frac{10x_1 + 10x_2 + 10x_3 - 50}{2} = \lambda_{s_5}^{w_1}
\]

\[
S(45x_1 + 56x_2 + 48x_3, 20, 000) = \lambda_{s_6}^{w_1} \iff \sum_{j=1}^{7} \lambda_{x_j}^{w_1} = 45x_1 + 56x_2 + 48x_3 - 20, 000 + \frac{10x_1 + 10x_2 + 10x_3 - 50}{2} = \lambda_{s_6}^{w_1}
\]

\[
S(0.5x_1 + 0.2x_2 + 0.425x_3, 120) = \lambda_{s_7}^{w_1} \iff \sum_{j=1}^{7} \lambda_{x_j}^{w_1} = 0.5x_1 + 0.2x_2 + 0.425x_3 - 120 + \frac{0.1x_1 + 0.1x_2 + 0.1x_3 - 50}{2} = \lambda_{s_7}^{w_1}
\]

\[
S(0.25x_1 + 0.36x_2 + 0.48x_3, 150) = \lambda_{s_8}^{w_1} \iff \sum_{j=1}^{7} \lambda_{x_j}^{w_1} = 0.25x_1 + 0.36x_2 + 0.48x_3 - 150 + \frac{0.1x_1 + 0.1x_2 + 0.1x_3 - 50}{2} = \lambda_{s_8}^{w_1}
\]

\[
S(0.24x_1 + 0.84x_2 + 0.74x_3, 160) = \lambda_{s_9}^{w_1} \iff \sum_{j=1}^{7} \lambda_{x_j}^{w_1} = 0.24x_1 + 0.84x_2 + 0.74x_3 - 160 + \frac{0.1x_1 + 0.1x_2 + 0.1x_3 - 50}{2} = \lambda_{s_9}^{w_1}
\]

\[
S(0.84x_1 + 0.37x_2 + 0.14x_3, 500) = \lambda_{s_{10}}^{w_1} \iff \sum_{j=1}^{7} \lambda_{x_j}^{w_1} = 0.84x_1 + 0.37x_2 + 0.14x_3 - 500 + \frac{0.1x_1 + 0.1x_2 + 0.1x_3 - 50}{2} = \lambda_{s_{10}}^{w_1}
\]

\[
S(0.14x_1 + 0.6x_2 + 0.74x_3, 100) = \lambda_{s_{11}}^{w_1} \iff \sum_{j=1}^{7} \lambda_{x_j}^{w_1} = 0.14x_1 + 0.6x_2 + 0.74x_3 - 100 + \frac{0.1x_1 + 0.1x_2 + 0.1x_3 - 50}{2} = \lambda_{s_{11}}^{w_1}
\]

