SIR Analysis for OFDM Transmission in the Presence of CFO, Phase Noise and Doubly Selective Fading


*Department of Electronics and Telecommunications Engineering, The University of Danang, University of Science and Technology, Vietnam
†Department of Electrical and Computer Engineering, McGill University, Montreal, Canada

E-mail: ndnvien@dut.udn.vn, nlhung@dut.udn.vn, ttchien@ac.udn.vn, tho.le-ngoc@mcgill.ca

Abstract—This paper presents an analysis of the detrimental effects of carrier frequency offset (CFO), phase noise (PHN) and doubly-selective fading on orthogonal frequency division multiplexing (OFDM) transmission performance. In particular, we derive a closed-form expression for the signal-to-interference ratio (SIR) at an OFDM receiver in the presence of CFO, PHN and doubly-selective fading. Simulation and analytical results under various OFDM system settings are in good agreement and reveal the contributions of these channel impairments in the SIR degradation.

Index Terms—Carrier frequency offset, phase noise, time-selective channel, signal-to-interference, orthogonal frequency division multiplexing.

I. INTRODUCTION

Recently, orthogonal frequency division multiplexing (OFDM) has been recognized as a promising solution to facilitate the explosive growth in broadband data traffic of wireless multimedia services [1]. However, the superior advantages of OFDM only exist under the condition of perfect synchronization and quasi-static fading channel [2]. Unfortunately, such system conditions are rarely attainable in practice, whereas imperfect synchronization and time-selective fading most likely happen in OFDM transmissions [3]. In particular, synchronization impairments (e.g., CFO and PHN) give rise to inter-carrier interference (ICI) that would significantly degrade the performance of OFDM transmissions [3], [4]. In addition, the presence of high-speed moving subscribers (in 4G mobile networks) causes time-selective channel response that also leads to ICI in OFDM systems [5].

For inter-carrier interference management, accurate estimates of CFO, phase noise and channel responses are needed to compensate those channel impairments at OFDM receivers. In a noisy channel, existing estimation techniques always produce imperfect estimates of those channel impairments [6], [7]. Reducing the discrepancy between the estimated and actual values needs an increase in transmission overhead (i.e., spectral efficiency loss in pilot-aided estimation [8]) or in computational complexity (in blind estimation [9]). After compensating those channel impairments by using the imperfect estimates, residual channel impairments still cause inter-carrier interference that would reduce the SIR of the system [10]. An analytical SIR expression that indicates their effects on the residual SIR would be useful for system performance analysis and system design specifications.

In the literature, most of existing studies consider one or two of these channel impairments in system analysis. In particular, the CFO effect on OFDM systems has been extensively studied in [3] while the investigation of phase noise has been addressed in [4]. Besides imperfect synchronization conditions, the effect of time-selective channels has been considered in [5], [11]. Combined time-selective fading and phase noise effects on OFDM systems have been analyzed in [12], [14]. In addition, the effect of CFO and time-selective channels in SIR analysis has been well documented in [13], [15] while the impacts of CFO and phase noise have been investigated in [16].

SIR expression can be derived by using the ambiguity function of the Gaussian pulse (i.e., a modulation pulse used at transmitter) and the scattering function [17]. In particular, the scattering function characterizes statistics of wide-sense stationary uncorrelated scattering (WSSUS) channels. With these functions, signal powers and interference-plus-noise powers can be calculated separately (as shown in Eqs. (18) and (19)).

Different from [3] - [5], [11]-[17], this paper considers the effect of CFO, phase noise and time-selective channel responses in deriving an exact expression of SIR. The SIR analysis helps to reveal the impacts of these channel impairments on the OFDM system performance. Under various OFDM system settings and channel conditions, this paper provides several numerical results to illustrate the tightness of the derived SIR expression as compared to simulation results.

The rest of this paper is organized as follows. Section II describes the considered OFDM system and the modeling of the aforementioned channel impairments. An exact expression of SIR is developed in Section III. Numerical results and related discussions are located in Section IV. Finally, Section V provides some concluding remarks.

Notations: $E[\cdot]$ stands for expectation operator, $J_0(\cdot)$ is the zeroth-order Bessel function of the first kind.

