Compact probabilistic metrics on bounded closed intervals of distribution functions

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Abstract

A Menger space (a special type of probabilistic metric spaces) is said to be compact if its strong uniformity is compact. We construct, in a natural way, Menger T-metrics on the set of distribution functions, one for each copula T (a special type of continuous t-norms). We show that, on each bounded closed interval of distribution functions, the strong uniformities of all our spaces are induced by the modified Lévy metric. Hence, they are compact. We establish an alternative description and a number of good properties of our Menger spaces, which may render them well-behaved examples for workers in the field.

Keywords: Probabilistic metric space; Menger space; Strong uniformity; Compactness

1. Introduction

Menger’s T-metrics (T-metrics, for short, a special type of probabilistic metrics), where T stands for a t-norm on the unit interval, take values in the set $\Delta^+$ of distance distribution functions. So, naturally, this $\Delta^+$ and the larger set $\Delta$ of distribution functions have become main work domains in the theory and applications of Menger T-metric spaces; including the fields of fuzzy analysis, fuzzy differential equations and fuzzy topology. There, questions on convergence and accumulation points are posed repeatedly. Part of that can be handled adequately through ordinary metrics on $\Delta$, notably the compact metric $d_L$ due to Lévy. But, in some situations, it is also desirable to have some sorts of metrics on $\Delta$ that interact well with the basic operations on $\Delta$, such as the sup-T addition and multiplication. Menger T-metrics on $\Delta$ bear a promising potential for such interaction, in natural and

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straightforward ways. Prominent among them are certain Menger $T$-metrics $\mathcal{F}^T_{\delta}$ on $\Delta$ due to Höhle. Unfortunately, the strong uniformities of those $\mathcal{F}^T_{\delta}$ fail to be compact, for some continuous t-norms $T$ such as Min. This limits their usefulness.

We propose to remedy this shortcoming by modifying Höhle’s $T$-metrics $\mathcal{F}^T_{\delta}$ into smaller $T$-metrics $\mathcal{M}^T$ on $\Delta$, one for each $T$. Those $\mathcal{M}^T$ retain much of the good behaviour of $\mathcal{F}^T_{\delta}$, but with the added advantage that their strong uniformities are compact on all bounded closed intervals in $\Delta$, for all copulas $T$ (a special type of continuous t-norms).

In Sections 2–4 we recapitulate on basics of Menger spaces, the Lévy metric $d_L$ on $\Delta$ and Höhle’s $T$-metrics $\mathcal{F}^T_{\delta}$ on $\Delta$, respectively; setting up the machinery needed for our study. Then in Section 5 we define an indexed family $\{G^T_r\}_{r>0}$ of functions: $\Delta^2 \to \Delta^+$, and we study its properties. This helps, in Section 6, in the smooth introduction of the function $\mathcal{M}^T: \Delta^2 \to \Delta^+$, and in proving that it is a Menger $T$-metric smaller than $\mathcal{F}^T_{\delta}$. We also prove the compactness of the strong uniformity of $\mathcal{M}^T$ on every bounded closed interval in $\Delta$, by showing that it coincides there with the compact uniformity of the Lévy metric. In Section 7, we provide an alternative, computationally efficient formula for $\mathcal{M}^T$. And we supply eight propositions to depict aspects of good behaviour of $\mathcal{M}^T$. Our conclusions follow in Section 8.

2. Menger spaces

The set $\Delta$ of distribution functions consists of all monotone left continuous functions $\eta$, from the set $R$ of real numbers to the unit interval $I = [0,1]$. Several important subsets of $\Delta$ are defined by

$$
\Delta_0 = \{ \eta \in \Delta \mid \exists t \in R: \eta(t) = 0 \},$

$$
\Delta^+ = \{ \eta \in \Delta \mid \eta(0) = 0 \},$

$$
\mathcal{D} = \{ \eta \in \Delta \mid \inf \eta = 0 \text{ and } \sup \eta = 1 \},$

$$
\mathcal{D}_0 = \mathcal{D} \cap \Delta_0,$n

$$
\mathcal{D}^+ = \mathcal{D} \cap \Delta^+.$$

For each pair $a < b$ in $R$, we take

$$
\mathcal{D}[a,b] = \{ \eta \in \Delta \mid \eta(a) = 0 \text{ and } \eta(b+) = 1 \},$$

which we call a bounded closed interval in $\Delta$.

