Graph Minors
XV. Giant Steps

Neil Robertson*

Department of Mathematics, Ohio State University,
231 West 18th Avenue, Columbus, Ohio 43210

and

P. D. Seymour

Bellcore, 445 South Street, Morristown, New Jersey 07960

Received March 15, 1992

Let $G$ be a graph with a subgraph $H$ drawn with high representativity on a surface $\Sigma$. When can the drawing of $H$ be extended “up to 3-separations” to a drawing of $G$ in $\Sigma$ if we permit a bounded number ($\kappa$ say) of “vortices” in the drawing of $G$, that is, local areas of non-planarity? (The case $\kappa = 0$ was studied in the previous paper of this series.) For instance, if there is a path in $G$ with ends in $H$, far apart, and otherwise disjoint from $H$, then no such extension exists. We are concerned with the converse; if no extension exists, what can we infer about $G$? It turns out that either there is a path as above, or one of two other obstructions is present.

1. INTRODUCTION

The current objective of this sequence of papers is to prove a result about the structure of the graphs not containing a fixed minor ($K_n$, say). The proof will be completed in the next paper, but here we accomplish a substantial part. Before we go into the details of what we are going to prove in this paper, it may be helpful if we sketch the bigger picture. Suppose that $G$ has no $K_n$ minor. If $G$ has small tree-width, we are done; so let us assume it has large tree-width (at least some enormous function of $n$). Hence it has a planar subgraph $H_0$ with large tree-width, by a theorem of an earlier paper. Take a drawing of $H_0$ in a sphere $\Sigma_0$. The pair $H_0$, $\Sigma_0$ can be regarded as a degenerate case of the following: a subgraph $H$ of $G$, drawn in a surface $\Sigma$, with large representativity (that is, every simple

* This work was partially performed under a consulting agreement with Bellcore.
closed curve in the surface that does not bound a disc meets the drawing many times). Our initial pair $H_0, \Sigma_0$ is a degenerate case, because for the sphere $\Sigma_0$ representativity does not make sense, but we showed in earlier papers that in the sphere, high tree-width is the appropriate analogue for high representativity. Let us make a sequence $H_i, \Sigma_i (i = 0, 1, 2, \ldots)$ of such pairs, at each stage sacrificing a large amount of the remaining representativity if necessary for an increase in genus (we permit $\Sigma_i$ to be non-orientable, in which case its “genus” is half the number of crosscaps). Since $H_0, \Sigma_0$ has large representativity, the initial few terms in our sequence still have high representativity. Now every graph with high representativity that can be drawn in a surface in which $K_n$ can be drawn has a $K_n$ minor. Hence the process stops in the initial few terms, before $\Sigma_i$ has grown complex enough that $K_n$ can be drawn in it. We therefore have a pair $H, \Sigma$ (say), where $\Sigma$ has bounded genus and $H$ is drawn in it with very large representativity, so that no improvement in $\Sigma$ can be made.

How does the remainder of $G$ attach to the subgraph $H$? The theorem that we are going to prove here says roughly that either

(i) we can delete a small area of $H$, and replace it by some other subgraph of $G$, to obtain a pair $H', \Sigma'$ where $\Sigma'$ is obtained by adding a crosscap to $\Sigma$ and $H'$ still has large representativity—but this contradicts the choice of $H, \Sigma$;

(ii) we can choose another $H', \Sigma$ so that $H'$ can be drawn in $\Sigma$ with crossings, but with a large constant number of crossings, all pairwise “far apart” and still with high representativity in the appropriate sense—but this implies that $G$ has a $K_n$ minor, a contradiction;

(iii) we can rearrange a small area of $H$ so that some path of $G$ has both ends far apart in $H$ and is otherwise disjoint from $H$; or,

(iv) $G$ has the desired structure (roughly, all of $G$ can be drawn in $\Sigma$ “up to 3-separations,” except for a bounded number of local areas of non-planarity, called “vortices”).

Thus, we are done in every case except (iii). That case needs work, and to handle it we actually need a more complicated optimization of $H, \Sigma$ than is described here; but we hope this provides some motivation for the present result.

This paper is closely related to [5], and uses most of the same terminology. In particular, the reader should see [5, Sections 2-4] for the meaning of the following terms and notation: $G_1 \cup G_2$, $G_1 \cap G_2$, separation, order of a separation, tangle, order of a tangle, free, open and closed disc, surface, bd($\Sigma$), O-arc, line, ends of a line, drawing, $U(H)$, region, $R(H)$, atom, $A(H)$, radical drawing, respectfulltangle, metric of a tangle, $\lambda$-zone, clearing, 2-cell, rigid, dial, natural order, society, $\Omega$, segregation, 3-segregation,
\( V(\mathcal{F}) \), arrangement, \( \mathcal{F} \)-local, bridge, \( H \)-path in \( G \) (called an “\( H \)-pathing” in [5], due to a typographical error).

Let \( (G, \Omega) \) be a society, and let the elements of \( \Omega \) be \( v_1, \ldots, v_n \), numbered in order under \( \Omega \). A transaction in \( (G, \Omega) \) is a set \( \mathcal{P} \) of mutually disjoint paths of \( G \), each with distinct ends both in \( \Omega \), such that for some \( i, j \) with \( 0 \leq i, j \leq n \), each member of \( \mathcal{P} \) has exactly one end in \( \{v_{i+1}, \ldots, v_j\} \). For \( p \geq 0 \), a \( p \)-vortex is a society with no transaction of cardinality \( > p \).

A segregation \( \mathcal{S} \) of a graph \( G \) is of type \((p, \kappa)\), where \( p, \kappa \geq 0 \) are integers, if there are at most \( \kappa \) members \((A, \Omega)\) in \( \mathcal{S} \) with \( |\Omega| > 3 \), and each is a \( p \)-vortex. Thus, segregations of type \((p, 0)\) are 3-segregations, for any \( p \).

If \( \mathcal{F}^* \) is a tangle in \( G \), a segregation \( \mathcal{S} \) of \( G \) is \( \mathcal{F}^* \)-central if for each \((A, \Omega)\) in \( \mathcal{S} \) there is no \((A^*, B^*)\) in \( \mathcal{F}^* \) with \( B^* \subseteq A \). We denote the order of a tangle \( \mathcal{F} \) by \( \text{ord}(\mathcal{F}) \). Let \( \mathcal{F}^* \) be a tangle in \( G \), and let \( \Sigma \) be a surface with \( bd(\Sigma) = \emptyset \). A triple \( H, \eta, \mathcal{F} \) is a \( \Sigma \)-span of order \( \theta \) (in \( G \), with respect to \( \mathcal{F}^* \)) if

(i) \( H \) is a rigid drawing in \( \Sigma \)

(ii) \( \eta \) is an isomorphism from \( H \) to a subgraph \( \eta(H) \) of \( G \)

(iii) \( \mathcal{F} \) is a respectful tangle in \( H \) of order \( \theta \)

(iv) \( \theta \leq \text{ord}(\mathcal{F}^*) \), and \((A \cap \eta(H), B \cap \eta(H)) \in \eta(\mathcal{F}) \), for every \((A, B) \in \mathcal{F}^* \) of order \( < \theta \).

(\( \eta(\mathcal{F}) \) denotes the image of \( \mathcal{F} \) under \( \eta \)—we shall frequently use similar notation without further explanation.) Whenever possible we shall arrange that \( \eta \) is the identity function on \( H \), and so \( H \) itself is a subgraph of \( G \); and in that case we loosely say that \( H, \mathcal{F} \) is a \( \Sigma \)-span (without reference to \( \eta \) or \( \theta \)).

If \( H \) is a 2-cell drawing in \( \Sigma \), and \( \mathcal{F} \) is a respectful tangle in \( H \) with metric \( d \), then for \( \sigma \in \Sigma \) and \( h \in A(H) \), we define \( d(\sigma, h) = d(a, h) \) where \( a \in a \in A(H) \); and for \( \sigma_1, \sigma_2 \in \Sigma \) we define \( d(\sigma_1, \sigma_2) = d(a_1, a_2) \) where \( a_1 \in a_1 \in A(H) \) and \( a_2 \in a_2 \in A(H) \). If \( H' \) is another 2-cell drawing in \( \Sigma \) with a respectful tangle \( \mathcal{F}' \) with metric \( d' \), we say that \( \mathcal{F}' \) is a \( \lambda \)-compression of \( \mathcal{F} \) (where \( \lambda \geq 0 \) is an integer) if \( d'(\sigma_1, \sigma_2) \geq d(\sigma_1, \sigma_2) - \lambda \) for all \( \sigma_1, \sigma_2 \in \Sigma \), and \( \text{ord}(\mathcal{F}') = \text{ord}(\mathcal{F}) - \lambda \).

If \( \Sigma, H, \eta, \mathcal{F}, H', \eta', \mathcal{F}', G^*, \mathcal{F}^* \) are as above, and \( z \in A(H) \), we say that \( H', \eta', \mathcal{F}' \) is obtained from \( H, \eta, \mathcal{F} \) by rearranging within \( \lambda \) of \( z \), where \( \lambda \geq 0 \) is an integer, if

(i) \( a \in A(H') \) for all \( a \in A(H) \) with \( d(z, a) > \lambda \), where \( d \) is the metric of \( \mathcal{F} \)

(ii) \( \eta(x) = \eta'(x) \) for every vertex or edge \( x \) of \( H \) with \( d(z, x) > \lambda \), and

(iii) \( \mathcal{F}' \) is a \((4\lambda + 2)\)-compression of \( \mathcal{F} \).
A $\Sigma$-span $H, \eta, \mathcal{T}$ is $\mu$-stepped if there is an $\eta(H)$-path in $G$ with ends $\eta(u), \eta(v)$ such that $d(u, v) \geq \mu$, where $d$ is the metric of $\mathcal{T}$. A $\Sigma$-span $H, \eta, \mathcal{T}$ is $(\lambda, \mu)$-flat, where $\lambda, \mu \geq 0$ are integers, if $\text{ord}(\mathcal{T}) \geq 4\lambda + \mu + 2$ and there is no $\mu$-stepped $\Sigma$-span $H', \eta', \mathcal{T}'$ which can be obtained from $H, \eta, \mathcal{T}$ by rearranging within $\lambda$ of any $z \in A(H)$.

Let $H, \eta, \mathcal{T}$ be a $\Sigma$-span. A region $r$ of $H$ is an eye (of $H, \eta, \mathcal{T}$ in $G$) if

(i) there is a circuit $C$ of $H$ with $U(C) = bd(r)$

(ii) there are distinct vertices $a, b, c, d$ of $C$ such that $a, b, c, d$ occur in $V(C)$ in that order, and $\{a, b, c, d\}$ is free with respect to $\mathcal{T}$

(iii) there are two disjoint $\eta(H)$-paths $P, Q$ in $G$ with ends $\eta(a), \eta(c)$ and $\eta(b), \eta(d)$ respectively.

Now let $r_1, ..., r_\kappa$ be eyes, with corresponding $\eta(H)$-paths $P_i, Q_i$ ($1 \leq i \leq \kappa$). We say that $r_1, ..., r_\kappa$ are independent eyes if the paths $P_i, Q_i$ can be chosen so that

(i) for $1 \leq i < j \leq \kappa$, $d(r_i, r_j) = \text{ord}(\mathcal{T})$, where $d$ is the metric of $\mathcal{T}$

(ii) for $1 \leq i < j \leq \kappa$, $P_i \cup Q_i$ is disjoint from $P_j \cup Q_j$.

Finally we can state our main result.

\textbf{(1.1)} For any surface $\Sigma$ with $bd(\Sigma) = \emptyset$, and any integers $\kappa, \phi, \mu \geq 0$, there are integers $0, \lambda, \rho \geq 0$ such that the following holds. Let $\mathcal{T}^*$ be a tangle in a graph $G$, such that some $\Sigma$-span of order $\geq 0$ is $(\lambda, \mu)$-flat. Then either

(i) there is a $\Sigma$-span of order $\geq \phi$ with $\geq \kappa$ independent eyes, or

(ii) there is a $\Sigma'$-span of order $\geq \phi$, where $\Sigma'$ is a surface obtained by adding a crosscap to $\Sigma$, or

(iii) there is a $\mathcal{T}^*$-central segregation of $G$ of type $(\rho, \kappa)$ with an arrangement in $\Sigma$.

Result (1.1) is a consequence of the following.

\textbf{(1.2)} For any surface $\Sigma$ with $bd(\Sigma) = \emptyset$, and any integers $k \geq 0, \phi \geq 8, \mu \geq 12$ and even, $\lambda_{k+1} \geq 4$, and $\theta_{k+1} \geq 4\lambda_{k+1} + \mu + 2$, there are integers $\rho_k, \lambda_k \geq 0$ and $\theta_k \geq 4\lambda_k + \mu + 2$ such that the following holds. Let $\mathcal{T}^*$ be a tangle in a graph $G$, and suppose that there is a $(\lambda_k, \mu)$-flat $\Sigma$-span of order $\geq \theta_k$ with $k$ eyes mutually at distance $\geq \theta_k$. Then either

(i) there is a $(\lambda_{k+1}, \mu)$-flat $\Sigma$-span of order $\theta_{k+1}$ with $k + 1$ eyes, mutually at distance $\theta_{k+1}$, or
(ii) there is a $\Sigma$-span of order $\phi$, where $\Sigma'$ is a surface obtained by adding a crossover to $\Sigma$, or

(iii) there is a $\mathcal{F}^*$-central segregation of $G$ of type $(\rho_k, k)$ with an arrangement in $\Sigma$.

Proof of (1.1), assuming (1.2). Let $\Sigma, \kappa, \phi, \mu$ be as in (1.1). We may assume that $\mu \geq 12$ and is even, for every $(\lambda, \mu)$-flat $\Sigma$-span is also $(\lambda, \mu')$-flat for all $\mu' \geq \mu$ (see (5.1)). We may also assume that $\phi \geq 8$, for if (1.1) holds for $\phi \geq 8$ then it holds for all $\phi$. Define $\theta_{k+1} = \max(\phi, \mu + 16)$ and $\zeta_{k+1} = 4$; and for $k = \kappa, \kappa - 1, \ldots, 0$, define $\theta_k, \zeta_k, \rho_k$ inductively by (1.2). Let $\theta = \theta_0$, $\zeta = \zeta_0$ and $\rho = \max(\rho_0, \ldots, \rho_\kappa)$. We claim that $\theta, \zeta, \rho$ satisfy (1.1).

For let $G, \mathcal{F}$ be as in (1.1); then there is a $(\lambda_0, \mu)$-flat $\Sigma$-span of order $\geq \theta_0$. Choose $k \geq 0$ with $k \leq \kappa + 1$ maximum such that there is a $(\lambda_k, \mu)$-flat $\Sigma$-span of order $\geq \theta_k$, with $k$ eyes mutually at distance $\geq \theta_k$. Suppose first that $k = \kappa$. Since (1.2)(i) does not hold, it follows from (1.2) that either (1.2)(ii) or (1.2)(iii) holds, and hence either (1.1)(ii) or (1.1)(iii) holds (because a segregation of type $(\rho_k, \kappa)$ is also of type $(\rho, \kappa)$). We may therefore assume that $k = \kappa + 1$. Let $H, \eta, \mathcal{F}$ be a $(\lambda_{k+1}, \mu)$-flat $\Sigma$-span of order $\theta_{k+1}$, with $\kappa + 1$ eyes $r_1, \ldots, r_{\kappa+1}$ mutually at distance $\theta_{k+1}$. Since $H, \eta, \mathcal{F}$ is $(4, \mu)$-flat, and hence $(0, \mu)$-flat (see (5.1)), there is no $\eta(H)$-path in $G$ with ends $\eta(u), \eta(v)$, where $u \in V(H) \cap r_i, v \in V(H) \cap r_j$ for $i \neq j$, since $d(u, v) \geq \mu$ because

$$
\mu + 2 \leq \theta_{k+1} = d(r_i, r_j) \leq d(u, r_i) + d(u, v) + d(v, r_j) \leq d(u, v) + 2.
$$

Hence $r_1, \ldots, r_{\kappa+1}$ are independent eyes, and so (1.1)(i) holds.

2. TANGLES AND TRANSACTIONS

To prove (1.2) we need at some stage to prove that a certain segregation is $\mathcal{F}^*$-central, and to do so we shall use the following lemma.

(2.1) Let $\mathcal{F}$ be a tangle in a graph $G$, let $\rho \geq 1$ with $\text{ord}(\mathcal{F}) \geq 5\rho + 2$, and let $\mathcal{F}$ be a segregation of $G$ of type $(\rho, \kappa)$ for some $\kappa$. Suppose that for every $(C, \Omega) \in \mathcal{F}$ there is no $(A, B) \in \mathcal{F}$ of order $\leq 2\rho + 1$ with $B \subseteq C$. Then $\mathcal{F}$ is $\mathcal{F}^*$-central.

Proof. Let $(C, \Omega) \in \mathcal{F}$, and suppose, for a contradiction, that $(A, B) \in \mathcal{F}$ and $B \subseteq C$. If $|\Omega| \geq 4$ then $(C, \Omega)$ is a $\rho$-vortex since $\mathcal{F}$ is of type $(\rho, \kappa)$. If $|\Omega| \leq 3$ then $(C, \Omega)$ has no transaction of cardinality $\geq 2$, and again it is a $\rho$-vortex since $\rho \geq 1$. Hence in either case, from [1, Theorem (8.1)] we may enumerate $\Omega = \{t_1, \ldots, t_n\}$ in order so that there are subgraphs $X_1, \ldots, X_n$ of $C$ satisfying (1)-(4) below:
(1) \( X_1 \cup \cdots \cup X_n = C \).

(2) For \( 1 \leq i < j \leq n \), \( E(X_i \cap X_j) = \emptyset \) and \( |V(X_i \cap X_j)| \leq \rho \).

(3) For \( 1 \leq i \leq j \leq k \leq n \), \( X_i \cap X_k \subseteq X_j \).

(4) For \( 1 \leq i \leq n \), \( t_i \in V(X_i) \).

For any subgraph \( X \) of \( G \), let \( \hat{X} \) be the unique minimal subgraph so that \( (X, \hat{X}) \) is a separation of \( G \), and let \( N(X) = V(X \cap \hat{X}) \). Thus, \( N(X) \) is the set of all \( v \in V(X) \) incident with an edge of \( G \) not in \( E(X) \). We define \( X_i \) and \( X_{n+1} \) to be the null graph.

(5) For \( 1 \leq i \leq n \), \( N(X_i) \subseteq V(X_i \cap X_{i-1}) \cup V(X_i \cap X_{i+1}) \cup \{ t_i \} \), and consequently \( |N(X_i)| \leq 2\rho + 1 \).