II. SYSTEM MODEL

A. Transmitted Signal Model

Consider an OFDM system using $N$-point fast Fourier transform (FFT) for multicarrier transmission. After IFFT and
cyclic prefix (CP) insertion, the transmitted baseband samples in an OFDM symbol can be written as
\[ x_n = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} X_k \exp \left( j \frac{2\pi kn}{N} \right), \tag{1} \]
where \( n \in \{0, \ldots, N-1\} \), \( X_k \) is the \( k \)th data-modulated subcarrier in the considered OFDM symbol. It is assumed that the power of transmitted subcarriers has been normalized to one, i.e., \( E \left[ |X_k|^2 \right] = 1 \).

**B. Doubly Selective Channel Model**

In the considered OFDM transmission, the transmitted baseband signal \( x_n \) undergoes a time- and frequency-selective (doubly selective) channel. For the multipath fading channel between the transmit antenna and the receive antenna, the \( l \)th (time-selective) channel tap gain that includes the effect of transmit-receive filters and doubly selective propagation is denoted by \( h_{l,n} \) where \( n \) stands for the index of the time-domain sample. The correlation of the channel coefficients for path \( l \) can be described as [19]
\[ E \left[ h_{l,n} h_{l,n+m}^* \right] = J_0(2\pi mf_d T_s/N) \sigma_i^2, \tag{2} \]
where \( f_d \) is the Doppler shift (\( f_d = \frac{v l_0}{c_0} \), \( v \) is the mobile speed, \( c_0 \) denotes the carrier frequency, \( c_0 \) is the speed of light), \( T_s \) is the OFDM symbol duration, \( \sigma_i^2 \) is the power-delay-profile (PDP) of the considered channel. \( L \) is the number of resolvable paths.

In the presence of doubly selective fading, carrier frequency offset and phase noise, the complex baseband received signal in an OFDM symbol can be written by [12], [13], [16]
\[ y_n = e^{j\phi_n} \sum_{l=0}^{L-1} x_{n-l} h_{l,n} + z_n, \tag{3} \]
where \( \phi_n \) denotes phase noise, and \( z_n \) is the additive white Gaussian noise (AWGN) sample with variance of \( \tilde{N}_0 \). As a result, the signal-to-noise ratio (SNR) can be determined by \( SNR = \frac{1}{\tilde{N}_0} \).

**C. Effect of Phase Noise**

Phase noise \( \phi(t) \) denotes rapid, short-term, random fluctuations in the phase of the transmitted and receiver oscillators which are caused by time domain instabilities [12], [4]. Phase noise can be described as a continuous Brownian motion process. In this paper, we consider discrete Brownian motion for phase noise modeling, i.e., \( \phi_n = \phi(n T_s) \). As a result, we have \( \phi_{n+1} = \phi_n + \sigma_n \), where \( \sigma_n \) denotes mutually independent Gaussian random variables having zero mean and variance \( \sigma_n^2 = 2\pi T_s / N \), \( \beta \) stands for the two-sided 3 dB line-width of the Lorentzian power density spectrum of the free-running carrier generator [4]. This model is valid when the \( \beta \) is small as compared to the subcarrier spacing \( 1/T_s \) [18].

As shown in [4], the autocorrelation function of \( \phi_n \) can be computed by
\[ E[e^{j\phi_n} e^{-j\phi_n'}] = e^{-\pi \beta T_s |n-n'|/N}. \tag{4} \]

**D. Received Signal in the Frequency Domain**

After performing FFT at OFDM receiver, the \( k \)th received subcarrier \( Y_k \) can be expressed by
\[ Y_k = G_{k,k} X_k + \sum_{k'=0}^{N-1} G_{k,k'} X_{k'} + Z_k, \tag{5} \]
where \( k = 0, \ldots, N-1 \), \( Z_k \) is the noise sample in the frequency domain. \( G_{k,k'} \) can be defined by
\[ G_{k,k'} = \frac{1}{N} \sum_{l=0}^{L-1} \sum_{n=0}^{N-1} h_{l,n} e^{j\pi (n k'-nk-1k'+nnx) / N} e^{j\phi_n}. \tag{6} \]

The first term in the right-hand side of (5) is the desired subcarrier in the frequency domain. The second term in the left hand side of (5) is an ICI signal which denotes power leakage from other sub-carriers to the desired subcarrier \( X_k \). As observed in (5) and (6), the ICI component exists due to the presence of CFO, PHN and time-selective channel response. To obtain quantitative impacts of the channel impairments on OFDM transmission, we will derive an expression of SIR in the next section.