We also say that a distribution function is crisp if it takes values in $\{0,1\}$ only.

We have $\mathcal{D}[a,b], \mathcal{D}^+ \subseteq \mathcal{D}_0 \subseteq \mathcal{D} \subseteq \Delta$, and $\mathcal{D}[a,b], \Delta^+ \subseteq \Delta_0 \subseteq \Delta$. The elements of $\Delta^+$ are called distance distribution functions (ddfs). Since a ddf $\eta$ satisfies $\eta(t) = 0 \forall t \leq 0$, we may consider it as a function on the set $R^+$ of positive real numbers only.

We take the partial order $\preceq$ on $\Delta$ to be the opposite of the pointwise partial order of distribution functions as real functions. The partially ordered set $(\Delta, \preceq)$ is a complete lattice. We denote its join by $\sqcup$, and its meet by $\sqcap$. The extended real line $R^+ = R \cup \{-\infty, +\infty\}$ is order-embedded into
by sending every real \( r \) onto the crisp distribution function \( \varepsilon_r \), given by

\[
\varepsilon_r(s) = \begin{cases} 
0, & s \leq r, \\
1, & s > r,
\end{cases}
\]

and by sending \(-\infty\) onto the constant function \( \varepsilon_{-\infty} = 1 \) (the bottom element of \( \Delta \)), and \( +\infty \) onto the constant function \( \varepsilon_{\infty} = 0 \) (the top element of \( \Delta \)). This way, the chain \( \{0\} \cup \mathbb{R}^+ \) is order-embedded into \((\mathcal{D}^+, \leq)\).

All subsets of \( \text{PSOH} \) defined above are sublattices of \((\text{PSOH}, \sqcup, \sqcap)\). The sublattice \((\mathcal{D}^+, \sqcup, \sqcap)\) is complete, whereas the sublattice \((\mathcal{D}^+, \sqcup, \sqcap)\) is nonempty-meet-complete and conditionally join-complete, with bottom element \( \varepsilon_0 \) the constant function 1 on \( \mathbb{R}^+ \). The sublattice \((\mathcal{D}[a,b], \sqcup, \sqcap)\) is complete, with bottom element \( \varepsilon_a \) and top element \( \varepsilon_b \).

A triangular norm (t-norm) \([21,39]\) is a binary operation on \( I \) that is monotone in each argument, commutative, associative and has identity element 1.

For each left continuous t-norm \( T \), the \( T \)-addition \( \oplus_T \) on \( \text{PSOH} \) (it is denoted by \( \text{PST}_T \) in [39]) is the binary operation defined for all \( PDC_1, PDLE \in \text{PSOH} \) by \([39]\)

\[
(PDC_1 \oplus_T PDLE)(s) = \sup \{ PDC_1(t) T \varepsilon_j(s - t) \mid 0 < t < s \}, \quad s \in R.
\]

It is clear that \( \oplus_T \) is well-defined on \( \text{PSOH} \). It is commutative, associative, monotone in each argument (with respect to \( \leq \)) and has identity element \( \varepsilon_0 \). The subsets \( \mathcal{D}^+ \), \( \mathcal{D}_0 \), \( \mathcal{D} \), \( \Delta^+ \), \( \Delta_0 \) and the set of crisp ddfs are closed under \( \oplus_T \). In particular \( \varepsilon_a \oplus_T \varepsilon_b = \varepsilon_{a+b} \) for all \( a, b \in \mathbb{R} \), and \( \oplus_T \) becomes given on \( \Delta^+ \) by, \( \forall \eta, \zeta \in \Delta^+ \):

\[
(\eta \oplus_T \zeta)(s) = \sup \{ \eta(t) T \zeta(s - t) \mid 0 < t < s \}, \quad s > 0.
\]

**Lemma 2.1.** Let \( T \) be a left continuous t-norm. For all nonempty subsets \( \{\eta_k \mid k \in K\} \) and \( \{\zeta_j \mid j \in J\} \) of \( \Delta^+ \), we have

\[
\sqcap \{\eta_k \mid k \in K\} \oplus_T \sqcap \{\zeta_j \mid j \in J\} = \sqcap \{\eta_k \oplus_T \zeta_j \mid (k, j) \in K \times J\}.
\]