Subproof. The second claim follows from the first and (2). To prove the first claim, let \( v \in N(X_i) \), incident with an edge \( e \) of \( G \) not in \( E(X_i) \). If \( v \in V(X_j) \) for some \( j \) with \( 1 \leq j < i \), then by (3) \( v \in V(X_{j-1}) \) and hence \( v \in V(X_j \cap X_{j-1}) \). If \( v \in V(X_j) \) for some \( j \) with \( i < j \leq n \) then similarly \( v \in V(X_j \cap X_{j+1}) \). If \( v \not\in V(X_j) \) for all \( j \neq i \) with \( 1 \leq j \leq n \), then \( e \not\in E(X_j) \) for \( 1 \leq j \leq n \), and so \( e \not\in E(C) \); hence \( v \in N(C) \subseteq \hat{X} \) and so \( v = t_j \) for some \( j \). Hence \( v \in V(X_j) \) by (4), and so \( j = i \) and \( v = t_i \), as required.

Let us say that \( X \subseteq G \) is small if \( (X, \hat{X}) \in \mathcal{F} \).

(6) \( X_1, \ldots, X_n \) are small.

Subproof. Let \( 1 \leq i \leq n \). By (5), \( |N(X_i)| \leq 2\rho + 1 \), and so one of \( (X_i, \hat{X}_i), (\hat{X}_i, X_i) \in \mathcal{F} \), since \( \text{ord}(\mathcal{F}) \leq 2\rho + 2 \). But from the hypothesis, \( (X_i, \hat{X}_i) \not\in \mathcal{F} \), and so \( X_i \) is small as required.

(7) If \( Z \subseteq X \cup Y \subseteq G \) and \( X, Y \) are small and \( |N(Z)| < \text{ord}(\mathcal{F}) \), then \( Z \) is small.

Subproof. Since \( (X, \hat{X}), (Y, \hat{Y}) \in \mathcal{F} \) it follows that \( (\hat{Z}, Z) \not\in \mathcal{F} \) by the second tangle axiom, since

\[
G = Z \cup \hat{Z} \subseteq X \cup Y \cup \hat{Z}.
\]

Since \( (Z, \hat{Z}) \) has order \( < \text{ord}(\mathcal{F}) \) it follows from the first axiom that \( (Z, \hat{Z}) \in \mathcal{F} \) and hence \( Z \) is small. This proves (7).

We recall that \( (A, B) \in \mathcal{F} \) and \( B \subseteq C = X_1 \cup \cdots X_n \). Let \( R = V(A \cap B) \). For \( 1 \leq i \leq n \), let
\[ P_i = \bigcup \{X_j : 1 \leq j \leq i\} \]
\[ Q_i = \bigcup \{X_j : i \leq j \leq n\}, \]

and let \( P_0 \) and \( Q_{n+1} \) be the null graph.

(8) For \( 0 \leq i \leq n \), if \( |V(P_i) \cap R| < \text{ord}(\mathcal{F}) - \rho \) then \( P_i \cap B \) is small.

Subproof. The result holds if \( i = 0 \), and we assume that \( i \geq 1 \) and proceed by induction on \( i \). Since \( V(P_{i-1}) \cap R \subseteq V(P_i) \cap R \) and hence \( |V(P_{i-1}) \cap R| < \text{ord}(\mathcal{F}) - \rho \), it follows from the inductive hypothesis that \( P_{i-1} \cap B \) is small. Since \( X_i \) is small by (6), and \( P_i \cap B \subseteq (P_{i-1} \cap B) \cup X_i \), it suffices by (7) to show that \( |N(P_i \cup B)| < \text{ord}(\mathcal{F}) \). To show this we shall prove that

\[ N(P_i \cap B) \subseteq (V(P_i) \cap R) \cup (X_i \cap X_{i+1}) \]

for since \( |V(P_i) \cap R| < \text{ord}(\mathcal{F}) - \rho \) and \( |V(X_i \cap X_{i+1})| \leq \rho \), the desired inequality is a consequence. Thus, let \( e \in N(P_i \cap B) \), incident with an edge \( e \) of \( G \) not in \( E(P_i \cap B) \). If \( e \notin E(B) \) then \( e \in N(B) \subseteq R \) as required. If \( e \in E(B) \), then since \( e \notin E(P_i \cap B) \) and \( B \subseteq C \) it follows that \( e \in E(X_i) \) for some \( j \) with \( i < j \leq n \). Hence \( e \in V(X_i) \), and since \( e \in V(P_i) \), it follows from (3) that \( e \in V(X_i \cap X_{i+1}) \). This proves that

\[ N(P_i \cap B) \subseteq V(P_i) \cap R \cup (X_i \cap X_{i+1}) \]

as claimed, and (8) follows.

Now choose \( j \) with \( 0 \leq j \leq n \), maximum such that \( |V(P_j) \cap R| < \text{ord}(\mathcal{F}) - 2\rho - 1 \). (This is possible, since \( P_0 \) is null.)

(9) \( j < n \), and \( (P_j \cap B) \cup X_{j+1} \) is small, and \( |P_{j+1} \cap R| \geq \text{ord}(\mathcal{F}) - 2\rho - 1 \).

Subproof. Now \( P_j \cap B \) is small by (8). Since \( P_n \cap B \) is not small because \( (A, B) \in \mathcal{F} \), it follows that \( j < n \). Now

\[ N((P_j \cap B) \cup X_{j+1}) \subseteq N(P_j \cap B) \cup N(X_{j+1}) \]

and so \( |N((P_j \cap B) \cup X_{j+1})| < \text{ord}(\mathcal{F}) \), since \( |N(P_j \cap B)| < \text{ord}(\mathcal{F}) - 2\rho - 1 \) by hypothesis and \( |N(X_{j+1})| \leq 2\rho + 1 \) by (5). From (7) we deduce that \( (P_j \cap B) \cup X_{j+1} \) is small. Finally, from the maximality of \( j \) it follows that \( |P_{j+1} \cap R| \geq \text{ord}(\mathcal{F}) - 2\rho - 1 \). This proves (9).

Choose \( k \) with \( 1 \leq k \leq n + 1 \), minimum such that \( |V(Q_k) \cap R| < \text{ord}(\mathcal{F}) - 2\rho - 1 \). Then, similarly,
\[ k > 1, \text{ and } (Q_k \cap B) \cup X_{k-1} \text{ is small, and } |Q_{k-1} \cap R| \geq \text{ord}(\mathcal{F}) - 2p - 1. \]

Suppose that \( k \leq j + 3. \) Then
\[ B \subseteq ((P_j \cap B) \cup X_{j+1}) \cup ((Q_k \cap B) \cup X_{k-1}) \]
contrary to (7). Thus \( k \geq j + 4, \) and hence \( P_{j+1} \cap Q_{k-1} \subseteq X_{j+1} \cap X_{k-2} \) and so \( V(P_{j+1} \cap Q_{k-1}) \leq p. \) But by (9) and (10),
\[
2(\text{ord}(\mathcal{F}) - 2p - 1) \leq |P_{j+1} \cap R| + |Q_{k-1} \cap R| = |P_{j+1} \cup Q_{k-1} \cap R| + |P_{j+1} \cap Q_{k-1} \cap R| \leq |R| + p < \text{ord}(\mathcal{F}) + p
\]
since \( R = V(A \cup B) \) and \((A, B) \in \mathcal{F}. \) Consequently, \( \text{ord}(\mathcal{F}) < 5p + 2, \) a contradiction. Hence there is no such \((A, B), \) as required. \( \square \)

3. RIGIDITY

In this section we prove a lemma that will permit us to modify a rigid drawing and infer that the result remains rigid. First, we observe the following.

(3.1) Let \( H \) be a rigid drawing in a surface \( \Sigma, \) and let \( \mathcal{F} \) be a tangle in \( H \) of order \( \geq 3. \) Let \( \Delta \subseteq \Sigma \) be a closed disc with \( \text{bd}(\Delta) \cap H \subseteq \mathcal{V}(H) \) and \( |\text{bd}(\Delta) \cap \mathcal{V}(H)| < 2. \) Then \((H \cap A, H \cap \Sigma - \Delta) \in \mathcal{F} \) if and only if \( \Delta \) is a dial for \( \text{bd}(\Delta). \)

Proof. Since \( H \) is rigid, there is a dial \( \Delta' \) for \( \text{bd}(\Delta). \) Let \( A = H \cap \Delta', \) \( B = H \cap \Sigma - \Delta'. \) Then \( A \) has no circuit, and so \((B, A) \notin \mathcal{F} \) by \([2, \text{Theorem (2.10)}]. \) Since \((A, B)\) has order \( \leq 2, \) and \( \text{ord}(\mathcal{F}) \geq 3, \) it follows that \((A, B) \in \mathcal{F}. \) Now either \( A = \Delta' \) or \( A = \Sigma - \Delta', \) and in either case the result follows. \( \square \)

The main result of this section is the following cumbersome lemma, which we shall have several occasions to use.

(3.2) Let \( \Sigma \) be a surface with \( \text{bd}(\Sigma) = \emptyset, \) and let \( \Sigma_1, \Sigma_2 \subseteq \Sigma \) be surfaces with \( \Sigma_1 \cup \Sigma_2 = \Sigma \) and \( \Sigma_1 \cap \Sigma_2 = \text{bd}(\Sigma_1) = \text{bd}(\Sigma_2). \) Let \( H \) be a drawing in \( \Sigma \) with \( \Sigma_1 \cap \Sigma_2 \subseteq \mathcal{U}(H), \) and let \( X \subseteq \mathcal{V}(H) \cap \Sigma_1 \cap \Sigma_2. \) Let \( \Sigma_1 \) be a surface with \( \Sigma_1 \subseteq \Sigma_1 \) and \( \text{bd}(\Sigma_1) = \emptyset, \) such that each O-arc in \( \text{bd}(\Sigma_1) \) bounds an open disc in \( \Sigma_1 \) disjoint from \( X \) (thus, \( \Sigma_1 \) is obtained from \( \Sigma_1 \) by “pasting” a disc onto each O-arc in \( \text{bd}(\Sigma_1) \)). Suppose that
F the same component

We shall show that there is a dial 2

Moreover, there is a path |

Every component of 

Then H is rigid.

Proof. Let \( F \subseteq \Sigma \) be an \( O \)-arc with \( F \cap U(H) \subseteq V(H) \) and \( |F \cap V(H)| \leq 2 \). We shall show that \( \Sigma \) includes a dial for \( F, H \). From (ii) we may assume that \( F \not\subseteq K \). Suppose that \( F \subseteq \Sigma \). From (ii), there is a dial \( \Delta \subseteq \Sigma \) for \( F, H \cap \Sigma \).

Every component of \( \Sigma_1 - \Sigma_1 \) is bounded by an \( O \)-arc in \( \Sigma_1 \cap \Sigma_2 \subseteq U(H) \), and hence \( A \) includes no such component, since \( (H \cap \Sigma_1) \cap A \) has no circuit. Since \( bd(A) \subseteq \Sigma_1 \) it follows that \( A \subseteq \Sigma_1 \subseteq \Sigma \), and hence \( A \) is a dial in \( \Sigma \) for \( F, H \) as required. We may assume then that \( F \not\subseteq \Sigma_1 \).

Since \( F \not\subseteq \Sigma_1 \) and \( F \not\subseteq \Sigma_2 \), it follows that \( |F \cap \Sigma_1 \cap \Sigma_2| \geq 2 \). Since \( F \cap U(H) \subseteq V(H) \) and \( \Sigma_1 \cap \Sigma_2 \subseteq U(H) \) and \( |F \cap V(H)| \leq 2 \), we deduce that \( F \cap \Sigma_1 \cap \Sigma_2 = \{ u, v \} \subseteq V(H) \) say, and \( F = F_1 \cup F_2 \) where \( F_1, F_2 \) are lines with ends \( u, v \) and \( F_i \subseteq \Sigma_i \) (\( i = 1, 2 \)). For \( i = 1, 2 \), let \( r_i \) be the region of \( H \) in \( \Sigma \) with \( F_i \subseteq r_i \cup \{ u, v \} \). Then \( r_i \subseteq \Sigma_i \) (\( i = 1, 2 \)). From (i), \( u \) and \( v \) lie in the same component \( C \) of \( \Sigma_1 \cap \Sigma_2 \). Let \( A \) be the component of \( \Sigma_1 - \Sigma_1 \) bounded by \( U(C) \). By adding a line joining \( u, v \) within \( A \cup \{ u, v \} \) we extend \( F_i \) to an \( O \)-arc in \( \Sigma_1 \), and thereby deduce from (ii) that there is a path \( P \) of \( C \) between \( u \) and \( v \) such that \( |V(P) \cap X| \leq 2 \), and \( F \cup U(P) \) bounds an open disc in \( \Sigma_1 \) included in \( r_i \). From (iv), \( F \cup U(P) \) bounds an open disc in \( \Sigma_2 \) included in \( r_2 \). It follows that the union of the closures of these open discs is a dial in \( \Sigma \) for \( F, H \), as required.

If \( F \) is a tangle of order \( \theta \) in a graph \( H \), and \( 1 \leq \theta \leq \theta \), the set \( F' \) of all members of \( F \) of order \( < \theta \) is a tangle in \( H \) of order \( \theta' \), and we call it the \( \theta' \)-truncation of \( F \).

We apply (3.2) to deduce the following.

(3.3) Let \( \Sigma \) be a surface and let \( \phi \geq 3 \) be an integer. Let \( F^* \) be a tangle in a graph \( G \), and let \( H, \eta, F \) be a \( \Sigma \)-span. Let \( \Lambda \subseteq \Sigma \) be a region of \( H \) such that \( H \cap bd(A) \) is a circuit, and let \( a_1, ..., a_\phi, b_1, ..., b_\phi \) be distinct vertices of this circuit, in order, such that \( \{ a_1, ..., a_\phi, b_1, ..., b_\phi \} \) is free with respect to \( F \). For \( 1 \leq i \leq \phi \) let \( P_i \) be an \( \eta(H) \)-path with ends \( \eta(a_i), \eta(b_i) \), such that
\[ P_1, \ldots, P_\delta \text{ are mutually disjoint. Then there is a } \Sigma' \text{-span of order } \phi, \text{ where } \Sigma' \text{ is the surface obtained by adding a crosscap to } \Sigma. \]

**Proof.** Let \( \Sigma_1 = \Sigma - A \). We may assume that \( \Sigma' = \Sigma_1 \cup \Sigma_2 \), where \( \Sigma_2 \) is homeomorphic to a Möbius band and \( \Sigma_1 \cap \Sigma_2 = bd(\Sigma_1) = bd(\Sigma_2) \). We may assume also that \( \eta \) is the identity, and \( P_1, \ldots, P_\delta \) are drawings in \( \Sigma_2 \), so that \( (H \cap bd(A)) \cup P_1 \cup \cdots \cup P_\delta \) is a drawing in \( \Sigma_2 \). Let \( H' = H \cup P_1 \cup \cdots \cup P_\delta \); then \( H' \) is a drawing in \( \Sigma' \). Let \( \mathcal{T}' \) be the tangle in \( H' \) of order \( \theta \) induced by \( \mathcal{T} \), and let \( \mathcal{T}'' \) be the \( \phi \)-truncation of \( \mathcal{T}' \). We claim that \( H', \mathcal{T}' \) is a \( \Sigma' \)-span of order \( \phi \). To show this, we must show

(1) The following statements (i)-(iii) hold:

(i) \( H' \) is a rigid drawing in \( \Sigma' \)

(ii) \( \mathcal{T}' \) is a respectful tangle in \( H' \) of order \( \phi \)

(iii) \( \phi \leq \text{ord}(\mathcal{T}'') \), and \( (A \cap H', B \cap H') \in \mathcal{T}' \) for every \( (A, B) \in \mathcal{T}'' \) of order \( < \phi \).

Let \( \mathcal{T}_1 \) be the \( \phi \)-truncation of \( \mathcal{T} \). Then \( \mathcal{T}' \) is the tangle in \( H' \) induced by \( \mathcal{T}_1 \). We claim

(2) Every \( X \subseteq \{a_1, \ldots, a_\delta, b_1, \ldots, b_\delta\} \) with \( |X| \leq \phi \) is free with respect to \( \mathcal{T}_1 \).

**Subproof.** If \( X \) is not free, then since \( |X| \leq \phi \) and \( \mathcal{T}_1 \) has order \( \phi \), there exists \( (A, B) \in \mathcal{T}_1 \) of order \( < |X| \) with \( X \subseteq V(A) \). Then \( (A, B) \in \mathcal{T} \), and so \( X \) is not free with respect to \( \mathcal{T} \), contrary to [2, Theorem (12.2)]. This proves (2).

From (2) and [4, Theorem (5.2)] we deduce that \( \mathcal{T}' \) is respectful, so (1)(ii) holds. Next we show that (1)(iii) holds. For \( \text{ord}(\mathcal{T}) \leq \text{ord}(\mathcal{T}'') \) since \( H, \eta, \mathcal{T} \) is a \( \Sigma \)-span, and \( 2\phi \leq \text{ord}(\mathcal{T}) \) since \( \{a_1, \ldots, a_\delta, b_1, \ldots, b_\delta\} \) is free with respect to \( \mathcal{T} \). Thus \( \phi \leq 2\phi \leq \text{ord}(\mathcal{T}'') \). If \( (A, B) \in \mathcal{T}'' \) has order \( < \phi \), then \( (A \cap H', B \cap H') \) is a separation of \( H' \) of order \( < \phi \), and

\[
((A \cap H') \cap H, (B \cap H') \cap H) = (A \cap H, B \cap H) \in \mathcal{T}
\]

because \( H, \eta, \mathcal{T} \) is a \( \Sigma \)-span. Hence \( (A \cap H', B \cap H') \in \mathcal{T}'' \) by definition of \( \mathcal{T}'' \), and hence \( (A \cap H', B \cap H') \in \mathcal{T}' \) as required. Thus (1)(iii) holds.

To prove (1) it remains to show that \( H' \) is rigid, and to show this we shall use (3.2). Let \( X = \{a_1, \ldots, a_\delta, b_1, \ldots, b_\delta\} \).

(3) For every \( O \)-arc \( F \subseteq \Sigma \) with \( F \cap U(H') \subseteq V(H') \) and \( |F \cap V(H')| \leq 2 \), there is a dial \( A \subseteq \Sigma \) for \( F, H' \cap \Sigma_1 \) with \( |X \cap A| \leq |F \cap V(H')| \).
Subproof. Now $H' \cap \Sigma_1 = H$, and since $H$ is rigid there is a dial $A \subseteq \Sigma$ for $F, H$. By (3.1) it follows that $(H \cap A, H \cap \Sigma - A) \in \mathcal{F}$. Since $X$ is free with respect to $\mathcal{F}$, we deduce that

$$|X \cap A| = |X \cap V(H \cap A)| \leq |V((H \cap A) \cap (H \cap \Sigma - A))| = |V(H) \cap F| = |F \cap V(H')|.$$ 

This proves (3).