**III. SIR Formulation**

The average SIR can be determined by
\[ SIR = \frac{E \left[ |G_{k,k} X_k|^2 \right]}{E \left[ \left( \sum_{k' \neq k} G_{k,k'} X_{k'} \right)^2 \right]} \tag{7}. \]

To attain an expression of SIR from (7), one can determine the autocorrelation function of channel frequency response as follows
\[ E \left[ G_{k,k'} G_{r,r'}^* \right] = \frac{1}{N^2} \sum_{l=0}^{L-1} \sum_{n=0}^{N-1} \sum_{n'=0}^{N-1} E[h_{l,n} h_{l,n'}^*] e^{j2\pi (n k'-nk-1k'+nnx) / N} e^{j\phi_n} e^{-j\phi_{n'}}. \tag{8} \]

For SIR formulation, we consider the computation of \( E \left[ |G_{k,k'}|^2 \right] \) as follows:
\[ E \left[ |G_{k,k'}|^2 \right] = \frac{1}{N^2} \sum_{l=0}^{L-1} \sum_{n=0}^{N-1} \sum_{n'=0}^{N-1} E[h_{l,n} h_{l,n'}^*] e^{j2\pi (n k'-nk-1k'+nnx) / N} e^{-\pi \beta T_s |n-n'| / N}. \tag{9} \]

Based on (2) and (4), one can obtained
\[ E \left[ |G_{k,k'}|^2 \right] = \frac{1}{N^2} \sum_{l=0}^{L-1} \sum_{r=1-N}^{N-1} (N-|r|) J_0 \left( \frac{2\pi f_d r T_s}{N} \right) \times \frac{\sigma_i^2}{N} e^{-2\pi \beta T_s |r|} e^{-2\pi \beta T_s |n-n'|}, \tag{10} \]
where \( r = n - n' \) and \( \Delta = k - k' \). Since \( (N-|r|) \) and \( J_0(2\pi f_d r T_s/N) \) are both even functions, (10) can be rewritten
\[ E \left[ |G_{k,k'}|^2 \right] = \frac{1}{N^2} \sum_{l=0}^{L-1} \sum_{r=1-N}^{N-1} (N-|r|) J_0 \left( \frac{2\pi f_d r T_s}{N} \right) \times \cos \left( \frac{2\pi r \Delta}{N} \right) \cos \left( \frac{2\pi r r}{N} \right) e^{-\pi T_s |r| / N}. \tag{11} \]
For a normalized PDP \( \sum_{i=0}^{L-1} \sigma_i^2 = 1 \), (11) can be simplified as

\[
E \left[ |G_{k,k'}|^2 \right] = \frac{1}{N^2} \left\{ N + 2 \sum_{r=1}^{N-1} (N-r) J_0 \left( \frac{2\pi f_d \tau T_s}{N} \right) \times \cos \left( \frac{2\pi r\Delta}{N} \right) \cos \left( \frac{2\pi \varepsilon}{N} \right) e^{-\frac{2\pi \varepsilon}{N}} \right\},
\]

(12)

As a result, the numerator of (7) can be obtained by

\[
E \left[ |G_{k,k'}X_k|^2 \right] = \frac{1}{N^2} \left\{ N + 2 \sum_{r=1}^{N-1} (N-r) J_0 \left( \frac{2\pi f_d \tau T_s}{N} \right) \times \cos \left( \frac{2\pi r\Delta}{N} \right) \cos \left( \frac{2\pi \varepsilon}{N} \right) e^{-\frac{2\pi \varepsilon}{N}} \right\},
\]

(13)

Assuming transmitted symbols \( X_k \) to be statistically independent, the ICI power can be calculated as

\[
E \left[ \sum_{k' \neq k} G_{k,k'}X_k \right]^2 = \frac{1}{N^2} \sum_{\Delta=1}^{N-1} \left\{ N + 2 \sum_{r=1}^{N-1} (N-r) \right\} J_0 \left( \frac{2\pi f_d \tau T_s}{N} \right) \times \cos \left( \frac{2\pi r\Delta}{N} \right) \cos \left( \frac{2\pi \varepsilon}{N} \right) e^{-\frac{2\pi \varepsilon}{N}} \}
\]

(14)

Substituting (13) and (14) into (7), we can obtain the SIR expression (15) as shown at the bottom of the page.