**Proof.** We have for all \( s > 0 \):

\[
(\sqcap \{\eta_k \mid k \in K\} \oplus_T \sqcap \{\zeta_j \mid j \in J\})(s)
\]

\[
= \sup_{k,j} \left\{ \sup_{t \in \mathbb{R}} \eta_k(t) T \sup_{j \in J} \zeta_j(s - t) \mid 0 < t < s \right\}
\]

\[
= \sup_{k,j} \eta_k(t) T \sup_{j \in J} \zeta_j(s - t) \mid 0 < t < s
\]

\[
= \sup_{k,j} (\eta_k \oplus_T \zeta_j)(s)
\]

\[
= (\sqcap \{\eta_k \oplus_T \zeta_j \mid (k, j) \in K \times J\})(s).
\]

This proves (4). \( \Box \)
We quote the following famous definition from [25,39], with slight modification of notation:

**Definition 2.1** (Menger [25], Schweizer and Sklar [39]). Let $T$ be a left continuous t-norm. A *Menger probabilistic $T$-pseudo-metric* $F$ on a set $X$ is a function $F: X^2 \to \mathbb{A}^+$ that satisfies, for all $x, y, z \in X$:

1. (PM1) $F(x, x) = \varepsilon_0$ (self-coincidence),
2. (PM2) $F(x, y) = F(y, x)$ (symmetry),
3. (PM3) $F(x, y) \odot_T F(y, z) \geq F(x, z)$ ($T$-triangle inequality).

Such $F$ will be a *Menger probabilistic $T$-metric* if it further satisfies for all $x, y \in X$:

4. (PM4) If $x \neq y$ then $F(x, y) \neq \varepsilon_0$ (separation).

The pair $(X, F)$ is called a *Menger probabilistic $T$-(pseudo)-metric space*, or a *Menger $T$-space*, for short.

Menger spaces are a particular type of the more general *probabilistic (pseudo)-metric spaces*, whereby the binary operations $\odot_T$ can be replaced by operations from a larger class [39,40].

The *strong uniformity* (on $X$) of a Menger $T$-space $(X, F)$ is defined through the uniform basis consisting of the following vicinities:

$$U(\varepsilon) = \{ (x, y) \in X^2 \mid F(x, y)(\varepsilon) > 1 - \varepsilon \}, \quad \varepsilon > 0. \quad (5)$$

Schweizer and Sklar [39] establish that this is indeed a uniformity on $X$, provided that the t-norm $T$ satisfies the following two conditions:

$$\sup_{0 < t < 1} tTt = 1, \quad (6)$$
$$\sup_{0 < t < 1} tTs = s \quad \forall s \in I, \quad (7)$$

see Theorem 12.1.5 in [39]. In this article, we impose the stronger assumption that $T$ is left continuous.

The *strong topology* on $X$ [39] (also called the $(\varepsilon, \lambda)$-topology [40]) of $(X, F)$ is the topology induced by the strong uniformity. It is defined, equivalently, through either one of the following two neighbourhood bases at each point $x \in X$:

$$N_x(\varepsilon) = \{ y \in X \mid F(x, y)(\varepsilon) > 1 - \varepsilon \} \subseteq X, \quad \varepsilon > 0, \quad (8)$$
$$N_x(\lambda, \varepsilon) = \{ y \in X \mid F(x, y)(\varepsilon) > 1 - \lambda \} \subseteq X, \quad \lambda, \varepsilon > 0. \quad (9)$$

**Theorem 2.1** (Schweizer and Sklar [39]). *The strong topology of $(X, F)$ is a Hausdorff topology on $X$, and the strong uniformity of $(X, F)$ is a Hausdorff uniformity on $X$, provided that (PM4), (6) and (7) hold.*

**Proof.** See Theorem 12.1.2 and the comments following it in [39], and Theorem 12.1.5 in [39]. \( \Box \)

The uniform basis described in (5) can be replaced by the countable basis $\{ U(1/n) \subseteq X^2 \mid n$ is a positive integer}. Therefore, under the assumptions (PM4), (6) and (7), the strong uniformity, and
hence the strong topology, are *metrizable* (Theorem 12.1.6 in [39]). Although the metric inducing them is not given an explicit formula in [39], that uniformity is induced in the usual way from the semimetric $\hat{d}$ on $X$ defined by

$$\hat{d}(x, y) = \inf\{\varepsilon > 0 \mid \mathcal{F}(x, y)(\varepsilon) > 1 - \varepsilon\}, \quad (x, y) \in X^2$$  \hspace{1cm} (10)