(4) For every O-arc $F \subseteq \Sigma_2$ with $F \cap U(H') \cap V(H')$ and $|F \cap V(H')| \leq 2$ there is a dial $A \subseteq \Sigma_2$ for $F, H'$.

This is easy to see, because since $\phi \geq 3$ and $|F \cap V(H')| \leq 2$, there exists $i$ such that $F \cap U(P_i) = \emptyset$, and hence $F$ bounds a closed disc in $\Sigma_2$.

(5) If $F \subseteq \Sigma_2$ is a line with ends $u, v \in \Sigma_1 \cap \Sigma_2 \cap V(H')$ and is otherwise disjoint from $U(H')$, and $P$ is a path of $H' \cap \Sigma_1 \cap \Sigma_2$ with ends $u, v$ and with $|V(P) \cap X| \leq 2$, then the O-arc $U(P) \cup F$ bounds an open disc in $\Sigma_2$ disjoint from $U(H)$.

Subproof. Since $|U(P) \cap X| \leq 2$ and $F \cap U(H') \subseteq U(P)$, it follows that $U(P) \cup F$ meets at most two of $P_1, \ldots, P_k$, and hence it bounds a disc in $\Sigma_2$. Again, the claim follows easily.

From (3), (4), and (5) and the fact that $\Sigma_1 \cap \Sigma_2$ has only one component, it follows from (3.2) that $H'$ is rigid in $\Sigma'$. This proves (1) and completes the proof of the theorem.

A second application of (3.2) is the following.

(3.4) Let $\Sigma$ be a surface and let $k \geq 3$ be an integer. Let $F^*$ be a tangle in a graph $G$, and let $H, \mathcal{F}$ be a $\Sigma$-span of order $0$. Let $A \subseteq \Sigma$ be a region of $H$ such that $H \cap bd(A)$ is a circuit, and let $v_1, \ldots, v_{2k}$ be distinct vertices of this circuit, in order, such that $\{v_1, \ldots, v_{2k}\}$ is free with respect to $\mathcal{F}$. Let $H' \subseteq \Sigma$ be a drawing in $\Sigma$ such that $H' \cap (\Sigma - A) = H$ and $H' = H \cup P_1 \cup \cdots \cup P_{k-1}$, where $P_i$ is a path with ends $v_i, v_{2i-1}$ for $1 \leq i \leq k-1$. Let $\mathcal{T}'$ be a tangle in $H'$ induced by $\mathcal{F}$; then $H'$, $\mathcal{T}'$ is a $\Sigma$-span of order $0$. Moreover, let $r_1, r_2 \subseteq A$ be the regions of $H'$ incident with $v_1, v_2$, respectively; then $d'(r_1, r_2) \geq 2k - 6$, and $d'(v_1, v_2) \geq 2k - 8$, where $d'$ is the metric of $\mathcal{F}'$.

Proof. To show that $H'$, $\mathcal{T}'$ is a $\Sigma$-span of order $0$, we must show

1. The following statements (i)-(iii) hold:
   (i) $H'$ is a rigid drawing in $\Sigma$
   (ii) $\mathcal{T}'$ is a respectful tangle of order $0$, and
   (iii) $d'(v_1, v_2) \geq 2k - 8$.

2. For every O-arc $F \subseteq \Sigma_2$ with $F \cap U(H') \cap V(H')$ and $|F \cap V(H')| \leq 2$ there is a dial $A \subseteq \Sigma_2$ for $F, H'$.

3. If $F \subseteq \Sigma_2$ is a line with ends $u, v \in \Sigma_1 \cap \Sigma_2 \cap V(H')$ and is otherwise disjoint from $U(H')$, and $P$ is a path of $H' \cap \Sigma_1 \cap \Sigma_2$ with ends $u, v$ and with $|V(P) \cap X| \leq 2$, then the O-arc $U(P) \cup F$ bounds an open disc in $\Sigma_2$ disjoint from $U(H)$.

Subproof. Since $|U(P) \cap X| \leq 2$ and $F \cap U(H') \subseteq U(P)$, it follows that $U(P) \cup F$ meets at most two of $P_1, \ldots, P_k$, and hence it bounds a disc in $\Sigma_2$. Again, the claim follows easily.

From (3), (4), and (5) and the fact that $\Sigma_1 \cap \Sigma_2$ has only one component, it follows from (3.2) that $H'$ is rigid in $\Sigma'$. This proves (1) and completes the proof of the theorem.
(iii) \( \theta \leq \text{ord}(\mathcal{F}) \), and \((A \cap H', B \cap H') \in \mathcal{F}'\) for every \((A, B) \in \mathcal{F}^*\) of order \( \theta \).

(1)(ii) is clear, and (1)(iii) is proved in the same way as in the proof of (3.3). It remains to show (1)(i). Let \( \Sigma_1 = \Sigma - A, \Sigma_2 = A \) and \( X = \{v_1, \ldots, v_{2k}\} \). We shall verify the hypotheses of (3.2). Now (3.2)(i) is trivial since \( \Sigma_1 \cap \Sigma_2 \) is connected; and (3.2)(ii) is proved as in the proof of (3.3). For (3.2)(iii), let \( F \subseteq \Sigma_2 \) be an \( O \)-arc with \( F \cap U(H') \subseteq \cup \{H'\} \) and \( \mid F \cap U(H') \mid \leq 2 \); then the closed disc in \( \Sigma_2 \) bounded by \( F \) is a dial for \( F \) because \( H' \cap \Sigma_2 \) is the union of \( H \cap bd(A) \) and the disjoint paths \( P_1, \ldots, P_{k-1} \). For (3.2)(iv), let \( u, v, F, P \) be as in (3.2)(iv). If \( F \cap r_1 \cup \{u, v\} \neq \emptyset \), then (3.2)(iv) is satisfied, and so we may assume that \( F \subseteq r_1 \cup \{u, v\} \). Let the four vertices in \( X \) incident with \( r_1 \) be \( v_i, v_{i+1}, v_{2k-i-1}, v_{2k-i} \), where \( 1 \leq i \leq k - 2 \). Now there is a path of \( H \cap \Sigma \) between \( u \) and \( v \), passing through at most two vertices in \( X \); and so \( u, v \) both lie in the same component of \( H \cap bd(r_1) \) (because of the vertices \( v_i, v_{2k} \in X \)) and again (3.2)(iv) is satisfied.

By (3.2), \( H' \) is rigid. Hence \( H', \mathcal{F}' \) is a \( \Sigma \)-span of order \( \theta \). Let \( d' \) be the metric of \( \mathcal{F}' \), and let \( e_1, e_2 \) be edges of \( P_1, P_{k-1} \) respectively. Now \( \text{ord}(\mathcal{F}') \leq 2k > 2(k - 1) \), and so by [4, Theorem (6.1)], \( d'(e_1, e_2) \geq 2k - 2 \). Hence \( d'(r_1, r_2) \geq 2k - 6 \), since \( d'(r_1, e_1) \leq 2 \) and \( d'(r_2, e_2) \leq 2 \); and \( d'(v_1, v_2) \geq 2k - 8 \), since \( d'(v_1, e_1) \leq 3 \) and \( d'(v_2, e_2) \leq 3 \).

Our third application of (3.2) is the following, which will be needed in a future paper.

(3.5) Let \( \Sigma \) be a surface and let \( \phi \geq 3 \) be an integer. Let \( \mathcal{F}^* \) be a tangle in a graph \( G \), and let \( H, \eta, \mathcal{F} \) be a \( \Sigma \)-span. Let \( A_1, A_2 \subseteq \Sigma \) be distinct regions of \( H \) so that \( H \cap bd(A_i) \) is a circuit \( C_i \) \((i = 1, 2) \), and no region of \( H \) is incident with a vertex in \( C_1 \) and with a vertex in \( C_2 \). Let \( a_1, \ldots, a_\phi \) be distinct vertices of \( C_1 \) in order, and let \( b_1, \ldots, b_\phi \) be distinct vertices of \( C_2 \) in order (under some orientations of \( C_1 \) and \( C_2 \)), such that \( \{a_1, \ldots, a_\phi\} \) and \( \{b_1, \ldots, b_\phi\} \) are free with respect to \( \mathcal{F} \). For \( 1 \leq i \leq \phi \) let \( P_i \) be an \( \eta(H) \)-path with ends \( \eta(a_i) \), \( \eta(b_i) \), such that \( P_1, \ldots, P_\phi \) are mutually disjoint. Then there is a \( \Sigma \)-span of order \( \phi \), where \( \Sigma \) is a surface obtained by adding a handle to \( \Sigma \).

Proof. We may assume that \( \eta \) is the identity. Let \( \Sigma_1 = \Sigma - (A_1 \cup A_2) \). Let \( \Sigma_2 \) be homeomorphic to a closed cylinder, with boundary \( U(C_1) \cup U(C_2) \), such that \( \Sigma' = \Sigma_1 \cup \Sigma_2 \) is a surface obtained from \( \Sigma \) by adding a handle, and there are \( \phi \) disjoint lines in \( \Sigma_2 \) with ends \( a_i, b_i \) \((1 \leq i \leq \phi) \). We may therefore assume that \( P_1, \ldots, P_\phi \) are drawings in \( \Sigma_2 \) and hence in \( \Sigma' \). Let \( H' = H \cup P_1 \cup \cdots \cup P_\phi \), let \( \mathcal{F}' \) be the tangle in \( H' \) of order \( \theta \) induced by \( \mathcal{F} \), and let \( \mathcal{F}'' \) be the \( \phi \)-truncation of \( \mathcal{F}' \). We claim that
$H'$ is a $\Sigma$-span of order $\phi$. To show this, it follows by an argument similar to that in the proof of (3.3), using [4, Theorem (5.1)] in place of [4, Theorem (5.2)], that $\mathcal{T}'$ is respectful and that it suffices to prove that $H'$ is rigid. To show that $H'$ is rigid we verify (3.2)(i)(iv). Now (3.2)(i) holds by hypothesis. For (3.2)(ii), let $X = \{a_1, ..., a_{\phi}, b_1, ..., b_{\phi}\}$.

(1) For every $O$-arc $F \subseteq \Sigma$ with $F \cap U(H') \subseteq V(H')$ and $|F \cap V(H')| \leq 2$, there is a dial $A \subseteq \Sigma$ for $F, H' \cap \Sigma$, with $|X \cap A| \leq |F \cap V(H')|$.

Subproof. As in the proof of (3) in (3.3), there is a dial $2 \subseteq \Sigma$ for $F, H$, and $(H \cap \Delta, H \cap \Sigma - \Delta) \in \mathcal{T}$. Since $|F \cap V(H')| \leq 2$ and no region of $H$ is incident with a vertex of $C_1$ and a vertex of $C_2$, we may therefore assume that $F \cap U(C_2) = \emptyset$. Since $\{b_1, ..., b_{\phi}\}$ is free with respect to $\mathcal{T}$ and $\phi \geq 3$, it follows that $\{b_1, ..., b_{\phi}\} \not\subseteq \Delta$, and so $A \cap \Delta = \emptyset$. Moreover, since $\{a_1, ..., a_{\phi}\}$ is free with respect to $\mathcal{T}$, we deduce that $|\{a_1, ..., a_{\phi}\} \cap A| \leq |F \cap V(H')|$. Consequently $|X \cap A| \leq |F \cap V(H')|$. This proves (1).

(3.2)(ii) follows, and (3.2)(iii) is easy to see. (3.2)(iv) is proved as in step (5) of (3.3). From (3.2) we therefore deduce that $H'$ is rigid in $\Sigma'$, as required.

4. REARRANGEMENT

In this section, we develop an assortment of lemmas about compressions and rearrangement. If $\mathcal{T}$ is a respectful tangle of order $\theta$ in a 2-cell drawing $H$ in $\Sigma$, and $K$ is a radial graph of $G$, we define $\text{ins}(C)$, for each circuit $C$ of $K$, to be the closed disc $\Delta \subseteq \Sigma$ bounded by $U(C)$ such that

$$(H \cap \Delta, H \cap \Sigma - \Delta) \in \mathcal{T}.$$  

(This exists since $\mathcal{T}$ is respectful.) For any closed walk $W$ of $K$ of length $< 2\theta$, we define $\text{ins}(W)$ to be the union of $U(K')$ and $\text{ins}(C)$ taken over all circuits $C$ of $K'$, where $K' \subseteq K$ is the subdrawing formed by the vertices and edges in $W$. We call $\text{ins}$ the slope corresponding to $\mathcal{T}$. For tangles $\mathcal{T}'$, $\mathcal{T}''$, $\mathcal{T}_0$ etc. we shall consistently denote the corresponding slopes by $\text{ins}'$, $\text{ins}''$, $\text{ins}_0$ etc. without further definition, and we use a similar convention for metrics ($d'$, $d''$, $d_0$ etc.).

(4.1) Let $\mathcal{T}$ be a respectful tangle of order $\theta$ in a 2-cell drawing $H$ in a surface $\Sigma$. Let $H_1, H_2$ be 2-cell drawings in $\Sigma$, and for $i = 1, 2$ let $\mathcal{T}_i$ be a $\lambda_i$-compression of $\mathcal{T}$ in $H_i$. Let $H_1, H_2$ have radial drawings $K_1, K_2$. Then
for every $W$ which is a closed walk of a common subdrawing of $K_1$ and $K_2$, if $\text{ins}_1(W)$, $\text{ins}_2(W)$ are both defined then they are equal. In particular, if $H_1 = a$ and $\lambda_1 \geq \lambda_2$, then $\mathcal{F}_1 \subseteq \mathcal{F}_2$.

Proof. Since $\text{ins}_1(W)$ is determined by $\text{ins}_1(C)$ for the circuits $C$ of the subdrawing formed by the vertices and edges in $W$, it suffices to show that $\text{ins}_1(C) = \text{ins}_2(C)$ for every $C$ with $|E(C)| < 2 \min(\theta - \lambda_1, \theta - \lambda_2)$ which is a circuit of both $K_1$ and $K_2$. Choose $v \in V(C)$. By [3, Theorem (8.12)], there exists $\sigma \in \Sigma$ with $d(v, \sigma) = 0$, and hence with $d_i(v, \sigma) \geq \theta - \lambda$ for $i = 1, 2$. Hence $\sigma \notin \text{ins}_1(C)$ and $\sigma \notin \text{ins}_2(C)$, and so $\text{ins}_1(C) = \text{ins}_2(C)$, since there is at most one closed disc in $\Sigma$ bounded by $U(C)$ not containing $\sigma$.

In particular, let $H_1 = a$ and $\lambda_1 \geq \lambda_2$; and hence we may assume that $K_1 = K_2 = K$. Then $\text{ins}_1(C) = \text{ins}_2(C)$ for every circuit $C$ of $K$ with $< 2(\theta - \lambda_1)$ edges, and hence $\mathcal{F}_1 \subseteq \mathcal{F}_2$ by [3, Theorem (6.3)].

(4.2) Let $H, H'$ be 2-cell drawings in a surface $\Sigma$, and let $Z \subseteq \Sigma$ include every atom of $H$ which is not in $A(H)$. (Hence $Z$ includes every atom of $H'$ not in $A(H)$.) Let $\mathcal{F}, \mathcal{F}'$ be respectful tangles in $H, H'$ respectively, such that, for some $\lambda \geq 0$, one of $\mathcal{F}, \mathcal{F}'$ is a $\lambda$-compression of the other. Then for all $x, y \in Z$, if $d(x, y) < d(x, \sigma)$ for all $\sigma \in Z$, and $d(x, y) < \text{ord}(\mathcal{F})$, then $d'(x, y) \leq d(x, y)$.

Proof. Let $K$ be a radial drawing of $H$. Since $d(x, y) < \text{ord}(\mathcal{F})$, there is a closed walk $W$ of $K$ of length $2d(x, y)$ such that $\text{inst}(W) \cap a \cap b \neq \emptyset$, where $a, b \in A(H)$ include $x, y$ respectively. Thus $K$ may be chosen so that $x, y \in \text{inst}(W)$. Now no atom of $H$ which intersects $\text{inst}(W)$ also intersects $Z$, since $d(x, y) < d(x, \sigma)$ for all $\sigma \in Z$. Consequently every atom of $H$ which intersects $\text{inst}(W)$ is an atom of $H'$, and we may therefore choose a radial drawing $K'$ of $H'$ such that $W$ is a walk of $K'$. If $d(x, y) \geq \text{ord}(\mathcal{F}')$ then certainly $d(x, y) \geq d'(x, y)$ as required; and so we may assume that $d(x, y) < \text{ord}(\mathcal{F}')$, and $\text{ins}'(W)$ is defined. By (4.1), $\text{ins}'(W) = \text{ins}'(W)$, and so $x, y \in \text{ins}'(W)$, and $d'(x, y) \leq d(x, y)$ as required.

Let $H$ be a 2-cell drawing in $\Sigma$, let $\mathcal{F}$ be a respectful tangle in $H$ of order $\theta$ with metric $d$, let $\lambda > 0$ be an integer with $\lambda < \theta$, let $\sigma \in A(H)$, and let $H'$ be the subdrawing of $H$ formed by all edges of $H$ with $d(\sigma, e) > \lambda$, and their ends (or $H' = H$, if $E(H) = \emptyset$). We define $H - z^\perp H'$ (we shall only use this notation when the hidden parameters $\Sigma, \mathcal{F}$ are obvious). [4, Theorem (7.11)] implies that

(4.3) If $H, \Sigma, \mathcal{F}$, $\theta$, $d$, $\lambda$, $z$ are as above then $H - z^\perp$ is 2-cell.

[4, Theorem (7.10)] implies
Let $T$ be a respectful tangle of order $0$ in a 2-cell drawing $H$ in a surface $\Sigma$. Let $z \in A(H)$, let $\lambda \geq 2$ be an integer with $\theta > 4\lambda + 2$, and let $H' \subseteq H$ be a 2-cell subdrawing such that $a \in A(H')$ for every $a \in A(H)$ with $d(z, a) > \lambda$. Then there is a unique respectful tangle $T'$ in $H'$ which is a $(4\lambda + 2)$-compression of $T$, and it includes $(A \cap H', B \cap H')$ for every $(A, B) \in T$ of order $< \theta - 4\lambda - 2$.

Let $T$ be a respectful tangle of order $0$ in a 2-cell drawing $H$ in a surface $\Sigma$. Let $z \in A(H)$, let $\lambda \geq 2$ be an integer with $\theta > 4\lambda + 2$, and let $H'$ be a 2-cell drawing in $\Sigma$ such that $a \in A(H')$ for every $a \in A(H)$ with $d(z, a) > \lambda$. Then

(i) there is a unique $(4\lambda + 2)$-compression $T'$ of $T$ in $H'$

(ii) $H - z^\epsilon \subseteq H'$, and $T'$ is the tangle induced in $H'$ from the unique $(4\lambda + 2)$-compression of $T$ in $H - z^\epsilon$

(iii) if $H' \subseteq H$ then $(A \cap H', B \cap H') \in T'$ for every $(A, B) \in T$ of order $< \theta - 4\lambda - 2$.