\[
S(0.1x_1 + 0.33x_2 + 0.47x_3, 200) = \lambda_{s_{12}}^{w_1} \iff \sum_{j=1}^{7} \lambda_{x_j}^{w_1} = 0.1x_1 + 0.33x_2 + 0.47x_3 - 200 + \frac{0.1x_1 + 0.1x_2 + 0.1x_3 - 50}{2} = \lambda_{s_{12}}^{w_1}
\]
S(0.12x_1 + 0.64x_2 + 0.78x_3, 150) = \lambda_{62}^S \iff 0.12x_1 + 0.64x_2 + 0.78x_3 - 150 + \frac{0.1x_1 + 0.1x_2 + 0.1x_3 - 50}{2} = \lambda_{62}^S \\
S(0.68x_1 + 0.23x_2 + 0.41x_3, 400) = \lambda_{72}^S \iff 0.68x_1 + 0.23x_2 + 0.41x_3 - 400 + \frac{0.1x_1 + 0.1x_2 + 0.1x_3 - 50}{2} = \lambda_{72}^S \\
I(10,000, 35x_1 + 75x_2 + 87x_3) = \lambda_{11}^I \iff 35x_1 + 75x_2 + 87x_3 - 10,000 + \frac{50 - (10x_1 + 10x_2 + 10x_3)}{2} = \lambda_{11}^I \\
I(18,000, 62x_1 + 25x_2 + 50x_3) = \lambda_{21}^I \iff 62x_1 + 25x_2 + 50x_3 - 18,000 + \frac{50 - (10x_1 + 10x_2 + 10x_3)}{2} = \lambda_{21}^I \\
I(22,000, 37x_1 + 40x_2 + 48x_3) = \lambda_{31}^I \iff 37x_1 + 40x_2 + 48x_3 - 22,000 + \frac{50 - (10x_1 + 10x_2 + 10x_3)}{2} = \lambda_{31}^I \\
I(12,000, 48x_1 + 62x_2 + 84x_3) = \lambda_{12}^I \iff 48x_1 + 62x_2 + 84x_3 - 12,000 + \frac{50 - (10x_1 + 10x_2 + 10x_3)}{2} = \lambda_{12}^I \\
I(20,000, 67x_1 + 33x_2 + 61x_3) = \lambda_{22}^I \iff 67x_1 + 33x_2 + 61x_3 - 20,000 + \frac{50 - (10x_1 + 10x_2 + 10x_3)}{2} = \lambda_{22}^I \\
I(20,000, 45x_1 + 56x_2 + 48x_3) = \lambda_{32}^I \iff 45x_1 + 56x_2 + 48x_3 - 20,000 + \frac{50 - (10x_1 + 10x_2 + 10x_3)}{2} = \lambda_{32}^I \\
I(120, 0.5x_1 + 0.2x_2 + 0.425x_3) = \lambda_{41}^I \iff 0.5x_1 + 0.2x_2 + 0.425x_3 - 120 + \frac{50 - (0.1x_1 + 0.1x_2 + 0.1x_3)}{2} = \lambda_{41}^I \\
I(150, 0.25x_1 + 0.36x_2 + 0.48x_3) = \lambda_{51}^I \iff 0.25x_1 + 0.36x_2 + 0.48x_3 - 150 + \frac{50 - (0.1x_1 + 0.1x_2 + 0.1x_3)}{2} = \lambda_{51}^I \\
I(160, 0.24x_1 + 0.84x_2 + 0.74x_3) = \lambda_{61}^I \iff 0.24x_1 + 0.84x_2 + 0.74x_3 - 160 + \frac{50 - (0.1x_1 + 0.1x_2 + 0.1x_3)}{2} = \lambda_{61}^I \\
I(500, 0.84x_1 + 0.37x_2 + 0.14x_3) = \lambda_{71}^I \iff 0.84x_1 + 0.37x_2 + 0.14x_3 - 500 + \frac{50 - (0.1x_1 + 0.1x_2 + 0.1x_3)}{2} = \lambda_{71}^I \\
I(100, 0.14x_1 + 0.6x_2 + 0.74x_3) = \lambda_{42}^I \iff 0.14x_1 + 0.6x_2 + 0.74x_3 - 100 + \frac{50 - (0.1x_1 + 0.1x_2 + 0.1x_3)}{2} = \lambda_{42}^I \\
I(200, 0.1x_1 + 0.33x_2 + 0.47x_3) = \lambda_{52}^I \iff 0.1x_1 + 0.33x_2 + 0.47x_3 - 200 + \frac{50 - (0.1x_1 + 0.1x_2 + 0.1x_3)}{2} = \lambda_{52}^I \\
I(150, 0.12x_1 + 0.64x_2 + 0.78x_3) = \lambda_{62}^I \iff 0.12x_1 + 0.64x_2 + 0.78x_3 - 150 + \frac{50 - (0.1x_1 + 0.1x_2 + 0.1x_3)}{2} = \lambda_{62}^I \\
I(400, 0.68x_1 + 0.23x_2 + 0.41x_3) = \lambda_{72}^I \iff 0.68x_1 + 0.23x_2 + 0.41x_3 - 400 + \frac{50 - (0.1x_1 + 0.1x_2 + 0.1x_3)}{2} = \lambda_{72}^I \\
\lambda_{ji}^I = \lambda_{ji}(w_i) \geq 0; \quad \lambda_{ji}^S = \lambda_{ji}(w_i) \geq 0; \quad i = 1, \ldots, 7; \quad j = 1, \ldots, 3; \quad h = 1, 2

Solving this model, we obtain the optimal solution \(X^* = (425, 410, 93)\). With the results of this example we can see that the proposed method could be applied easily to model a real-life problem and solve it efficiently. With fuzziness and randomness characteristics of the models, we can overcome main obstacle in data collection process due to impreciseness and fluctuating nature of the data in any situation. Therefore, the proposed method will be a powerful tool to help top management making efficient business decisions.

6. Conclusions

In this paper, two new measures of superiority and inferiority between triangular fuzzy numbers/triangular fuzzy stochastic variables are developed to convert the fuzzy (stochastic) linear program into the conventional deterministic linear program. Using the penalty method for any violation of the constraints instead of ranking operations, the proposed method provides a simple deterministic LP model, which can be solved easily by standard optimization packages. In addition, the computation efficiency is advantage of the proposed approach because the obtained model has few additional constraints and variables.

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References