(combine (12.1.9), (4.3.4) and (12.1.3) in [39]); that is,

$$\mathcal{U}(\varepsilon) = \{(x, y) \in X^2 \mid \hat{d}(x, y) < \varepsilon\}, \quad \varepsilon > 0.$$  \hspace{1cm} (11)

It is with respect to this strong uniformity that the notions of convergence, Cauchy sequences and completeness in $(X, \mathcal{F})$ are meant in some literature [6,7,10–12,29,30,32]. It follows from (11) that these are the same notions as the corresponding ones with respect to $\hat{d}$ (Theorem 12.1.7 in [39]). This endows them with the following simple descriptions, derived from (10):

A sequence $(x_n)$ in $(X, \mathcal{F})$ is said to *converge* to a point $x \in X$ if for every $\varepsilon > 0$ there exists a positive integer $n_\varepsilon$ such that

$$\forall n \geq n_\varepsilon: \mathcal{F}(x_n, x)(\varepsilon) > 1 - \varepsilon.$$  \hspace{1cm} (12)

A sequence $(x_n)$ in $(X, \mathcal{F})$ is called a *Cauchy sequence* in $(X, \mathcal{F})$ if for every $\varepsilon > 0$ there exists a positive integer $n_\varepsilon$ such that

$$\forall n, m \geq n_\varepsilon: \mathcal{F}(x_n, x_m)(\varepsilon) > 1 - \varepsilon.$$  \hspace{1cm} (13)

It follows from the $T$-triangle inequality and (6) that every convergent sequence in $(X, \mathcal{F})$ is Cauchy [39]. The Menger $T$-space $(X, \mathcal{F})$ is said to be *complete* if every Cauchy sequence in it converges.

We adhere to this tradition, and say that $(X, \mathcal{F})$ is *compact* ($\text{Hausdorff}$) whenever $X$ is compact ($\text{Hausdorff}$) under the strong topology of $\mathcal{F}$.

3. The modified Lévy metric on $\Delta$

P. Lévy defined his metric on the set $\Delta$ in a note appended to Fréchet [8]. A useful modification of the Lévy metric was introduced by Sibley [41], and we use the following slight modification of Sibley’s definition, due to Schweizer and Sklar [39]:

**Definition 3.1** (Schweizer and Sklar [39]). The *modified Lévy metric* is the real function $d_L$ on $\Delta \times \Delta$ defined by, for all $(\eta, \zeta) \in \Delta \times \Delta$:

$$d_L(\eta, \zeta) = \inf \left\{ h > 0 \mid \forall x \in \left(-\frac{1}{h}, \frac{1}{h}\right): \eta(x - h) - h \leq \zeta(x) \right\} \leq \eta(x + h) + h \text{ and } \zeta(x - h) - h \leq \eta(x) \leq \zeta(x + h) + h.$$  \hspace{1cm} (14)

**Lemma 3.1** (Schweizer and Sklar [39]). If $d_L(\eta, \zeta) > 0$, then the infimum in (14) is attained.
Theorem 3.1 (Schweizer and Sklar [39], Sibley [41]). The function $d_L$ is a metric on $\Delta$. The metric space $(\Delta, d_L)$ is compact, and hence complete.

Lemma 3.2 (Schweizer and Sklar [39]). The restriction of $d_L$ to $\Delta^+$ is also given by, for all $(\eta, \zeta) \in \Delta^+ \times \Delta^+$:

$$d_L(\eta, \zeta) = \inf \left\{ h > 0 \mid \forall x \in \left(0, \frac{1}{h}\right) : \zeta(x) \leq \eta(x + h) + h \text{ and } \eta(x) \leq \zeta(x + h) + h \right\}.$$  

(15)

Also, whenever $d_L(\eta, \zeta) > 0$, the infimum in (15) is attained.