Proof. By (4.3) and (4.4), there is a unique $(4\lambda + 2)$-compression of $T$ in $H - z^\epsilon$, $T_0$ say. Since $d(z, e) > \lambda$ for every edge $e$ of $H - z^\epsilon$ and $H - z^\epsilon$ has no isolated vertices (unless $E(H) = \emptyset$, when $\lambda = 0$ and $H = H'$), it follows that $H - z^\epsilon \subseteq H'$. Let $T''$ be the tangle in $H'$ induced by $T_0$. Then $T''$ is a $(4\lambda + 2)$-compression, and hence $T'' = T'$, by (4.1). This proves (i) and (ii), and (iii) follows from (4.4).

Let $T, 0, H, \Sigma, z, \lambda, H', T'$ be as in (4.5). Let $T^*$ be a tangle in a graph $G$, and let $H, \eta, T$ be a $\Sigma$-span in $G$ with respect to $T^*$. Let $H'$ be rigid, and let $\eta'$ be an isomorphism from $H'$ to a subgraph of $G$, such that $\eta'(x) = \eta(x)$ for every vertex or edge $x$ of $H$ with $d(z, x) > \lambda$. Then $H', \eta'$, $T'$ is a $\Sigma$-span of order $\theta - 4\lambda - 2$, obtained from $H, \eta, T$ by rearranging within $\lambda$ of $z$.

Proof. We must verify conditions (i)–(iv) in the definition of $\Sigma$-span. Conditions (i) and (ii) are trivial, and since $T'$ is a $(4\lambda + 2)$-compression it is by definition respectful, and so (iii) holds. It remains to show (iv). Let $(A^*, B^*) \in T^*$ with order $< \theta - 4\lambda - 2$. We must show that

$$(\eta'^{-1}(A^* \cap \eta'(H')), \eta'^{-1}(B^* \cap \eta'(H'))) \in T'.
$$

Let $H'' = H - z^\epsilon$, and let $T''$ be the unique $(4\lambda + 2)$-compression of $T$ in $H''$. By (4.5)(ii), $T'$ is the tangle induced in $H'$ by $T''$; and so it suffices to show that

$$(\eta'^{-1}(A^* \cap \eta'(H')) \cap H'', \eta'^{-1}(B^* \cap \eta'(H')) \cap H'') \in T''.
$$
But
\[ \eta^{-1}(A^* \cap \eta(H^*)) \cap H^* = \eta^{-1}(A^* \cap \eta(H^*)) = \eta^{-1}(A^* \cap \eta(H^*)) \]
since \( H^* \subseteq H' \) by (4.5(ii)); and \( \eta^{-1}(A^* \cap \eta(H^*)) = \eta^{-1}(A^* \cap \eta(H^*)) \),
because the restriction of \( \eta \) and \( \eta' \) to \( H^* \) are equal. Similarly,
\[ \eta^{-1}(B^* \cap \eta(H^*)) \cap H^* = \eta^{-1}(B^* \cap \eta(H^*)) , \]
and so it suffices to show that
\[ (\eta^{-1}(A^* \cap \eta(H^*)), \eta^{-1}(B^* \cap \eta(H^*))) \in \mathcal{F}^* . \]

Let \( A = \eta^{-1}(A^* \cap \eta(H)), \ B = \eta^{-1}(B^* \cap \eta(H)) \). Then \( (A, B) \in \mathcal{F} \) since
\( H, \eta, \mathcal{F} \) is a \( \Sigma \)-span. Moreover, \( (A, B) \) has order at most that of \( (A^*, B^*) \)
and hence less than \( \theta - 4 \lambda - 2 \). By (4.5(iii)), \( (A \cap H^*, B \cap H^*) \in \mathcal{F}^* \). But
\[ A \cap H^* = \eta^{-1}(A^* \cap \eta(H)) \cap H^* = \eta^{-1}(A^* \cap \eta(H) \cap \eta(H^*)) \]
\[ = \eta^{-1}(A^* \cap \eta(H^*)) \]
and similarly \( B \cap H^* = \eta^{-1}(B^* \cap \eta(H^*)) \), and so
\[ (\eta^{-1}(A^* \cap \eta(H^*)), \eta^{-1}(B^* \cap \eta(H^*))) \in \mathcal{F}^* \]
as required. \( \square \)

We shall also need the following.

\[ (4.7) \text{ Let } \mathcal{F} \text{ be a respectful tangle of order } \theta \text{ in a } 2 \text{-cell drawing } H \text{ in a surface } \Sigma. \text{ Let } z \in A(H), \text{ let } \lambda \geq 2 \text{ be an integer, and let } X \subseteq V(H) \text{ be free with respect to } \mathcal{F}. \text{ Suppose that } d(z, v) > 2 |X| + 5 \lambda + 2 \text{ for all } v \in X. \text{ Now let } H' \text{ be a } 2 \text{-cell drawing in } \Sigma \text{ such that } a \in A(H') \text{ for every } a \in A(H) \text{ with } d(z, a) > \lambda; \text{ and let } \mathcal{F}' \text{ be a } (4 \lambda + 2) \text{-compression of } \mathcal{F} \text{ in } H'. \text{ Then } X \text{ is free with respect to } \mathcal{F}'. \]

\textbf{Proof}. Suppose not; then there exists \( (A, B) \in \mathcal{F}' \) with \( |X \cap V(A)| > |V(A \cap B)| \). Choose such \( (A, B) \) with \( A \) minimal.

1) \( A \) is connected.

\textbf{Subproof}. Suppose that \( A = A_1 \cup A_2 \), where \( A_1, A_2 \) are non-null and \( A_1 \cap A_2 \) is null. Then \( (A_1, A_2 \cup B), (A_2, A_1 \cup B) \in \mathcal{F}, \) and by the minimality of \( A \),
\[ |X \cap V(A_1)| \leq |V(A_1 \cap (A_2 \cup B))| = |V(A_1 \cap B)| \]
\[ |X \cap V(A_2)| \leq |V(A_2 \cap (A_1 \cup B))| = |V(A_2 \cap B)| . \]
Adding, we deduce that \(|X \cap V(A)| \leq |V(A \cap B)|\), a contradiction. This proves (1).

Now \(|X \cap V(A)| > |V(A \cap B)| \geq 0\), and so we may choose \(v \in X \cap V(A)\).

(2) For every \(u \in V(A)\), \(d'(u, v) \leq 2 |X| - 2\).

Subproof. By (1) there is a path \(P\) of \(A\) from \(u\) to \(v\). By [4, Theorem (4.1)], \(|V(A \cap B)| \geq (1/2) d'(u, v)\), and so \(d'(u, v) \leq 2 |V(A \cap B)| \leq 2 |X| - 2\). This proves (2).

(3) Every \(v \in V(A)\) belongs to \(V(H)\) and every edge of \(H'\) incident with \(u\) is an edge of \(H\).

Subproof. Let \(e \in E(H')\) be incident with \(u \in V(A)\). Let \(\sigma \in \Sigma\) with \(\sigma \in e\). By (2), \(d'(\sigma, v) \leq 2 |X| - 2\). Hence \(d(\sigma, v) \leq 2 |X| + 4 \lambda + 2\), since \(\mathcal{F}'\) is a \((4 \lambda + 2)\)-compression of \(\mathcal{F}\). Since \(d(z, \sigma) > \lambda\), \(d(z, v) > 2 |X| + 5 \lambda + 2\), it follows that \(d(z, u) > \lambda\). Let \(a\) be the atom of \(H\) with \(\sigma \in a\). Then \(d(z, a) > \lambda\), and so \(a \in A(H')\). Hence \(a = e\), and so \(e \in E(H)\). Similarly, by setting \(\sigma = u\) and applying (2), we deduce that \(u \in V(H)\). This proves (3).

From (3), \(A\) is a subgraph of \(H\).

(4) For all \(u \in V(A)\), every edge of \(H\) incident with \(u\) is an edge of \(H'\).

Subproof. Let \(e \in E(H)\) be incident with \(u\). Since \(d'(u, v) \leq 2 |X| - 2\) and hence \(d(u, v) \leq 2 |X| - 2 + 4 \lambda + 2\), it follows that \(d(z, v) > 2 |X| + 5 \lambda + 2\). Hence \(d(z, e) > \lambda\), and so \(e \in E(H')\). This proves (4).

By (3) and (4), there is a subgraph \(C\) of \(H\) such that \((A, C)\) is a separation of \(H\) and \(A \cap B = A \cap C\). Now \(H_0 = H - v^*\) is a subgraph of both \(H\) and \(H'\); let \(\mathcal{F}_0\) be the \((4 \lambda + 2)\)-compression of \(\mathcal{F}\) in \(H_0\). Since \(\theta > 4 \lambda + 2\), it follows from (4.4) that \((A \cap H_0, B \cap H_0) \in \mathcal{F}_0\), since \((A, B)\) has order \(< |X| \leq \theta - 4 \lambda - 2\). Since by (4.5) \(\mathcal{F}'\) is the tangle in \(H'\) induced by \(\mathcal{F}_0\), and \(C \cap H_0 = B \cap H_0\), it follows that \((A, C) \in \mathcal{F}'\). But

\[
|X \cap V(A)| > |V(A \cap B)| = |V(A \cap C)|
\]

and so \(X\) is not free with respect to \(\mathcal{F}\), by [2, Theorem (12.2)], a contradiction. This completes the proof. 

5. FLATNESS

In this section we prove that, if \(\lambda\) is large enough, rearranging a \((\lambda, \mu)\)-flat \(\Sigma\)-span produces another \(\Sigma\)-span which is still reasonably flat. First, we need the following.
(5.1) Let $\mathcal{F}$ be a tangle in $G$, and let $H, \eta, \mathcal{F}$ be a $(\lambda, \mu)$-flat $\Sigma$-span of order $0$. Let $0 \leq \lambda_1 \leq \lambda$, let $\mu_1 \geq \mu$ and let $1 \leq \theta_1 \leq \theta$, such that $\theta_1 \geq 4\lambda_1 + \mu_1 + 2$; and let $\mathcal{T}_i$ be the $\theta_i$-truncation of $\mathcal{F}$. Then $(H, \eta, \mathcal{T}_i)$ is $(\lambda_1, \mu_1)$-flat.

Proof. Certainly $(H, \eta, \mathcal{T}_i)$ is a $\Sigma$-span of order $\theta_1$, and $\theta_1 \geq 4\lambda_1 + \mu_1 + 2$. Suppose it is not $(\lambda_1, \mu_1)$-flat. Let $H', \eta', \mathcal{T}_i'$ be obtained from $H, \eta, \mathcal{T}_i$ by rearranging within $\lambda_1$ of some $z \in A(H)$, and let $P$ be an $\eta'(H')$-path in $G$ with ends $\eta'(u), \eta'(v)$, where $d'(u, v) \geq \mu_1$.

(1) $a \in A(H')$ for every $a \in A(H)$ with $d(z, a) > \lambda$.

Subproof. Now $d(z, a) = \min(\lambda_1, d(z, a))$, and since $\theta_1 > \lambda_1$ and $d(z, a) > \lambda \geq \lambda_1$ it follows that $d(z, a) > \lambda_1$. Hence $a \in A(H')$, since $H', \eta', \mathcal{T}_i$ is obtained from $H, \eta, \mathcal{T}_i$ by rearranging within $\lambda_1$ of $z$. This proves (1).

Since $\mathcal{T}_i$ is a $(4\lambda_1 + 2)$-compression of $\mathcal{T}_i$, and $\mathcal{T}_1$ is a $(\theta - \theta_1)$-compression of $\mathcal{T}$, it follows that $\mathcal{T}_1$ is a $(\theta - \theta_1 + 4\lambda_1 + 2)$-compression of $\mathcal{T}$. Let $\mathcal{T}'$ be the $(4\lambda + 2)$-compression of $\mathcal{T}$ in $H'$ (this exists by (4.5)). By (4.1), if $\theta - 4\lambda_1 - 2 \geq \theta_1 - 4\lambda_1 - 2$ then $\mathcal{T}_1' \subseteq \mathcal{T}'$, and if $\theta - 4\lambda_1 - 2 \geq \theta_1 - 4\lambda_1 - 2$ then $\mathcal{T}' \subseteq \mathcal{T}_1$.

(2) $d'(u, v) \geq \mu$.

Subproof. Certainly $d'(u, v) \geq \mu_1 \geq \mu$. If $\mathcal{T}' \subseteq \mathcal{T}_1'$ then $d'(u, v) = \min(\theta - 4\lambda_1 - 2, d'(u, v)) \geq \mu$

since $\theta \geq 4\lambda_1 + \mu$ (because $H, \eta, \mathcal{T}$ is $(\lambda, \mu)$-flat). If $\mathcal{T}_1' \subseteq \mathcal{T}'$ then $d'(u, v) \geq d'(u, v) \geq \mu$. This proves (2).

From (1) and (2), $H', \eta', \mathcal{T}'$ is a $\mu$-stepped $\Sigma$-span, obtained from $H, \eta, \mathcal{T}$ by rearranging within $\lambda$ of $z$, which is impossible since $H, \eta, \mathcal{T}$ is $(\lambda, \mu)$-flat. The result follows.

The main result of this section is the following.

(5.2) Let $\mathcal{F}$ be a tangle in a graph $G$, let $\Sigma$ be a surface, and let $H, \eta, \mathcal{F}$ be a $(\lambda, \mu)$-flat $\Sigma$-span of order $0$. Let $\lambda_1, \lambda_2 \geq 4$ and $\mu \geq 0$ be integers such that $\lambda \geq 6\lambda_1 + 5\lambda_2 + 3\mu + 4$. Let $z_1 \in A(H)$, and let $H', \eta', \mathcal{F}'$ be a $\Sigma$-span obtained from $H, \eta, \mathcal{F}$ by rearranging within $\lambda_1$ of $z_1$. Then $H', \eta', \mathcal{F}'$ is $(\lambda_2, \mu)$-flat.

Proof. Let $\mathcal{F}$ have order $\theta$; then $\mathcal{F}'$ has order $\theta - 4\lambda_1 - 2$. Let $z_2 \in A(H')$, and let $H', \eta', \mathcal{F}''$ be obtained from $H', \eta', \mathcal{F}'$ by rearranging within $\lambda_2$ of $z_2$; and suppose, for a contradiction, that $H', \eta', \mathcal{F}''$ is $\mu$-stepped. Let $P$ be an $\eta'(H')$-path in $G$ with ends $\eta'(u), \eta'(v)$, where
Since $\mathcal{F}'$ is a $(4\lambda_1 + 2)$-compression of $\mathcal{F}$, and $\mathcal{F}''$ is a $(4\lambda_2 + 2)$-compression of $\mathcal{F}'$, we deduce

(1) For all $\sigma_1, \sigma_2 \in \Sigma$, $d'(\sigma_1, \sigma_2) \geq d(\sigma_1, \sigma_2) - 4\lambda_1 - 2$ and $d''(\sigma_1, \sigma_2) \geq d'(\sigma_1, \sigma_2) - 4\lambda_2 - 2$.

Let $Z_1$ be the union of all atoms $a$ of $H$ with $d(z_1, a) \leq \lambda_1$, and let $Z_2$ be the union of all atoms $a$ of $H'$ with $d'(z_1, a) \leq \lambda_2$. Then, again immediately from the definitions, we have

(2) $Z_1$ includes every $a \in A(H) - A(H')$, and $Z_2$ includes every $a \in A(H') - A(H)$.

Choose $\tau \in \Sigma$ with $\tau \in z_2$, and let $z_2 \in A(H)$ with $\tau \in z_2$.

(3) $d(\tau, z_2) > \lambda - \lambda_1$.

Subproof. Suppose not. Let $\mathcal{F}^0$ be the $(\theta - 4\lambda - 2)$-truncation of $\mathcal{F}''$. (This exists, since $ord(\mathcal{F}''') = \theta - 4(\lambda_1 + \lambda_2) - 4 \geq \theta - 4\lambda - 2 > 1$.) We claim that $H''$, $\eta''$, $\mathcal{F}^0$ is a $\Sigma$-span obtained from $H, \eta, \mathcal{F}$ by rearranging within $\lambda$ of $z_2$, and it is $\mu$-stepped. Certainly it is a $\Sigma$-span, because $H''$, $\eta''$, $\mathcal{F}''$ is a $\Sigma$-span and $\mathcal{F}^0 \subseteq \mathcal{F}''$; and it is $\mu$-stepped, because

$$d''(u, v) = \max(\theta - 4\lambda - 2, d''(u, v)) \geq \mu$$

since $\theta \geq 4\lambda + \mu + 2$ and $d''(u, v) \geq \mu$. To see that $H''$, $\eta''$, $\mathcal{F}^0$ is obtained from $H, \eta, \mathcal{F}$ by rearranging within $\lambda$ of $z_2$, we must check that

(i) $a \in A(H'')$ for all $a \in A(H)$ with $d(z_1, a) > \lambda$.

(ii) $\eta(x) = \eta''(x)$ for every vertex or edge $x$ of $H$ with $d(z_1, x) > \lambda$, and

(iii) $\mathcal{F}^0$ is a $\lambda$-compression of $\mathcal{F}$.

To show (i), let $a \in A(H)$ with $d(z_1, a) > \lambda$. Since by hypothesis $d(z_1, z_2) \leq \lambda - \lambda_1$, it follows that $d(z_1, a) > \lambda_1$, and so $a \cap Z_1 = \emptyset$. Consequently, by (2), $a \in A(H')$. Now $d(\tau, a) > \lambda$, and so $d'(\tau, a) > \lambda - 4\lambda_1 - 2$ since $\mathcal{F}'$ is a $(4\lambda_1 + 2)$-compression of $\mathcal{F}$. Since $\lambda - 4\lambda_1 - 2 \geq \lambda_2$, it follows that $a \cap Z_2 = \emptyset$ and so $a \in A(H'')$, by (2). This proves (i), and (ii) follows immediately. For (iii), let $\sigma_1, \sigma_2 \in \Sigma$. Then

$$d''(\sigma_1, \sigma_2) = \min(\theta - 4\lambda - 2, d''(\sigma_1, \sigma_2))$$

$$\geq \min(\theta - 4\lambda - 2, d(\sigma_1, \sigma_2) - 4(\lambda_1 + \lambda_2) - 4)$$

by (1); and

$$\min(\theta - 4\lambda - 2, d(\sigma_1, \sigma_2) - 4(\lambda_1 + \lambda_2) - 4) \geq d(\sigma_1, \sigma_2) - 4\lambda - 2$$
because \( \theta \geq d(\sigma_1, \sigma_2) \) and \( \lambda_1 + \lambda_2 < \lambda \). This proves (iii), and hence proves that \( H', \eta' \), \( F' \) is obtained from \( H, \eta, F \) by rearranging within \( \lambda \) of \( z_1 \).