![Fig. 1. SIR as a function of CFO, PHN for different speed values](image)

Using (15), the plots of SIR as a function of CFO, PHN for different speed values reveal the contributions of the channel impairments on the OFDM system performance. It is observed that CFO is a dominant factor in SINR values when CFO values are close to 0.5. For instance, under \( \varepsilon = \pm 0.5 \), SIR values almost unchanged under different values of PHN and mobile speeds. For CFO value smaller than 0.1, phase noise becomes a dominant factor in SIR values. By using Fig. 1 and (15), one can determine allowable ranges of CFO, PHN level and mobile speeds to satisfy a target SIR.

\[
SIR(f_d T_s, \varepsilon, \beta T_s) = \frac{\left\{ N + 2 \sum_{r=1}^{N-1} (N-r) J_0 \left( \frac{2\pi f_d \tau T_s}{N} \right) \cos \left( \frac{2\pi \varepsilon}{N} \right) e^{-\frac{2\pi \varepsilon}{N}} \right\}}{\sum_{\Delta=1}^{N-1} \left\{ N + 2 \sum_{r=1}^{N-1} (N-r) \cos \left( \frac{2\pi \Delta}{N} \right) J_0 \left( \frac{2\pi f_d \tau T_s}{N} \right) \cos \left( \frac{2\pi \varepsilon}{N} \right) e^{-\frac{2\pi \varepsilon}{N}} \right\}},
\]

(15)

Fig. 2 illustrates the level surfaces of the SIR as a function of \( f_d T_s \), \( \varepsilon \), and \( \beta T_s \). Each level surfaces represent \( SIR(f_d T_s, \varepsilon, \beta T_s) = C \) for a target SIR value of \( C \) (in dB) where \( f_d T_s \) stands for normalized Doppler frequency (NDF). Similar to observations from Fig. 1, one can find that PHN becomes the dominant factor in SIR values when CFO is smaller than 0.1 as shown in Fig. 2.

**IV. ILLUSTRATIVE RESULTS**

In this section, computer simulation was conducted to evaluate the tightness of the derived SIR expression in the presence of time-selective channels, phase noise and carrier frequency offset. The time-selective multipath fading channel with \( L = 5 \) resolvable paths and an exponentially decaying power-delay profile is generated by using Jakes’ model [19]. The simulation uses 10,000 channel realizations (with different channel responses) for Monte Carlo simulation. In the simulation, the considered OFDM system uses 512-point FFT, carrier frequency of \( f_c = 3.5 \) GHz, sampling frequency of 5.6 MHz and CP length of 40 samples. For each transmission burst (of each simulation trial), phase noise \( \phi(nT_s) \) is a random process following a continuous-path Brownian motion and the normalized CFO \( \varepsilon \) is a random value uniformly distributed in the range \( [-\varepsilon_0, \varepsilon_0] \) where \( \varepsilon_0 = 0.5 \).

To verify the validity of SIR analysis, numerical results of (15) versus PHN level \( \beta T_s \) are shown in Fig. 3. SIR curves are provided under different CFO values. It is observed that the SIR decreases as synchronization impairments increases. In addition, Fig. 3 shows a good agreement between simulated and theoretical results (15).

To illustrate the need of considering the joint effect of CFO, PHN and Doppler spread in SIR analysis, Fig. 4 shows...
V. CONCLUSION

This paper formulated an exact SIR expression for OFDM transmissions in the presence of phase noise, carrier frequency offset, and time-selective channels. The analytical results using (15) are in a good agreement with the simulation results over wide ranges of mobile speeds, CFO, and PHN.

ACKNOWLEDGMENT

This research is funded by Vietnam National Foundation for Science and Technology Development (NAFOSTED) under grant number 102.05-2012.07.

REFERENCES