Definition 3.2 (Schweizer and Sklar [39], Sibley [41]). A sequence $(\eta_n)$ in $\Delta$ is said to converge weakly to $\eta \in \Delta$ if the sequence $(\eta_n(t))$ converges to $\eta(t)$ at each continuity point $t$ of $\eta$.

Theorem 3.2 (Schweizer and Sklar [39], Sibley [41]). A sequence $(\eta_n)$ converges in $(\Delta, d_L)$ to $\eta \in \Delta$ if and only if it converges weakly to $\eta$.

The subsets $\mathcal{D}[a, b]$, $\Delta^+$, and $\mathcal{D}[0, 1]$ are clearly closed in $(\Delta, d_L)$. Hence, they are compact under the modified Lévy metric. It is clear that on bounded closed intervals, $d_L$ becomes equivalent to the original Lévy metric of [8]; that is, on each $\mathcal{D}[a, b]$ we can replace $d_L$ by the larger metric

$$\tilde{d}_L(\eta, \zeta) = \inf \{ h > 0 \mid \forall x \in \mathbb{R} : \eta(x - h) - h \leq \zeta(x) \leq \eta(x + h) + h \}$$

and still come out with the same compact Hausdorff topology on $\mathcal{D}[a, b]$.

Corollary 3.1. On a bounded closed interval $\mathcal{D}[a, b]$ in $\Delta$, the uniformity induced by the modified Lévy metric has uniform basis $\mathcal{V} = \{ \mathcal{V}(h) \mid h > 0 \}$ whose vicinities are given $\forall h > 0$ by

$$\mathcal{V}(h) = \{ (\eta, \zeta) \in (\mathcal{D}[a, b])^2 \mid \forall x \in \mathbb{R} : \eta(x - h) - h \leq \zeta(x) \leq \eta(x + h) + h \}.$$ 

The uniform space $(\mathcal{D}[a, b], [\mathcal{V}])$ is compact and Hausdorff.

4. Höhle’s $T$-metric on $\Delta$

Several fuzzy uniformities have been proposed on either of the sets $\Delta$, $\mathcal{D}$ or $\mathcal{D}[0, 1]$, see [3,5,15,18,19,22,23,27,36,37]. Note that the sets $\mathcal{D}[0, 1]$ and $\Delta$ have been generalized by Hutton [18] and Gantrner et al. [9], respectively, by allowing truth values from a general completely distributive lattice $L$ with order-reversing involution, rather than the lattice $[0, 1]$ in particular. Those
fuzzy uniformities are usually fuzzy metrizable in some sense or another, resulting in standard examples of what workers in fuzzy set theory prefer to call fuzzy real lines. In one principal trend, Hutton, in his theory of fuzzy uniformities [19], introduced a canonical fuzzy metric on $D[0,1]$ in terms of fuzzy $r$-neighbourhoods of fuzzy subsets, $r$ running on all positive reals. Its explicit realization as a function from $L^X \times L^X$ into $[0,\infty)$ has been achieved by Erceg [5], and Rodabaugh [35] has shown that this realization depends on the complete distributivity of the underlying lattice $L$ of truth values. Recently, Rodabaugh [38] also gave an axiomatic foundation for Hutton fuzzy uniformities using tensor products, which include t-norms on $[0,1]$ when $L=[0,1]$.

But, we are concerned in this article with our modifications of Höhle’s fuzzy real lines [15] (one for each $T$), which are defined by means of certain Menger metrics on $\Delta$, and quoted below.

Throughout this section, $T$ is a left continuous t-norm on $[0,1]$.

**Definition 4.1** (Höhle [15]). The function $F_T^T : \Delta \times \Delta \to \Delta^+$ is given, $\forall (\eta, \zeta) \in \Delta \times \Delta$, by

$$F_T^T(\eta, \zeta) = \sqcap \{ \xi \in \Delta^+ | \eta \oplus_T \xi \geq \zeta \text{ and } \xi \oplus_T \zeta \geq \eta \}. \quad (18)$$

It should be noted that Höhle discourses in the more general setting of a complete Brouwerian lattice $L$ of truth values, whereas we consider here the special case that $L=[0,1]$ only. This naturally defined function $F_T^T$ is indeed a Menger $T$-metric on $\Delta$ [15], which can be verified directly. In contrast, the Hutton–Erceg fuzzy metric on $D[0,1]$ is not a probabilistic metric. As such, it seems to differ fundamentally from all $F_T^T$. Nevertheless, Höhle [16] has shown that in the case $T$ is the Łukasiewicz conjunction $W$ on the unit interval, his $W$-uniformities become a special type of Hutton fuzzy uniformities. It is therefore an open question whether Menger $W$-metrics too can be reformulated in terms of Hutton–Erceg fuzzy metrics!