But it is \( \mu \)-stepped, a contradiction since \( H, \eta, F \) is \( (\lambda, \mu) \)-flat. This proves (3).

(4) For all \( \sigma_1 \in Z_1 \), \( d(\tau_1, \sigma_1) > \lambda - 2\lambda_1 \), and hence \( d'(\tau_1, \sigma_1) > \lambda - 6\lambda_1 - 2 \).

Subproof. By (3), \( \lambda - \lambda_1 < d(z_1, \tau_1) \leq d(z_1, \sigma_1) + d(\sigma_1, \tau_1) \leq \lambda_1 + d(\tau_1, \sigma_1) \), and so the first inequality holds. The second follows since \( F \) is rigid, hence (4.2) (applied to \( F \)). This proves (4).

(5) For all \( \sigma_2 \in Z_2 \), \( d(\sigma_2, \tau_1) \leq \lambda_2 \).

Subproof. For all \( \sigma_2 \in Z_1 \), \( d'(\sigma_2, \tau_1) > \lambda - 2\lambda_1 - 2 \), \( \lambda_2 \geq d'(\tau_1, \sigma_2) \) by (4).

Hence, by (4.2) (applied to \( F, F', Z_1 \)), \( d(\tau_1, \sigma_2) \leq d'(\tau_1, \sigma_2) \). This proves (5).

(6) For all \( \sigma_1 \in Z_1 \) and \( \sigma_2 \in Z_2 \), \( d(\sigma_1, \sigma_2) > \lambda - 2\lambda_1 - \lambda_2 \).

Subproof. Since \( d(z_1, \tau_1) \leq d(z_1, \sigma_1) + d(\sigma_1, \sigma_2) + d(\sigma_2, \tau_1) \), the claim follows from (3) and (5).

In particular, from (6) \( Z_1 \cap Z_2 = \emptyset \) and so \( z_2 = z_5 \). Let \( H_0 \subseteq H \) be the subdrawing formed by the edges of \( H \) not in \( Z_1 \cup Z_2 \) and their ends. Let \( L_1, M_1, L_2, M_2 \) be the subdrawings of \( H, H', H'' \) formed by its edges in \( Z_1, Z_1, Z_2, Z_2 \) and their ends, respectively. Then \( H = H_0 \cup L_1 \cup L_2 \), \( H'' = H_0 \cup M_1 \cup M_2 \).

(7) \( L_1 \) and \( L_2 \) both have circuits.

Subproof. There is a region \( r \) of \( H \) with \( d(z_1, r) = 2 \); and hence \( d(z_1, e) \leq 4 \leq \lambda_1 \), and so \( e \in \Lambda_1 \), for every edge \( e \) of \( H \) incident with \( r \). Since \( H \) is a forest (because it has a tangle of order \( \geq 3 \)) this set of edges includes a circuit. Similarly \( L_2 \) has a circuit, since \( \lambda_1 \geq 4 \). This proves (7).

Let \( H'' = H_0 \cup L_1 \cup M_2 \).

(8) \( H'' \) is rigid and 2-cell.

Subproof. Let \( F \subseteq \Sigma \) be an \( O \)-arc with \( F \cap (H'' \cap V(H')) \cap V(H') \) and \( F \cap (H'' \cap V(H')) \cap V(H') \). Suppose first that \( F \cap \Lambda_1 = \emptyset \). Then \( F \cap (H'' \cap V(H)) \cap V(H) \) and \( F \cap V(H) \), and \( F \cap V(H') \), and \( F \cap V(H) \), and \( F \cap V(H') \), and \( F \cap V(H') \), and \( F \cap V(H') \). Suppose now that \( F \cap Z_1 = \emptyset \); then a similar argument applied to \( H' \) yields the desired dial. Finally, we assume that \( F \cap Z_1 \neq \emptyset \). But \( F \cap U(H'') \neq \emptyset \), and this is impossible by (6). This proves that \( H'' \) is rigid.
Since $bd(\Sigma) = \emptyset$ and $H''$ is non-null, it follows that $H''$ is 2-cell. This proves (8).

(9) Every component of $\eta'(M_1)$ meets $\eta(L_1)$, and every component of $\eta'(M_2)$ meets $\eta'(L_2)$.

Subproof. Let $C$ be a component of $M_1$ (so $\eta'(C)$ is a component of $\eta'(M_1)$). Now $M_1 \neq H'$ since $L_2$ is not null by (7), and since $H'$ is 2-cell and hence connected and $Z_1 \cap Z_2 = \emptyset$, there is a vertex $x \in V(M_1)$ incident with an edge $e \in H'$ not in $E(M_1)$, and with its other end not in $V(M_1)$. Then $e \in E(H)$ and so $x \in V(H)$, and since $x \in Z_2$ it follows that $x \in V(L_1)$. Moreover, $\eta(x) = \eta'(x)$ (because $\eta(N) = \eta'(e)$ and $\eta(y) = \eta'(y)$ where $e$ has ends $x, y$). Hence $\eta(x) \in V(\eta'(C) \cap V(\eta(L_1)))$, and so $\eta'(C)$ meets $\eta(L_1)$, as required. The argument for $\eta'(M_2)$ is similar. This proves (9).

Let $X_1$ be the union of $V(\eta'(L_1)) \cup V(\eta'(M_1))$ with the set
\[
\{ \eta(x) : x \in V(H) \text{ and } d(\sigma_1, x) < \mu \text{ for some } \sigma_1 \in Z_1 \}.
\]

Let $X_2$ be the union of $V(\eta'(L_2)) \cup V(\eta'(M_2))$ with
\[
\{ \eta(x) : x \in V(H) \text{ and } d(\sigma_2, x) < \mu \text{ for some } \sigma_2 \in Z_2 \}.
\]

Then $X_1, X_2 \subseteq V(G)$.

(10) Every path of $G$ from $X_1$ to $X_2$ has an internal vertex or edge in $\eta(H)$.

Subproof. Suppose that some path of $G$ from $X_1$ to $X_2$ has no internal vertex or edge in $\eta(H)$. By (9), there is an $\eta(H)$-path with ends $\eta(t_1), \eta(t_2)$ where $d(\sigma_i, t_i) < \mu$ for some $\sigma_i \in Z_i (i = 1, 2)$. Since $H, \eta, \mathcal{F}$ is $(0, \mu)$-flat by (5.1), it follows that $d(t_1, t_2) < \mu$, and so $d(\sigma_1, \sigma_2) < 3\mu < (2\lambda_1 + \lambda_2 - \lambda_2)$ contrary to (6). This proves (10).

It follows from (10) that $\eta(L_1)$ is disjoint from $\eta'(M_2)$. For each vertex or edge $x$ of $H_1$, define $\eta''(x) = \eta(x)$ unless $x$ belongs to $M_2$, when $\eta''(x) = \eta'(x)$. Then $\eta''$ is an isomorphism from $H''$ to the subgraph $\eta(H_0) \cup \eta(L_1) \cup \eta'(M_2)$ of $G$.

(11) $a \in A(H'')$ for every $a \in A(H)$ with $d(z_2, a) > \lambda_2$, and for every vertex or edge $x$ of $H''$, if $d(z_2, x) > \lambda_2$ then $\eta''(x) = \eta(x)$.

Subproof. If $a \in A(H)$ with $d(z_2, a) > \lambda_2$, then $a \cap Z_2 = \emptyset$ by (5), and so $a \in A(H'')$. The second claim follows similarly.

Let $\mathcal{F}''$ be the (unique, by (4.5)) $(4\lambda_2 + 2)$-compression of $\mathcal{F}$ in $H''$. Then from (4.6), (8) and (11), we have
(12) \( H'', \eta'', \mathcal{F}'') \) is a \( \Sigma \)-span of order \( \theta-4\lambda_2-2 \) obtained from \( H, \eta, \mathcal{F} \) by rearranging within \( \lambda_2 \) of \( z_2 \).

We recall that \( P \) is an \( \eta''(H'') \)-path with ends \( \eta'(u), \eta'(v) \), where \( d''(u, v) \geq \mu \).

(13) Either \( P \) meets \( \eta(L_2) \cup \eta'(M_2) \), or \( d(u, \sigma_2) < \mu \) for some \( \sigma_2 \in Z_2 \).

Subproof. Suppose that \( P \) does not meet \( \eta(L_2) \cup \eta'(M_2) \). Then \( u, v \in V(M_1 \cup H_0) \), and \( P \) is an \( \eta'(H') \)-path. Since \( H, \eta, \mathcal{F} \) is \( (\lambda, \mu) \)-flat, and \( H', \eta', \mathcal{F}' \) is obtained from \( H, \eta, \mathcal{F} \) by rearranging within \( \lambda \leq \lambda_1 \) of \( z \), it follows that \( d''(u, v) < \mu \). Since \( d''(u, v) > \mu \), we deduce from (4.2) (applied to \( \mathcal{F}', \mathcal{F}'' \) and \( Z_2 \)) that \( d''(u, v) > d''(u, \sigma_2) \) for some \( \sigma_2 \in Z_2 \), and consequently \( d''(u, \sigma_2) < \mu \). By (4), for all \( \sigma_2 \in Z_2 \),

\[
d''(\sigma_1, \sigma_2) \geq d'(\tau, \sigma_1) - d'(\tau, \sigma_2) > \lambda - 6\lambda_1 - \lambda_2 - 2 \geq \mu > d''(u, \sigma_2)
\]

and so by (4.2) (applied to \( \mathcal{F}', \mathcal{F}'' \) and \( Z_1 \)), \( d(u, \sigma_2) \leq d''(u, \sigma_2) < \mu \). This proves (13).

(14) Either \( P \) meets \( \eta(L_1) \cup \eta'(M_1) \), or \( d(u, \sigma_1) < \mu \) for some \( \sigma_1 \in Z_1 \).

Subproof. Suppose that \( P \) does not meet \( \eta(L_1) \cup \eta'(M_1) \). Then \( P \) is an \( \eta'(H') \)-path, and so \( d''(u, v) < \mu \) by (12), since \( \lambda_2 \leq \lambda \). Let \( W \) be a closed walk of a radial drawing of \( H'' \) with length \( < 2\mu \) and with \( u, v \in \text{ins}''(W) \).

Suppose first that \( Z_1 \cap \text{ins}''(W) = \emptyset \). Then \( W \) is a walk of a radial drawing of \( H'' \), and so \( \text{ins}''(W) = \text{ins}''(W) \) by (4.1), since \( \mathcal{F}' \) and \( \mathcal{F}'' \) are \( (4\lambda_2 + 2) \)- and \( (4\lambda_2 + 2) \)-compressions of \( \mathcal{F} \) respectively. Hence \( u, v \in \text{ins}''(W) \), and so \( d''(u, v) < d''(u, v) < \mu \), a contradiction. We deduce that \( Z_1 \cap \text{ins}''(W) \neq \emptyset \), and so there exists \( \sigma_1 \in Z_1 \) with \( d''(u, \sigma_1) < \mu \).

Now \( Z_2 \) includes all atoms of \( H \) which are not atoms of \( H'' \). Moreover, for all \( \sigma_2 \in Z_2 \), \( \mathcal{F}'' \) is a \( (4\lambda_2 + 2) \)-compression of \( \mathcal{F} \), and by (6),

\[
d''(\sigma_1, \sigma_2) \geq d(\sigma_1, \sigma_2) - 4\lambda_2 - 2 > (\lambda - 2\lambda_1 - \lambda_2) - 4\lambda_2 - 2 \geq \mu > d''(\sigma_1, u)
\]

Hence by (4.2) applied to \( \mathcal{F}, \mathcal{F}'' \) and \( Z_2 \), we deduce that \( d(\sigma_1, u) \leq d''(\sigma_1, u) < \mu \). This proves (14).

From (13) and (14) it follows that \( P \) meets \( X_1 \) and \( X_2 \); but this contradicts (10). We deduce that there is no such \( P \), that is, \( H'', \eta'', \mathcal{F}'' \) is not \( \mu \)-stepped. Since \( \mathcal{F}'' \) has order \( \theta - 4\lambda_1 - 2 \geq 4\lambda_2 + \mu + 2 \), it follows that \( H'', \eta'', \mathcal{F}'' \) is \( (\lambda_2, \mu) \)-flat, as required. \( \blacksquare \)

6. CROOKED TRANSACTIONS

If \( \mathcal{P} \) is a transaction in a society \((G, \Omega)\), we denote the set of ends of members of \( \mathcal{P} \) by \( V(\mathcal{P}) \). We say \( \mathcal{P} \) is crooked if for every \( P \in \mathcal{P} \) with ends
u, v say, there exists \( u', v' \in V(\mathcal{P}) \) so that \( u, u', v, v' \) are distinct and occur in that order in \( \Omega \).

Let \( \pi \) be an arrangement in a surface \( \Sigma \) of a segregation \( \mathcal{S} \) of \( G \). A subset \( X \subseteq \Sigma \) is said to be \( \pi \)-normal if

\[
X \cap \pi(\mathcal{A}, \Omega) \subseteq \{ \pi(v) \mid v \in \Omega \}
\]

for all \( \mathcal{A}, \Omega \in \mathcal{S} \). Let \( F \subseteq \Sigma \) be an \( \pi \)-normal O-arc. Let \( X = \{ v \in V(\mathcal{S}) \mid \pi(v) \in F \} \) and \( X' = \{ \pi(v) \mid v \in X \} \). Let \( \Omega' \) be a natural order of \( X' \) from \( F \), and let \( \Omega \) be the cyclic permutation with \( \Omega = X \) mapped to \( \Omega' \) by \( \pi \). We define \( \pi^{-1}(F) \) to be \( \Omega \) (or its reverse; the choice will not matter). We shall need the following theorem \([1, \text{Theorem } (11.11)]\).

(6.1) Let \( \pi \geq 2 \) be an integer, with \( \pi \neq 3 \), and let \( (G, \Omega) \) be a society with no crooked transaction of cardinality \( \pi \). Then there is a segregation \( \mathcal{S} \) of \( G \) of type \((3 \pi + 9, 1)\) with \( \Omega \subseteq V(\mathcal{S}) \), and an arrangement \( \pi \) of \( \mathcal{S} \) in a closed disc \( D \), such that \( \Omega = \pi^{-1}(bd(\mathcal{A})) \).

Let \( \mathcal{P} = \{ P_1, \ldots, P_n \} \) be a transaction in a society \((G, \Omega)\), where \( P_i \) has ends \( a_i, b_i (1 \leq i \leq \pi) \). If

\[
(i) \quad \pi \geq 2 \text{ and } a_\pi, a_{\pi-1}, \ldots, a_1, b_\pi, b_{\pi-1}, \ldots, b_1 \text{ occur in } \Omega \text{ in that order}, \mathcal{P} \text{ is a crosscap transaction}
\]

\[
(ii) \quad \pi \geq 3 \text{ and } a_\pi, a_{\pi-1}, \ldots, a_2, a_1, b_\pi, b_\pi, b_{\pi-1}, \ldots, b_1 \text{ occur in } \Omega \text{ in that order}, \mathcal{P} \text{ is a leap transaction}
\]

\[
(iii) \quad \pi \geq 4 \text{ and } a_\pi, a_{\pi-1}, \ldots, a_2, a_1, b_\pi, b_\pi, b_{\pi-1}, \ldots, b_3, b_2, b_1 \text{ occur in } \Omega \text{ in that order}, \mathcal{P} \text{ is a doublecross transaction}
\]

We shall need the following \([1, \text{Theorem } (7.1)]\).

(6.2) Let \( \phi \geq 2, \tau_1 \geq 3, \tau_2 \geq 4 \) be integers. Let \( \mathcal{P} \) be a crooked transaction in a society \((G, \Omega)\) such that \( |\mathcal{P}| \geq (\phi - 1)(2\tau_1 + \tau_2 - 7) \). Then there exists \( \mathcal{P} \subseteq \mathcal{P} \) such that either

\[
(i) \quad |\mathcal{P}| = \phi \text{ and } \mathcal{P} \text{ is a crosscap transaction, or}
\]

\[
(ii) \quad |\mathcal{P}| = \tau_1 \text{ and } \mathcal{P} \text{ is a leap transaction, or}
\]

\[
(iii) \quad |\mathcal{P}| = \tau_2 \text{ and } \mathcal{P} \text{ is a doublecross transaction.}
\]

7. THE MAIN PROOF

Now we shall prove (1.2). Throughout this section, let \( \Sigma \) be a surface with \( bd(\Sigma) = \emptyset \), and let \( k \geq 0, \phi \geq 8, \mu \geq 12, \lambda_{k+1} \geq 4 \) and \( \theta_{k+1} \geq 4\lambda_{k+1} + \mu + 2 \) be fixed integers, with \( \mu \) even. We shall show that they satisfy
To do so, we may assume that $\theta_{k+1}$ is even and $\theta_{k+1} > 6\mu + 14$, by increasing $\theta_{k+1}$ if necessary. Let $\pi = (\phi - 1)(4 + \mu + \theta_{k+1}/2)$, let $\beta = 2\pi + 8$, and let $\theta_k = 100\beta, \lambda_k = 20\beta, \rho_k = 3\pi + 9$. We claim that $\theta_k, \lambda_k, \rho_k$ satisfy (1.2). (For convenience, let us write $\theta, \theta', \lambda, \lambda', \rho$ for $\theta_k, \theta_{k+1}, \lambda_k, \lambda_{k+1}, \rho_k$ respectively.) To show this, let (again, throughout this section) $\mathcal{F}^*$ be a tangle in a graph $G$ such that there is a $(\lambda, \mu)$-flat $\Sigma$-span of order $\geq \theta$, with $k$ eyes mutually at distance $\geq \theta$. We must prove that one of the following statements (S1), (S2), (S3) holds.

(S1) There is a $(\lambda', \mu)$-flat $\Sigma$-span of order $\theta'$ with $k + 1$ eyes, mutually at distance $\theta'$.

(S2) There is a $\Sigma'$-span of order $\phi$, where $\Sigma'$ is a surface obtained by adding a crosscap to $\Sigma$.

(S3) There is a $\mathcal{F}^*$-central segregation of $G$ of type $(\rho, k)$ with an arrangement in $\Sigma$.

The proof proceeds in a series of lemmas. We begin with the following.

(7.1) There is a $(\lambda, \mu)$-flat $\Sigma$-span of order $\theta$, with $k$ eyes mutually at distance $\theta$.

Proof. Let $H, \mathcal{F}$ be a $(\lambda, \mu)$-flat $\Sigma$-span of order $\geq \theta$ with $k$ eyes $r_1, ..., r_k$ satisfying $d(r_i, r_j) \geq \theta$ (1 $\leq i < j \leq k$). Let $\mathcal{F}'$ be the $\theta$-truncation of $\mathcal{F}$. Then $H, \mathcal{F}'$ is a $\Sigma$-span, and it is $(\lambda, \mu)$-flat by (5.1), since $\theta \geq \lambda + \mu + 2$. Moreover, $r_1, ..., r_k$ are eyes of $H, \mathcal{F}'$ since $ord(\mathcal{F}') = \theta \geq 4$; and for $1 \leq i < j \leq k$, $d'(r_i, r_j) = \min(\theta, d(r_i, r_j)) = \theta$. The result follows.

Throughout this section, let $H, \mathcal{F}$ be a $(\lambda, \mu)$-flat $\Sigma$-span of order $\theta$, with metric $d$, and let $r_1, ..., r_k$ be eyes of $H, \mathcal{F}$ with $d(r_i, r_j) = \theta$ for $1 \leq i < j \leq k$. For $1 \leq i \leq k$, let $Z_i$ be the union of all atoms $a$ of $H$ with $d(r_i, a) < \theta' + 12\mu + 14$, and let $J_i$ be the union of all bridges $B$ of $H$ in $G$ with $V(B) \cap Z_i \neq \emptyset$. Let $J_0$ be the union of all bridges $B$ of $H$ in $G$ such that $V(B) \cap \bigcup Z_i = \emptyset$.

(7.2) For $1 \leq i < j \leq k$, if $\sigma_1 \in Z_i$ and $\sigma_2 \in Z_j$, then $d(\sigma_1, \sigma_2) \geq \theta - 2(\theta' + 12\mu + 14)$, and so $Z_i \cap Z_j \neq \emptyset$ and $J_i \cap J_j \subseteq H$.

Proof. We have

$$\theta = d(r_i, r_j) \leq d(r_i, \sigma_1) + d(\sigma_1, \sigma_2) + d(r_j, \sigma_2) \leq d(\sigma_1, \sigma_2) + 2(\theta' + 12\mu + 14)$$
and so the inequality follows. Hence $Z_i \cap Z_j = \emptyset$, since $\theta > 2(\theta' + 12\mu + 14)$. Finally, suppose that $J_i \cap J_j \not \subseteq H$. Then there is an $H$-path with one end, $u$, in $Z_i$, and the other, $v$, in $Z_j$. Hence

$$d(u, v) \geq \theta - 2(\theta' + 12\mu + 14) \geq \mu.$$  

But $H$, $\mathcal{F}$ is $(0, \mu)$-flat by (5.1), a contradiction. Thus $J_i \cap J_j \subseteq H$. □

Let $G_0 = H \cup J_0$.

(7.3) If there is a $(3\mu + 3)$-zone $A \subseteq \Sigma$ for $H$ such that $H \cap (\Sigma - A)$ is rigid and such that $A$ is an eye of $H \cap (\Sigma - A)$ in $G_0$ with respect to the tangle of order $\theta - 12\mu - 14$ obtained from $\mathcal{F}$ by clearing $A$, then (S1) holds.

**Proof.** Choose $z \in A(H)$ such that $A$ is a $(3\mu + 3)$-zone around $z$. Let $\mathcal{T}_i$ be the tangle of order $\theta - 12\mu - 14$ obtained from $\mathcal{F}$ by clearing $A$; then $A$ is an eye of $H \cap (\Sigma - A)$ in $G_0$ with respect to $\mathcal{T}_i$. Let $H' = H \cap (\Sigma - A)$. From the definition of clearing $A$, we have

1. For all $\sigma_1, \sigma_2 \in \Sigma$, $d(\sigma_1, \sigma_2) \geq d(\sigma_1, \sigma_2) - 12\mu - 14$, and so $\mathcal{T}_i$ is a $(12\mu + 14)$-compression of $\mathcal{F}$.

2. For $1 \leq i \leq k$, $A - Z_i$ contains a vertex of $H$.

**Subproof.** Since $A$ is an eye of $H \cap (\Sigma - A)$ in $G_0$ it follows that some vertex of $V(H) \cap A$ belongs to $V(J_0)$. But no such vertex belongs to $Z_i$. This proves (2).

3. For $1 \leq i \leq k$, $d(z, r_i) > \theta' + 9\mu + 11$.  

**Subproof.** From (2), there exists $v \in V(H)$ with $v \in A - Z_i$. Then $d(z, v) \leq 3\mu + 3$ since $A$ is a $(3\mu + 3)$-zone; but $d(r_i, v) > \theta' + 12\mu + 14$, since $v \not \in Z_i$. Since $d(r_i, z) \leq d(r_i, z) + d(z, v)$, it follows that $d(r_i, z) > \theta' + \mu + 11$. This proves (3).

In particular, from (3) it follows that $r_i \not \subseteq A$, and so $r_i$ is a region of $H'$, for $1 \leq i \leq k$. Also, $A$ is a region of $H'$.

4. For $1 \leq i \leq k$, $d(A, r_i) \geq \theta'$.  

**Subproof.** Again from (2), let $v \in V(H)$ with $v \in A - Z_i$. Choose $\sigma \in A$ with $d(v, \sigma) \leq 1$. Then $d(r_i, \sigma) \geq \theta' + 12\mu + 14$, since $d(r_i, v) > \theta' + 12\mu + 14$; and so, by (1), $d(\sigma, \sigma) \geq \theta'$. This proves (4).

5. For $1 \leq i \leq k$, $r_i$ is an eye of the $\Sigma$-span $H'$, $\mathcal{T}_i$.

**Subproof.** It suffices to show that if $a, b, c, d \in V(H) \cap bd(r_i)$ are distinct and $\{a, b, c, d\}$ is free with respect to $\mathcal{F}$ then $a, b, c, d \in V(H')$ and

\[\text{Codes: 2972 Signs: 1754. Length: 48 pic 0 pts, 190 mm}\]
\{a, b, c, d\} is free with respect to $T_1$. The first claim follows from (3). For the second, we observe that, for all $v \in \{a, b, c, d\}$, by (3),
\[
d(v, z) \geq d(r_v, z) - 1 \geq \theta' + 9\mu + 11 > 8 + 5(3\mu + 3) + 2.
\]
Hence, by (4.7) applied to $T$ and $T_1$ (taking $\lambda = 3\mu + 3$) and (1), we deduce that $\{a, b, c, d\}$ is free with respect to $T_1$. This proves (5).

From (4.6), $H'$, $T$ is a $\Sigma$-span of order $\theta - 12\mu - 14$, obtained from $H$, $T$ by rearranging within $3\mu + 3$ of $z$.

(6) The $\Sigma$-span $H'$, $T$ is $(\lambda', \mu)$-flat.

Subproof. From (5.2), since $H$, $T$ is $(\lambda, \mu)$-flat, it suffices to check that $3\mu + 3 \geq 4$, $\lambda' \geq 4$, $\lambda \geq 6(3\mu + 3) + 5\lambda' + 3\mu + 4$, and $\theta \geq 4\lambda + \mu + 2$. All these inequalities hold, and so (6) follows.

Let $F_2$ be the $\theta'$-truncation of $F_1$. Then $H'$, $F_2$ is a $\Sigma$-span, and by (5.1) and (6), it is $(\lambda', \mu)$-flat, since $\theta' \geq 4\lambda' + \mu + 2$.

(7) $r_1, \ldots, r_k, r_{k+1}$ are eyes of $H'$, $F_2$ in $G$, and for $1 \leq i < j \leq k + 1$,
\[
d_2(r_i, r_j) = \min(\theta', d_1(r_i, r_j)).
\]
Subproof. Since $\theta' \geq 4$ and $F_2$ is the $\theta'$-truncation of $F_1$, and since by (5) $r_1, \ldots, r_{k+1}$ are eyes of $H'$, $F_2$ in $G$, it follows that $r_1, \ldots, r_{k+1}$ are eyes of $H'$, $F_2$ in $G$. Now for $1 \leq i < j \leq k + 1$,
\[
d_2(r_i, r_j) = \min(\theta', d_1(r_i, r_j)).
\]
But if $j = k + 1$ then $d_1(r_i, r_j) \geq \theta'$ by (4); and if $j \geq k$ then
\[
d_2(r_i, r_j) = d_1(r_i, r_j) - 12\mu + 14 = \theta - 12\mu - 14 \geq \theta'
\]
by (1). In either case $d_2(r_i, r_j) \geq \theta'$, and so $d_2(r_i, r_j) = \theta'$. This proves (7).

From (7) we see that (S1) holds, as required.

Let $H$ be a subgraph of $G$, and let $H$ be a 2-cell drawing in $\Sigma$. Let $\alpha$ be an arrangement in $\Sigma$ of a segregation $\mathcal{F}$ of $G$. We say that $\alpha$ is compatible with $H$ if

(i) $\alpha(v) = v$ for all $v \in V(H) \cap V(\mathcal{F})$, and

(ii) for each $(A, \Omega) \in \mathcal{F}$, $U(H) \cap bd(\alpha(A, \Omega)) = V(H) \cap \Omega$, and $H \cap \alpha(A, \Omega) = H \cap A$.

(7.4) If (S1) does not hold, there is a $\mathcal{F}$-local 3-segregation $\mathcal{F}_0$ of $G_0$ and an arrangement $\alpha_0$ of it in $\Sigma$ compatible with $H$. 

Codes: 2698 Signs: 1604. Length: 45 pic 0 pts, 190 mm
Proof. Now $\mu \geq 12$ is even and $\theta \geq 16\mu + 17$. Moreover $d(u, v) < \mu$ for every $H$-path $P$ in $G_0$ with ends $u, v$ because $H, T$ is $(0, \mu)$-flat by (5.1). The result follows from (7.3) and [5, Theorem (10.1)] (taking $\beta = \mu$).

If $\pi$ is an arrangement in a surface $\Sigma$ of a segregation $\mathcal{S}$ of $G$, and $A \subseteq \Sigma$ is a closed disc with $bd(A) \pi$-normal, we define $\pi^{-1}(A)$ to be the subgraph of $G$ formed by

$$\bigcup \{ (A, (A, \Omega) \in \mathcal{S} \text{ and } \pi(A, \Omega) \subseteq A\}$$

together with all $v \in V(\mathcal{S})$ such that $\pi(v) \in A$.

Henceforth, in this section, we assume that $\mathcal{S}_0, \pi_0$ are as in (7.4). Let $\tau_1 = 4 + \mu/2$, $\tau_2 = 3 + \theta'/2$; then $\pi = (\phi - 1)(2\tau_1 + \tau_2 - 7)$. Let $\beta = 2\pi + 8$.

(7.5) For $1 \leq i \leq k$ there is a $(3\beta + 3)$-zone $A_i$ around $r_i$ and a closed disc $A_i \subseteq \Sigma$ with $bd(A_i) \pi_0$-normal, with the following properties:

(i) $A_1, \ldots, A_k$ are mutually disjoint, and for $1 \leq i \leq k$, $V(J_i \cap H) \subseteq A_i$.
(ii) $d(r_i, x) \leq 3\beta + 6$ for all $x \in A(H)$ with $x \cap A_i \neq \emptyset$.
(iii) $x \subseteq A_i - bd(A_i)$ for every $x \in A(H)$ with $d(r_i, x) < \beta$.
(iv) $H \cap (\Sigma - A_i)$ has a circuit, and $H \cap (\Sigma - A_i)$ is rigid.
(v) For $1 \leq i \leq k$, one of the following holds:

(a) there is a crooked transaction $\mathcal{P}$ of cardinality $\pi$ in $(G_0 \cup J, \Omega)$ (where $\Omega$ is a natural order of $V(H) \cap bd(A_i)$ from $bd(A_i)$) such that $V(\mathcal{P})$ is free with respect to the $(12\beta + 14)$-compression of $\mathcal{F}$ in $H \cap (\Sigma - A_i)$, and each $P \in \mathcal{P}$ meets $H \cap (\Sigma - A_i)$ precisely in the ends of $P$.
(b) there is no crooked transaction of cardinality $\pi$ in $(J_i \cup \pi_0^{-1}(A_i), \pi_0^{-1}(bd(A_i)))$.

Proof:

(1) For $1 \leq i \leq k$, every $v \in V(J_i \cap H)$ satisfies $d(r_i, v) \leq \theta' + 3\mu + 13$.

Subproof. Let $B$ be a bridge of $H$ in $G$ with $B \subseteq J_i$ and $v \in V(B)$. By definition of $J_i$, there exists $u \in V(B \cap H)$ with $d(r_i, u) \leq \theta' + 12\mu + 14$. Now since $B$ is a bridge, there is an $H$-path in $G$ with ends $u, v$. Since $H, T$ is $(0, \mu)$-flat it follows that $d(u, v) < \mu$, and so $d(r_i, v) < \theta' + 13\mu + 14$. This proves (1).

Now $\pi \geq 0, \pi \neq 1, 3, \pi = 2\pi + 8$ is even, and $\theta \geq 16\beta + 17$. Since $\theta' + 13\mu + 13 < 2\beta$, we deduce from [5, Theorem (9.6)] (with the substitutions $z \rightarrow r_i, G \rightarrow G_0 \cup J_i, J \rightarrow J_i, G' \rightarrow G_0, \mathcal{S} \rightarrow \mathcal{S}_0, \pi \rightarrow \pi_0$) that there is a $(3\beta + 3)$-zone $A_i$ around $r_i$, and a closed disc $A_i \subseteq \Sigma$ with $bd(A_i) \pi_0$-normal, satisfying (ii)–(v) of the theorem.
From (iii) we see that \( V(J_i \cap H) \subseteq A_i \) for \( 1 \leq i \leq k \), since \( \theta' + 13 + \mu + 13 < \beta \). Moreover, \( A_1, \ldots, A_k \) are mutually disjoint; for if, say \( \sigma \in A_i \cap A_j \) where \( i \neq j \), then \( d(r_j, \sigma), d(r_i, \sigma) \leq 3\beta + 6 \) by (ii), and so \( d(r_i, r_j) \leq 6\beta + 13 < \theta \), a contradiction. Hence (i) holds, as required.

Henceforth we assume that \( A_i, A_j, \Omega \) are as in (7.5). For \( 1 \leq i \leq k \), let \( \Omega^*_i = \pi^{-1}_0(bd(A_i)) \) and \( G^*_i = J_i \cup \pi^{-1}_0(A_i) \).

(7.6) \( G^*_i, \ldots, G^*_k \) are mutually disjoint.

Proof. Let \( 1 \leq i < j \leq k \), and suppose that \( v \in V(G_i^* \cap G_j^*) \). If \( v \not\in V(\pi^{-1}_0(A)) \) then \( v \in V(J_i) \) and \( v \not\in V(H) \), and so \( v \not\in V(H \cup J_0 \cup J_i) \); and consequently \( v \not\in V(G_i^*) \), a contradiction. We assume then that \( v \in V(\pi^{-1}_0(A)) \), and similarly \( v \in V(\pi^{-1}_0(A_i)) \). Now either \( v \in V(A) \) for some \( (A, \Omega) \in \mathcal{S}_0 \) with \( \pi_0(A, \Omega) \subseteq A_i \), or \( v \in V(\mathcal{S}'_0) \) and \( \pi_0(v) \in A_i \); and either \( v \in V(A') \) for some \( (A', \Omega') \in \mathcal{S}_0 \) with \( \pi_0(A', \Omega') \subseteq A_i \), or \( v \in V(\mathcal{S}) \) and \( \pi_0(v) \in A_i \).

Suppose that \( v \in V(\mathcal{S}_0) \). Since \( A_i \cap A_j = \emptyset \), we may assume that \( \pi_0(v) \not\in A_i \). Hence there exists \( (A, \Omega) \in \mathcal{S}_0 \) with \( \pi_0(A, \Omega) \subseteq A_i \) such that \( v \in V(A) \). Since \( v \in V(\mathcal{S}_0) \) and \( v \in V(\mathcal{S}'_0) \), it follows that \( v \in \Omega \), and hence \( \pi_0(v) = \pi_0(A, \Omega) \subseteq A_i \), a contradiction. Hence \( v \not\in V(\mathcal{S}_0) \).

We deduce that \( v \in V(A \cap A') \) for some \( (A, \Omega), (A', \Omega') \in \mathcal{S}_0 \) with \( \pi_0(A, \Omega) \subseteq A_i \) and \( \pi_0(A', \Omega') \subseteq A_i \). Since \( A_i \cap A_j = \emptyset \) it follows that \( (A, \Omega) \neq (A', \Omega') \), and so \( V(A \cap A') \subseteq \Omega \cap \Omega' \subseteq V(\mathcal{S}) \); and hence \( v \in V(\mathcal{S}) \), a contradiction. Thus there is no such \( v \), as required.

(7.7) If there is no crooked transaction of cardinality \( \pi \) in \( (G_i^*, \Omega^*_i) \) for \( 1 \leq i \leq k \), then (S5) holds.

Proof. By (6.1), there is a segregation \( \mathcal{S}' \) of \( G_i^* \) of type \( (3\pi + 9, 1) \) with \( \mathcal{S}' \subseteq V(\mathcal{S}) \), and an arrangement \( \pi_0 \) of \( \mathcal{S}' \) in a closed disc \( A \), such that \( \mathcal{S}_0 = \pi_0^{-1}(bd(A)) \). By applying a suitable homeomorphism, we may assume that \( A = A_i \), and \( \pi_0(v) = \pi_0(A, \Omega) \) for each \( v \in \mathcal{S}_0 \).

Let \( \mathcal{S}_0' = \{ (A, \Omega) \in \mathcal{S}_0 : \pi_0(A, \Omega) \not\subseteq A_1, \ldots, A_k \} \).

(1) For \( 1 \leq i \leq k \), if \( (A, \Omega) \in \mathcal{S}_0' \) then \( E(G_i^* \cap A) = \emptyset \) and \( V(G_i^* \cap A) \subseteq \mathcal{S}_0 \).

Subproof. Let \( x \) be a vertex or edge of \( G \) which belongs to \( G_i^* \cap A_0 \). Since \( G_i^* \cap A \subseteq (J_i \cup \pi_0^{-1}(A_i)) \cap G_0 \) and \( J_i \cap G_0 \subseteq \pi_0^{-1}(A_i) \), it follows that \( A \subseteq \pi_0^{-1}(A_i) \). Suppose that it is not the case that \( x \in V(G_0) \) and \( \pi_0(x) \in A_i \). Then there exists \( (A', \Omega') \in \mathcal{S}' \) with \( \pi_0(A', \Omega') \subseteq A_i \) such that \( x \) belongs to \( A' \), since \( x \) belongs to \( \pi_0^{-1}(A_i) \).
Now \((A', \Omega') \neq (A, \Omega)\) because \(\pi_d(A, \Omega) \not\subseteq A\), and so \(V(A \cap A') \subseteq \overline{\Omega} \cap \Omega'\), since \(S_0\) is a segregation. Hence \(x \in \Omega' \subseteq V(S_0)\), and so \(\pi_d(x) \in bd(\pi_d(A', \Omega')) \subseteq A\), contrary to our assumption. Hence \(x \in V(S_0)\) (in particular, \(x\) is a vertex), and \(\pi_d(x) \in A\). Now since \(V(A) \cap V(S_0) = \Omega\), it follows that \(x \in \Omega\), and so \(\pi_d(x) \in bd(\pi_d(A, \Omega))\). But \(\pi_d(A, \Omega) \cap A \subseteq bd(A)\), and \(\pi_d(x) \in A\), and so \(\pi_d(x) \in bd(A)\). Consequently \(x \in \Omega^*\). This proves (1).