This $F_T^T$ enjoys a good number of useful properties, some of which are shown in the next five lemmas.

**Lemma 4.1** (Höhle [15]). $F_T^T$ is the unique Menger $T$-pseudo-metric on $\Delta$ that satisfies the following two conditions:

- $F_T^T(\eta, e_0) = \eta$, $\forall \eta \in \Delta^+$.
- For all Menger $T$-pseudo-metric spaces $(X, F)$ and all $x, y, z, w \in X$: $F_T^T(F(x, z), F(y, w)) \leq F(x, y) \oplus_T F(z, w)$.

We frequently write $T$-metric to mean $Menger T$-metric. In [31] we find some applications related to the uniform continuity of the $T$-metric $F_T^T$.

**Lemma 4.2** (Hashem and Morsi [14]). Höhle’s $T$-metric $F_T^T$ satisfies:

- $\forall (\eta, \zeta) \in \Delta \times \Delta : \eta \oplus_T F_T^T(\eta, \zeta) \geq \zeta$ and $F_T^T(\eta, \zeta) \oplus_T \zeta \geq \eta$.
- Let $(X, F)$ be a Menger $T$-pseudo-metric space, and let $S$ be a nonempty subset of $X$. Define a function $F_S : X \to \Delta^+$ by $F_S(x) = F(x, S) = \sqcap \{ F(x, z) | z \in S \}$, $x \in X$. Then $\forall x, y \in X$: $F_T^T(F_S(x), F_S(y)) \leq F(x, y)$.
- $\forall a, b \in R$, $F_T^T(e_a, e_b) = e_{|b-a|}$. 

By virtue of the properties listed in the preceding two lemmas, Hohle’s T-metric spaces $(D[0,1], F^T_{\Delta})$ are used to characterize certain notions of fuzzy complete regularity [14,15] and fuzzy normality [26].

The residuation implication of $T$ (see [1,13,17]) is the binary operation $I_T$ on $I$ given by
\[
I_T(x, \gamma) = \sup\{\beta \in I \mid T(x, \beta) \leq \gamma\}, \quad x, \gamma \in I.
\]
The ddfs $F^T_{\Delta}(\eta, \zeta)$ have the following functional description:

**Lemma 4.3** (Hashem and Morsi [14]). Hohle’s T-metric $F^T_{\Delta}$ is also given, $\forall(\eta, \zeta) \in \Delta \times \Delta, \forall s \in R^+$, by
\[
F^T_{\Delta}(\eta, \zeta)(s) = \sup_{q < s, p \in E} \min\{I_T(\zeta(p), \eta(p + q)), I_T(\eta(p), \zeta(p + q))\}.
\]
It follows that $F^T_{\Delta}(\eta, \zeta) \in D^+$ whenever $\eta, \zeta$ are in $D_0$. A binary operation $\star$ on $R$ is said to be **right nonexpansive** if it satisfies $|s \star r - s \star t| \leq |r - t|$ $\forall s, r, t \in R$. Left nonexpansion is defined similarly. On the basis of (20), the following property has been derived in [3] for Hohle’s T-metric:

**Lemma 4.4** (El-Abyad et al. [3]). Let $\star$ be a right nonexpansive binary operation on $R$, and let $\odot_T$ be the sup-$T$ extension of $\star$ over $I^R$ (defined in a similar way as in (2)). Then $\odot_T$ is right nonexpansive in $\Delta$ with respect to Hohle’s T-metric; that is, for all $\eta, \zeta, \xi \in \Delta$ we have
\[
F^T_{\Delta}((\xi \odot_T \eta, \xi \odot_T \zeta) \leq F^T_{\Delta}(\eta, \zeta).
\]
Similar conclusion holds for left nonexpansive binary operations.