Let \(S = S_0' \cup S_1 \cup \ldots \cup S_k\).

(2) \(S\) is a segregation of \(G\).

**Subproof.** We must show

(i) \(A \subseteq G\) for each \((A, \Omega) \in S\), and \(\bigcup \{(A, \Omega) \in S\} = G\)

(ii) for distinct \((A, \Omega), (A', \Omega') \in S\), \(V(A \cap A') \subseteq \overline{\Omega} \cap \Omega'\) and \(E(A \cap A') = \emptyset\).

We first show (i). Certainly \(A \subseteq G\) for each \((A, \Omega) \in S\). Let \(x\) be a vertex or edge of \(G\); to show that \(\bigcup \{(A, \Omega) \in S\} = G\) it suffices to show that \(x\) belongs to \(A\) for some \((A, \Omega) \in S\). If \(x\) belongs to \(G^*_i\) for some \(i \geq 1\), then \(x\) belongs to \(A\) for some \((A, \Omega) \in S_i \subseteq S\), as required. If not, then \(x\) does not belong to \(J_1, \ldots, J_k\), and so \(x\) belongs to \(G_0\). Hence there exists \((A, \Omega) \in S_0\) such that \(x\) belongs to \(A\). For \(1 \leq i \leq k\), since \(x\) does not belong to \(G^*_i\) and \(\pi_d(A, \Omega) \subseteq G^*_i\), it follows that \(\pi_d(A, \Omega) \not\subseteq A_i\). Hence \((A, \Omega) \in S_i\). This proves (i).

For (ii), let \((A, \Omega), (A', \Omega') \in S\) be distinct. If they both belong to \(S_0'\), or for some \(i \geq 1\) they both belong to \(S_i\), then (ii) holds since \(S_0', S_1, \ldots, S_k\) are segregations. We may assume then that \((A, \Omega) \in S_i\) for some \(i \geq 1\), and \((A', \Omega') \in S_j\). If \((A', \Omega') \in S_i\) for some \(j \neq i\), then \(A \cap A' \subseteq G^*_i \cap G^*_j\), which is null by \((7.6)\), and (ii) holds. Otherwise, \((A', \Omega') \in S_0'\), and from (1), \(E(A \cap A') = \emptyset\) and

\[
V(A \cap A') \subseteq \overline{\Omega} \cap V(S_0) \cap V(S_0);
\]
and since \(V(A) \cap V(S) = \overline{\Omega}\) and \(V(A') \cap V(S_0) = \overline{\Omega}'\), it follows that (ii) holds. This proves (2).

(3) \(S\) is of type \((r, k)\).

**Subproof.** \(S_0\) is a 3-segregation, and each \(S_i\) is of type \((3r + 9, 1)\), and so \(S\) is of type \((3r + 9, k)\). This proves (3).

(4) \(S\) is \(S^*\)-central.

**Subproof.** Let \((A, \Omega) \in S\), and let \(B \subseteq G\) so that \((A, B)\) is a separation of \(G\) and \(V(A \cap B) = \overline{\Omega}\). Suppose that there exists \((A^*, B^*) \in S^*\) with \(B^* \subseteq A\). By (2.1) we may assume that \((A^*, B^*)\) has order \(\leq 16r + 19\). Since
\[ \theta > 6\pi + 19 \] and \( H, \mathcal{T} \) is a \( \Sigma \)-span, it follows that \((A^* \cap H, B^* \cap H) \in \mathcal{T}\). Consequently \((A \cap H, B \cap H) \notin \mathcal{T}_0\), because \((A^* \cap H) \cup (A \cap H) \subseteq (A^* \cap H) \cup (B^* \cap H) = H\), and so \((A, \Omega) \notin \mathcal{T}_0\), because \(\mathcal{F}_0\) is \(\mathcal{T}\)-local. Thus there exists \(i\) with \(1 \leq i \leq k\) such that \((A, \Omega) \in \mathcal{T}\). Then \(B^* \subseteq A \subseteq G^*_\Omega\). However, by [5, Theorem (7.3)], there exists \(e \in E(B^* \cap H)\) with
\[
d(r_e, e) \geq \theta - |V(A^* \cap H) \cap (B^* \cap H)| \geq \theta - |V(A^* \cap B^*)| \geq \theta - 6\pi - 19.
\]
Since \(e \in E(H)\), there exists a unique \((A', \Omega') \in \mathcal{T}_\Omega^\varepsilon\) with \(e \in E(A')\); and since \(e \in E(G^*_\Omega)\) it follows that \(\pi_0(A', \Omega') \subseteq \mathcal{A}\). Since \(\pi_0\) is compatible with \(H\), we deduce that \(e \in \mathcal{R}\), and so \(d(r_e, e) \leq 3\beta + 6\) by (7.5)(ii). But \(3\beta + 6 < \theta - 6\pi - 19\), a contradiction. This proves (4).

Define \(\pi(A, \Omega)\) for \((A, \Omega) \in \mathcal{T}\) as follows: if \((A, \Omega) \in \mathcal{T}_0\), let \(\pi(A, \Omega) = \pi_0(A, \Omega)\); and otherwise, choose \(i\) with \(1 \leq i \leq k\) such that \((A, \Omega) \in \mathcal{R}_i\), and let \(\pi(A, \Omega) = \pi_i(A, \Omega)\). For \(v \in V(\mathcal{R})\), define \(\pi(v)\) by: \(v \in \pi_i\) for some \((A, \Omega) \in \mathcal{T}_0\), let \(\pi(v) = \pi_i(v)\); and otherwise, choose \(i\) with \(1 \leq i \leq k\) such that \(v \in \pi_i\) for some \((A, \Omega) \in \mathcal{T}_i\), and let \(\pi(v) = \pi_i(v)\).

(5) For \(1 \leq i \leq k\), if \(v \in V(\mathcal{R})\) then \(\pi(v) = \pi_i(v)\).

Subproof. Suppose first that \(v \in \pi_i\) for some \((A, \Omega) \in \mathcal{T}_0\). Then \(\pi(v) = \pi_0(v)\); but by (1), \(v \in \pi_0\) and so \(\pi_0(v) = \pi(v)\) as required. We assume then that there is no such \((A, \Omega)\). Choose \(1 \leq j \leq k\) and \((A, \Omega) \in \mathcal{R}_j\) with \(v \in \pi_i\), such that \(\pi(v) = \pi_i(v)\). Then \(v \in V(G^*_\Omega \cap G^*)\), and so \(i = j\) by (7.6). Hence \(\pi_i(v) = \pi(v)\). This proves (5).

(6) \(\pi \in \mathcal{B}(\pi(A, \Omega))\) for each \((A, \Omega) \in \mathcal{T}\) and each \(v \in \mathcal{T}\).

Subproof. If \((A, \Omega) \in \mathcal{T}_0\), then \(\pi(A, \Omega) = \pi_0(A, \Omega)\) and \(\pi(v) = \pi_0(v) \in \mathcal{B}(\pi_0(A, \Omega))\), as required. We assume then that \((A, \Omega) \in \mathcal{T}\) for some \(i \geq 1\). By (5), it follows that \(\pi(v) = \pi_i(v) \in \mathcal{B}(\pi_i(A, \Omega))\). This proves (6).

(7) For distinct \((A, \Omega), (A', \Omega') \in \mathcal{T}\), if \(x \in \pi(A, \Omega) \cap \pi(A', \Omega')\) then \(x = \pi(v)\) for some \(v \in \pi_i \cap \pi_i'\). Subproof. If \((A, \Omega), (A', \Omega') \in \mathcal{T}_0\) then \(x \in \pi_0(A, \Omega) \cap \pi_0(A', \Omega')\), and so \(x = \pi_0(v) = \pi(v)\). For some \(v \in \mathcal{T} \cap \mathcal{T}'\) we may assume then that \(\pi(A, \Omega) \notin \mathcal{T}_0\). Choose \(i\) with \(1 \leq i \leq k\) such that \((A, \Omega) \in \mathcal{T}_i\) and \(\pi(A, \Omega) = \pi_i(A, \Omega) \subseteq \mathcal{A}\). Suppose that \((A', \Omega') \notin \mathcal{T}_i\). Then there exists \(j\) with \(1 \leq j \leq k\) such that \((A', \Omega') \in \mathcal{T}_j\) and \(\pi(A', \Omega') = \pi_j(A', \Omega')\); and since \(\pi_i(A', \Omega') \subseteq \mathcal{A}\) and \(\pi_j(A', \Omega') \subseteq \mathcal{A}\) and so \(\pi(A', \Omega') \subseteq \mathcal{A}\), it follows that \(i = j\). Hence there exists \(v \in \mathcal{R} \cap \mathcal{R}'\) with \(\pi_i(v) = \pi_j(v) = \pi(v)\) and so \(\pi_i(v) = \pi_j(v) = \pi(v)\), by (5). We may therefore assume that \((A', \Omega') \in \mathcal{T}_i\). Then \(\pi(A, \Omega) \subseteq \mathcal{A}\), and \(\pi(A', \Omega') \subseteq \mathcal{B}(\mathcal{A})\), and so \(x \in \mathcal{B}(\mathcal{A})\). Since \(\mathcal{B}(\mathcal{A})\) is \(\mathcal{B}\)-normal and \(x = \pi_0(A', \Omega')\), it follows that \(x = \pi_0(v)\) for some \(v \in \pi_i\); and hence \(v \in \pi_i\). Consequently \(v \in V(\mathcal{R})\), and \(\pi_i(v) = \pi_0(v) = x\). Since \(x \in \pi(A', \Omega')\) we deduce
that \( v \in \bar{Q} \), and so \( v \in \bar{Q} \cap \bar{Q}' \). Moreover, since \( v \in (A', \Omega') \) it follows that 
\[ \pi(v) = \pi_0(v) \] 
and so \( \pi(v) = x \). This proves (7).

(8) For distinct \( v, v' \in V(\mathcal{F}) \), \( \pi(v) \neq \pi(v') \).

Subproof. Choose \((A, \Omega), (A', \Omega') \in \mathcal{F} \) with \( v \in \bar{Q} \) and \( v' \in \bar{Q}' \). If \((A, \Omega), (A', \Omega') \in \mathcal{F}' \), then \( \pi(v) = \pi_0(v) \neq \pi_0(v') = \pi(v') \) as required. We may therefore assume that \((A, \Omega) \in \mathcal{F}' \) where \( 1 \leq i \leq k \). Consequently, \( \pi_i(v) = \pi(v) \), by (5). If \( v' \in V(\mathcal{F}) \) then \( \pi_i(v') \neq \pi(v) \), and \( \pi_i(v') = \pi(v') \) by (5), and so \( \pi(v) \neq \pi(v') \) as required. We assume then that \( v' \notin V(\mathcal{F}) \). If \( v' \in V(\mathcal{F}) \) for some \( j \neq i \) with \( 1 \leq j \leq i \), then by (5), \( \pi(v') = \pi_j(v') \in A_j \), and so \( \pi(v) \neq \pi(v') \) because \( \pi(v) \in A_i \). Hence we may assume that \( v' \notin V(\mathcal{F}) \) for \( 1 \leq j \leq k \). Thus \( (A', \Omega') \in \mathcal{F}' \). Since \( \mathcal{F}' \subseteq V(\mathcal{F}) \) it follows that \( v' \notin \Omega' \), and so \( \pi_0(v') \notin bd(A_i) \). But \( \pi_0(v') \in \pi_0(A', \Omega') \) and \( \pi_0(A', \Omega') \cap A_i \subseteq bd(A_i) \), and so \( \pi_0(v') \notin \Omega' \), as required. This proves (8).

(9) For all \((A, \Omega) \in \mathcal{F}, \Omega \) is mapped by \( \pi \) to a natural order of \( \pi(\bar{Q}) \) from \( bd(\pi(A_i)) \).

Subproof. This is trivial unless \( |\bar{Q}| \geq 4 \); and if \( |\bar{Q}| \geq 4 \) then \((A, \Omega) \in \mathcal{F}' \) for some \( i \) with \( 1 \leq i \leq k \), and \( \pi(A, \Omega) = \pi(A, \Omega) \), and the result follows by (5) since \( \pi_i \) is an arrangement of \( \mathcal{F} \). This proves (9).

From (6), (7), (8), (9), it follows that \( \pi \) is an arrangement of \( \mathcal{F} \) in \( \Sigma \). From (2), (3) and (4) we deduce that (S3) holds, as required.

In view of (7.7), we may assume henceforth that \((G^*_0, \Omega^*_0) \) has a crooked transaction of cardinality \( \pi \) for some \( i \), say \( i = 1 \). We recall that \( \Omega_1 \) is a natural order of \( V(H) \cap bd(A_i) \) from \( bd(A_i) \).

(7.8) There is a transaction \( \mathcal{P} \) in \((G_0 \cup J_1, \Omega_1) \) such that \( V(\mathcal{P}) \) is free with respect to the \((12|b| + 14)\)-compression of \( \mathcal{F} \) in \( H \cap (\Sigma - A_i) \), each \( P \in \mathcal{P} \) meets \( H \cap (\Sigma - A_i) \) precisely in the ends of \( P \), and one of the following holds:

(i) \( |\mathcal{P}| = \emptyset \) and \( \mathcal{P} \) is a crosscap transaction

(ii) \( |\mathcal{P}| = \tau_1 \) and \( \mathcal{P} \) is a leap transaction

(iii) \( |\mathcal{P}| = \tau_2 \) and \( \mathcal{P} \) is a doublecross transaction.

Proof. We are assuming that \((G^*_0, \Omega^*_0) \) has a crooked transaction of cardinality \( \pi \). By (7.5)(v), there is a crooked transaction \( \mathcal{P}' \) of cardinality \( \pi \) in \((G_0 \cup J_1, \Omega_1) \) such that \( V(\mathcal{P}') \) is free with respect to the \((12|b| + 14)\)-compression of \( \mathcal{F} \) in \( H \cap (\Sigma - A_i) \), and each \( P \in \mathcal{P} \) meets \( H \cap (\Sigma - A_i) \) precisely in the ends of \( P \). By (6.2), there exists \( \mathcal{P} \subseteq \mathcal{P}' \) satisfying (i), (ii) or (iii) above, and hence satisfying the theorem, since any subset of a free set is free by [2, Theorem (12.2)].

Henceforth \( \mathcal{P} \) is a transaction satisfying (7.8).
(7.9) If $\mathcal{P}$ is a crossecap transaction of cardinality $\phi$ then (S2) is true.

Proof. This is immediate from (3.3). 

Let $\mathcal{S}_1$ be the $(12\beta + 14)$-compression of $\mathcal{F}$ in $H \cap (\Sigma - A_1)$.

(7.10) $\mathcal{P}$ is not a leap transaction of cardinality $\tau_1$.

Proof. Suppose that $\mathcal{P}$ is a leap transaction of cardinality $\tau_1$, and let $\mathcal{P} = (P_1, ..., P_{\tau_1})$, where $P_i$ has ends $v_{i}, v_{2i-1}$, for $1 \leq i \leq \tau_1 - 1$, and $P_{\tau_1}$ has ends $v_{2\tau_1 - 1}$, and where $v_{1}, ..., v_{2\tau_1}$ occur in $\mathcal{S}_1$ in order. By (3.4) applied to $H \cap (\Sigma - A_1)$, $\mathcal{S}_1$, we deduce that there is a $\Sigma$-span $H', \eta', \mathcal{F}'$ of order $\theta - 12\beta - 14$, obtained from $H$, $\mathcal{S}_1$, by rearranging within $\beta$ of each $P_i$, and $P_{\tau_1}$ is an $\eta'(H')$-path in $G$, and $d'(v_i, v_{2\tau_1}) \geq 2\tau_1 - 8 = \mu$. Hence $H', \eta', \mathcal{F}'$ is $\eta$-stepped. But $\beta \leq \lambda$, and so $H, \mathcal{S}_1, \eta', \mathcal{F}'$ is $(\beta, \mu)$-flat by (5.1), a contradiction. Thus there is no such $\mathcal{P}$.

(7.11) If $\mathcal{P}$ is a doublecross transaction of cardinality $\tau_2$ then (S1) holds.

Proof. For convenience, we write $\tau$ for $\tau_2$. Let $\mathcal{S} = \{P_1, ..., P_{\tau_2}\}$, where $P_i$ has ends $v_i, v_{2i-3}, v_{2i-2}$, for $1 \leq i \leq \tau - 4$, and $P_{\tau_2}$ has ends $v_{2\tau_2-3}, v_{2\tau_2-2}, v_{2\tau_2-1}$, and $P_{\tau_1}$ has ends $v_{2\tau_1-3}, v_{2\tau_1-2}, v_{2\tau_1-1}$. By “drawing” $P_1, ..., P_{\tau_2-4}$ in $A_1$, we see that there is a drawing $H'$ in $\Sigma$ with $U(H') \cap \delta \mathcal{A}_1 \subseteq V(H')$, such that $H' \cap (\Sigma - A_1) = H \cap (\Sigma - A_1)$, and there is an isomorphism $\eta'$ from $H'$ to a subgraph of $G$, such that $\eta'(x) = x$ for every vertex or edge in $H'(\Sigma - A_1)$, and

$$\eta'(H') = (H \cap (\Sigma - A_1)) \cup P_1 \cup \cdots \cup P_{\tau_2-4}. $$

Let $Q_1, ..., Q_{\tau_2-4}$ be the paths of $H'$ such that $\eta'(Q_1) = P_i(1 \leq i \leq \tau - 4)$. Let $s_1$ be the region of $H'$ in $A$ incident with $v_{2\tau_2-4}, v_{2\tau_2-3}, ..., v_{2\tau_2-1}, v_{2\tau_2}$, and let $s_2$ be the region incident with $v_{2\tau_2}, v_{2\tau_2-1}, ..., v_{2\tau_2-4}$. Let $\mathcal{S}_1$ be the bangle in $H'$ induced by $\mathcal{S}_1$. By (3.4) applied to $H \cap (\Sigma - A_1)$, $H'$, we see that

1. $H', \eta', \mathcal{S}_1$ is a $\Sigma$-span of order $\theta - 12\beta - 14$, and $d'(s_1, s_2) \geq 2\tau - 6 = \theta'$.

2. $s_1$ and $s_2$ are eyes of $H', \eta', \mathcal{S}_1$.

Subproof. Now $\{v_1, ..., v_{2\tau_2}\}$ is free with respect to $\mathcal{S}_1$, and hence with respect to $\mathcal{F}_1$. Since any subset of a free set is free, it follows that $\{v_{2\tau_2-3}, v_{2\tau_2-2}, v_{2\tau_2-1}, v_{2\tau_2}\}$ is free with respect to $\mathcal{F}_1$. The region $s_1$ of $H'$ is bounded by a circuit formed by $P_i$ and part of the circuit bounding $A_1$; and hence, because of the $\eta'(H')$-paths $P_{\tau_2-3}, P_{\tau_2-2}$, it follows that $s_1$ is an eye. Similarly $s_2$ is an eye. This proves (2).

3. $r_2, ..., r_\tau$ are eyes of $H \cap (\Sigma - A_1), \mathcal{F}_1$. 
Subproof. Certainly \( r_2, \ldots, r_k \) are regions of \( H \cap (\Sigma - A_1) \), since \( A_1 \) is a \((3\beta + 3)\)-zone (with respect to \( \mathcal{T} \)) around \( r_1 \), and \( d(r_1, r_i) = \theta - 3\beta + 3 \) for \( i \geq 2 \). Let \( i \geq 2 \), and let \( a, b, c, d \in V(H) \cap bd(r_i) \) such that \( \{a, b, c, d\} \) is free with respect to \( \mathcal{T} \). We claim it is free with respect to \( \mathcal{T}_i \). For \( \mathcal{T}_i \) is a \((4(3\beta + 3) + 2)\)-compression of \( \mathcal{T} \),

\[
d(r_1, v) \geq d(r_i, r_1) - 1 = \theta - 1 > 8 + 5(3\beta + 3) + 2
\]

for all \( v \in \{a, b, c, d\} \); and \( x \in A(H \cap (\Sigma - A_1)) \) for every \( x \in A(H) \) with \( d(r_1, x) > 3\beta + 3 \), since \( A_1 \) is a \((3\beta + 3)\)-zone. Consequently, by (4.7), it follows that \( \{a, b, c, d\} \) is free with respect to \( \mathcal{T}_i \). This proves (3).

(4) \( r_2, \ldots, r_k \) are eyes of \( H', \eta', \mathcal{T}' \).

Subproof. Let \( 2 \leq i \leq k \). Since \( r_i \) is an eye of \( H \cap (\Sigma - A_1) \), \( \mathcal{T}_i \), there are two disjoint \( H \cap (\Sigma - A_1) \)-paths \( P, Q \) with ends in \( bd(r_i) \) in alternating order. We claim that \( P, Q \) are \( \eta'(\mathcal{T}') \)-paths. For if \( P \) is not an \( \eta'(\mathcal{T}') \)-path, then \( P \) meets one of \( P_1, \ldots, P_{r - 4} \), and so there is an \( H \cap (\Sigma - A_1) \)-path in \( G \) with one end, \( u, \) in \( bd(r_i) \) and the other, \( v, \) in \( bd(A_1) \). Since

\[
d_i(u, v) \geq d_i(r_i, v) - 1 \geq d_i(r_i, v) - (12\beta + 14) - 1
\]

and

\[
d(r_i, v) \geq d(r_i, r_1) - d(r_1, v) = \theta - d(r_1, v) \geq \theta - (3\beta + 3)
\]

it follows that \( d_i(u, v) \geq \theta - 15\beta - 18 \geq \mu \), and so \( H \cap (\Sigma - A_1), \mathcal{T}_i \), is \( \mu \)-stepped. But \( H \cap (\Sigma - A_1), \mathcal{T}_i \) is obtained from \( H, \mathcal{T} \) by rearranging within \( 3\beta + 3 \) of \( r_i \), and \( 3\beta + 3 < \lambda \) and \( H, \mathcal{T} \) is \((\lambda, \mu)\)-flat, a contradiction. This proves that \( P \) and similarly \( Q \) are \( \eta'(\mathcal{T}') \)-paths, and hence proves (4).

(5) For \( 2 \leq i < j < k \), \( d'(r_i, r_j) \geq \theta' \).

Subproof. \( d'(r_i, r_j) \geq d(r_i, r_j) - 12\beta - 14 = \theta - 12\beta - 14 \geq \theta' \).

(6) For \( 2 \leq i < k \), \( d'(s_1, r_i) \geq \theta' \) and \( d'(s_2, r_i) \geq \theta' \).

Subproof. \( d'(s_1, r_i) \geq d_1(A_1, r_i) \geq d_1(r_1, r_i) - 12\beta - 14 = \theta - 12\beta - 14 \geq \theta' \).

Similarly \( d'(s_2, r_i) \geq \theta' \).

(7) \( H', \eta', \mathcal{T}' \) is \((\lambda', \mu)\)-flat.

Subproof. Now \( 3\beta + 3, \lambda \geq 4 \) and \( \mu \geq 1 \), and \( \lambda \geq 6(3\beta + 3) + 5\lambda' + 3\mu + 4 \). Since \( H', \eta', \mathcal{T}' \) is obtained from \( H, \mathcal{T} \) by rearranging within \( 3\beta + 3 \) of \( r_1 \), and \( H, \mathcal{T} \) is \((\lambda, \mu)\)-flat, the result follows from (5.2). This proves (7).

Let \( \mathcal{T}' \) be the \( \theta' \)-truncation of \( \mathcal{T}' \). Then \( H', \eta', \mathcal{T}' \) is a \( \Sigma \)-span of order \( \theta' \). Since \( \theta' \geq 4 \) it follows from (2) and (4) that \( s_1, s_2, r_2, \ldots, r_k \) are eyes.
of $H', \eta', \mathcal{F}^*$. From (1), (5), and (6) it follows that for distinct $r, r' \in \{s_1, s_2, r_2, ..., r_k\}$,

$$d''(r, r') = \min(\theta', d'(r, r')) = \theta'.$$

By (7) and (5.1), $H', \eta', \mathcal{F}^*$ is $(\lambda', \mu)$-flat, since $\theta' \geq 4\lambda' + \mu + 2$. Hence (S1) holds, as required.

In view of (7.8), (7.9), (7.10) and (7.11) it follows that in all cases one of (S1), (S2), (S3) holds, and the proof of (1.2) is complete. Hence so is the proof of (1.1).

8. SOME MILD IMPROVEMENTS

In order to facilitate applying our main theorem (1.1), we make in this section two modifications to it. The first is concerned with replacing (1.1)(i) with a condition involving large clique minors, because our application is to graphs with no large clique minor. Thus, let $\mathcal{F}^*$ be a tangle in a graph $G$, and let $p \geq 0$ be an integer. We say that $\mathcal{F}^*$ controls a $K_p$ minor of $G$ if $\mathcal{F}^*$ has order $\geq p$ and there are $p$ non-null disjoint connected subgraphs $X_1, ..., X_p$ of $G$ such that

(i) for $1 \leq i < j \leq p$ there is an edge of $G$ with one end in $V(X_i)$ and the other in $V(X_j)$, and

(ii) there is no $(A, B) \in \mathcal{F}^*$ of order $< p$ such that $V(X_i) \subseteq V(A)$ for some $i$ ($1 \leq i \leq p$).

[4, Theorem (4.5)] implies (since the tangle in $G$ induced by $\mathcal{F}$ is a subset of $\mathcal{F}^*$) that

(8.1) For any surface $\Sigma$ with $bd(\Sigma) = \emptyset$, and any integer $p \geq 0$ there exists $\kappa, \phi \geq 0$ such that the following is true. Let $\mathcal{F}^*$ be a tangle in a graph $G$ of order $\geq p$, and let $H, \eta, \mathcal{F}$ be a $\Sigma$-span of order $\geq \phi$, with $\geq \kappa$ independent eyes. Then $\mathcal{F}^*$ controls a $K_p$ minor of $G$.

Hence we may modify (1.1) as follows.

(8.2) For any surface $\Sigma$ with $bd(\Sigma) = \emptyset$, and any integers $p, \phi, \mu \geq 0$, there are integers $\kappa, \theta, \lambda, \rho \geq 0$ such that the following holds. Let $\mathcal{F}^*$ be a tangle in a graph $G$ such that there is no $\mathcal{F}^*$-central segregation of $G$ of type $(\rho, \kappa)$ with an arrangement in $\Sigma$. Then either
(i) \( \mathcal{F}^* \) controls a \( K_p \) minor of \( G \), or

(ii) there is a \( \Sigma' \)-span of order \( \geq \phi \), where \( \Sigma' \) is a surface obtained by adding a crosscap to \( \Sigma \), or

(iii) no \( \Sigma \)-span of order \( \geq \theta \) is \((\lambda, \mu)\)-flat.

Proof. If the result holds for \( 4, \phi, \mu \) then it holds for \( p, \phi, \mu \) for all \( p \leq 4 \), because if \( \mathcal{F}^* \) controls a \( K_4 \) minor then it controls a \( K_p \) minor for all \( p \leq 4 \). Hence we may assume that \( p \geq 4 \). Let \( \kappa, \phi_1 \) satisfy (8.1) (with \( \phi \) replaced by \( \phi_1 \)), and let \( \phi_2 = \max(\phi, \phi_1) \). Let \( \theta, \lambda, p \) satisfy (1.1) (with \( \phi \) replaced by \( \phi_2 \)). We may assume that \( \theta \geq p \). Then we claim \( \kappa, \theta, \lambda, p \) satisfy (8.2). For let \( \mathcal{F}^* \), \( G \) be as in the theorem. If \( \mathcal{F}^* \) has order \( \leq p \) then (8.2)(iii) holds, since \( \theta \geq p \). Thus we may assume that \( \mathcal{F}^* \) has order \( \geq p \). By (1.1), one of (1.1)(i), (ii), (iii) holds. If (1.1)(i) holds then (8.2)(i) holds, by (8.1), and if (1.1)(ii) or (iii) holds, then (8.2)(ii) or (iii) holds.

One of the conclusions of (8.2) is that no \( \Sigma \)-span of order \( \geq \theta \) is \((\lambda, \mu)\)-flat, and the second modification we wish to make concerns this. If \( \mathcal{F}^* \) is a tangle in \( G \) and \( H, \eta, \mathcal{F} \) is a \( \Sigma \)-span of order \( \theta \), where \( \Sigma \) is a surface with \( \text{bd}(\Sigma) = \emptyset \), we say that \( H, \eta, \mathcal{F} \) is \((\lambda, \mu)\)-level if

(i) \( \theta \geq 4\lambda + \mu + 2 \)

(ii) \( H, \eta, \mathcal{F} \) is not \( \mu \)-stepped, and

(iii) for every \( \Sigma \)-span \( H', \eta', \mathcal{F}' \), obtained from \( H, \eta, \mathcal{F} \) by rearranging within \( \lambda \) of some \( z \in A(H) \), there is no \( \eta'(H') \)-path in \( G \) with ends \( \eta'(u), \eta'(v) \) with \( d'(u, v) \geq \mu \) and \( d(z, u), d(z, v) \leq \lambda \).

(The difference between (i)–(iii) here and the definition of \((\lambda, \mu)\)-flat is the requirement that \( d(z, u), d(z, v) \leq \lambda \).)

(8.3) Let \( \lambda, \mu \geq 0 \) be integers, let \( \mathcal{F}^* \) be a tangle in a graph \( G \), and let \( \Sigma \) be a surface with \( \text{bd}(\Sigma) = \emptyset \). Then every \( \Sigma \)-span which is \((\lambda, \mu)\)-level is \((\lambda, \mu)\)-flat.

Proof. Suppose that \( H, \eta, \mathcal{F} \) is a \( \Sigma \)-span of order \( \theta \) that is not \((\lambda, \mu)\)-flat; we shall show that it is not \((\lambda + \mu + 2, \mu)\)-level. If \( \theta < 4(\lambda + \mu + 2) + \mu + 2 \) then \( H, \eta, \mathcal{F} \) is not \((\lambda + \mu + 2, \mu)\)-level as required, and so we assume that \( \theta \geq 4\lambda + 5\mu + 10 \). Since \( H, \eta, \mathcal{F} \) is not \((\lambda, \mu)\)-flat and \( \theta \geq 4\lambda + \mu + 2 \), there is a \( \mu \)-stepped \( \Sigma \)-span \( H', \eta', \mathcal{F}' \) of order \( \theta - 4\lambda - 2 \), obtained from \( H, \eta, \mathcal{F} \) by rearranging within \( \lambda \) of some \( z \in A(H) \). Let \( P \) be an \( \eta'(H') \)-path with ends \( \eta'(a), \eta'(b) \), such that \( d'(a, b) \geq \mu \).
(1) If \(d(z, a), d(z, b) \leq \lambda + \mu + 2\), then \(H, \eta, \mathcal{T}\) is not \((\lambda + \mu + 2)\)-level.

Subproof. Let \(\mathcal{T}'\) be the \((\lambda + \mu + 2 - 2)\)-truncation of \(\mathcal{T}\); then \(H', \eta', \mathcal{T}'\) is a \(\Sigma\)-span of order \(\theta - 4(\lambda + \mu + 2)\), obtained from \(H, \eta, \mathcal{T}\) by rearranging within \(\lambda + \mu + 2\) of \(z\), and \(P\) is an \(\eta'(H')\)-path, and

\[
d'(a, b) = \min(\theta - 4(\lambda + \mu + 2) - 2, d'(a, b)) \leq \mu,
\]

and so condition (iii) in the definition of “level” is false. This proves (1).

From (1) we may, therefore, assume that

(2) \(d(z, a) \geq \lambda + \mu + 3\), and \(a \in V(H)\), and \(\eta(a) = \eta'(a)\).

Let \(e\) be the edge of \(P\) incident with \(\eta(a)\).

(3) \(e\) is not an edge of \(\eta(H)\).

Subproof. Suppose that \(e = \eta(f)\) for some \(f \in E(H)\). Since \(\eta(f)\) is incident with \(\eta(a)\) it follows that \(a\) is an end of \(f\) in \(H\), and so \(d(a, f) \leq 2\). From (2), it follows that \(d(z, f) \geq \lambda + \mu + 1\), and so \(f \in E(H')\) and \(\eta'(f) = \eta(f) = e\). But \(e \in E(P)\), and \(P\) is an \(\eta'(H')\)-path, a contradiction. This proves (3).

Let \(A\) be the union of all \(x \in A(H)\) with \(d(z, x) \leq \lambda\). We may assume that

(4) There is no \(v \in V(H)\) with \(v \neq a\) such that \(\eta(v) \in V(P)\).

Subproof. Suppose that there is such a vertex, and choose it so that the subpath \(Q\) of \(P\) between \(\eta(a)\) and \(\eta(e)\) is minimal. Then \(Q\) is an \(\eta(H)\)-path, by (3). If \(d(a, v) \geq \mu\) then \(H, \eta, \mathcal{T}\) is \(\mu\)-stepped and so not \((\lambda + \mu + 2, \mu)\)-level, as required, and so we suppose, for a contradiction, that \(d(a, v) \leq \mu - 1\). Consequently, \(d(z, v) \geq \lambda + 4\) and so \(v \in V(H')\) and \(\eta'(v) = \eta(e)\). Since \(P\) is an \(\eta'(H')\)-path it follows that \(Q = P, v = h\), and \(d(a, b) \leq \mu - 1\). Since \(d(z, a) \geq \lambda + \mu + 3\), and \(d(z, \sigma) \leq \lambda\) for all \(\sigma \in A\), it follows that \(d(a, \sigma) \geq \mu + 3 - d(a, b)\) for all \(\sigma \in A\). By (4.2), we deduce that \(d'(a, b) \leq d(a, b) < \mu\), a contradiction. We may, therefore, assume that (4) holds.

From (4), it follows that \(b \in A\); for otherwise \(b \in V(H)\) and \(\eta(b) = \eta'(b) \in V(P)\), contrary to (4). Since \(H'\) is 2-cell and hence connected, there is a path of \(H'\) from \(b\) to \(a\), and hence we may choose a minimal path \(Q\) of \(\eta'(H')\) from \(\eta'(b)\) to \(V(\eta(H))\). Let \(Q\) have ends \(\eta'(b), \eta(c)\) where \(c \in V(H)\). It follows that no edge of \(Q\) is in \(E(\eta(H))\), and no vertex of \(Q\) is in \(V(\eta(H))\) except \(\eta(c)\).

(5) \(Q\) has at least one edge.

Subproof. Otherwise \(\eta'(b) = \eta(c)\), and so \(\eta(c) \in V(P)\). By (4), \(e = a\), and so \(\eta'(b) = \eta(c) = \eta(a) = \eta'(a) \neq \eta'(b)\), a contradiction. This proves (5).
(6) $d(z, c) \leq \lambda + 1$.

Subproof. Let $f \in E(H')$ be such that $\eta'(f)$ is the edge of $Q$ incident with $\eta(c)$. If $f \not\in A$, then $f \in E(H)$ and $\eta(f) = \eta'(f) \in E(Q)$, a contradiction since no edge of $Q$ is in $E(\eta(H))$. Thus $f \subseteq A$, and so $c \in \hat{A}$. Since $c \in V(H)$ there exists $\sigma \in \hat{A}$ with $d(c, \sigma) \leq 1$, and consequently $d(z, c) \leq d(z, \sigma) + 1 \leq \lambda + 1$. This proves (6).

From (6) we deduce that $c \neq a$, since $d(z, a) > \lambda' \geq \lambda + 1$, and so $P \cup Q$ is an $\eta(H)$-path with ends $\eta(a), \eta(c)$. From (2) and (6),

$$d(a, c) \geq d(z, a) - d(z, c) \geq (\lambda + \mu + 3) - (\lambda + 1) = \mu + 2 \geq \mu,$$

and so $H, \eta, \mathcal{F}$ is $\mu$-stepped and not $(\lambda + \mu + 2, \mu)$-level. This completes the proof.

From (8.2) and (8.3) we obtain

(8.4) For any surface $\Sigma$ with $bd(\Sigma) = \emptyset$, and any integers $p, \phi, \mu \geq 0$, there are integers $\kappa, \theta, \lambda, \rho \geq 0$ such that the following holds. Let $\mathcal{F}^*$ be a tangle in a graph $G$ such that there is no $\mathcal{F}^*$-central segregation of $G$ of type $(\rho, \kappa)$ with an arrangement in $\Sigma$. Then either

(i) $\mathcal{F}^*$ controls a $K_p$-minor of $G$, or

(ii) there is a $\Sigma$-span of order $\geq \phi$, where $\Sigma'$ is a surface obtained by adding a crosscap to $\Sigma$, or

(iii) no $\Sigma$-span of order $\geq \theta$ is $(\lambda, \mu)$-level.

Proof. Choose $\kappa, \theta, \lambda', \rho$ to satisfy (8.2). Let $\lambda = \lambda' + \mu + 2$; and we claim that (8.4) holds. For let $\mathcal{F}^*, G$ be as (8.4). By (8.2), we may assume that no $\Sigma$-span of order $\geq \theta$ is $(\lambda', \mu)$-flat. By (8.3), no $\Sigma$-span of order $\geq \theta$ is $(\lambda, \mu)$-level, as required.

REFERENCES