In particular, $F^T_{\Delta}((\xi \odot_T \eta, \xi \odot_T \zeta) \leq F^T_{\Delta}(\eta, \zeta)$ (which can be derived directly from (18)), and for all $\varepsilon_0 \leq \xi \leq \varepsilon_1$, $F^T_{\Delta}(\varepsilon_1/h \odot_T \eta, \varepsilon_1/h \odot_T \xi) \leq F^T_{\Delta}(\eta, \xi)$, where $\odot_T$ is the sup-$T$ extension of the multiplication on $R$.

Multiplication of distribution functions by a crisp ddf preserves order and distributes over $\odot_T$. Therefore, we obtain from the last inequality above:

**Lemma 4.5.** Let $b \in R^+$. Then for every ddf $\xi \leq e_b$ and all $\eta, \zeta \in \Delta$ we have
\[
F^T_{\Delta}(\xi \odot_T \eta, \xi \odot_T \zeta) \leq e_b \odot_T F^T_{\Delta}(\eta, \zeta).
\]

**Proof.** $F^T_{\Delta}(\xi \odot_T \eta, \xi \odot_T \zeta) = F^T_{\Delta}(e_1/h \odot_T \xi \odot_T \eta, e_1/h \odot_T \xi \odot_T \zeta) \leq F^T_{\Delta}(e_b \odot_T \eta, e_b \odot_T \zeta)$ (because $e_0 \leq e_{1/h} \odot_T \xi \leq e_{1}$) $= e_b \odot_T F^T_{\Delta}(\eta, \xi)$, by direct verification in (20).

Unfortunately, for some continuous t-norms $T$, $(\Delta, F^T_{\Delta})$ and even $(D[a,b], F^T_{\Delta})$ are not compact in the sense of Section 2. For instance, the sequence $(\eta_n)$ in $D[0,1]$ defined by $\eta_n(t) = 1/n$, $0 < t \leq 1$, satisfies for all pairs $n < m$ of distinct positive integers: $F^\text{Min}_{\Delta}(\eta_n, \eta_m) = \eta_m$. Consequently, for all $n \neq m$ and $0 < \varepsilon < 1/2$ we have $F^\text{Min}_{\Delta}(\eta_n, \eta_m)(\varepsilon) = 1/\max(n,m) > 1 - \varepsilon$. It follows that this sequence contains no Cauchy subsequence, demonstrating that $(D[0,1], F^\text{Min}_{\Delta})$ is not compact.

We endeavour to correct this situation by modifying $F^T_{\Delta}$ into a smaller Menger T-metric on $\Delta$ (smaller in the partial order $\leq$).
We point out that \((D^+, \mathcal{F}_{\min}^D)\) is complete [4]. For interesting constructions on this Menger Min-space, see [2, 4, 24, 28, 33, 34].

5. A family of probabilistic semi-pseudo-metrics on \(A\)

**Definition 5.1.** For each left continuous \(T\)-norm \(T\) and each positive real number \(r\), let \(G^T_r : A \times A \to A^+\) be the mapping defined by, \(\forall (\eta, \xi) \in A \times A\)

\[
G^T_r(\eta, \xi) = \cap \{ \xi \in A^+ | \eta \oplus_T \xi \geq \xi + r \text{ and } \xi \oplus_T \xi \geq \eta + r \}, \tag{21}
\]

where \(\xi + r\) (and similarly \(\eta + r\)) is the distribution function defined by

\[
(\xi + r)(s) = \min\{\xi(s) + r, 1\}, \quad s \in R. \tag{22}
\]

Note that \(\xi + r \leq \zeta\). The next two propositions provide basic properties of the family of these mappings \(G^T_r\), for a fixed \(T\) and for all \(r > 0\).

**Proposition 5.1.** Let \(T\) be a left continuous \(T\)-norm. The following holds for all positive reals \(p, r\) and all distribution functions \(\eta, \xi\):

1. \(e_0 \leq G^T_r \leq \mathcal{F}_D^T\).
2. \(p < r \Rightarrow G^T_p \leq G^T_p(\eta, \xi)(s) \geq G^T_p(\eta, \xi)(s)\).
3. \(p < r \Rightarrow G^T_p(\eta, \xi)(p) \leq G^T_p(\eta, \xi)(r)\).
4. \(G^T_r(\eta, \eta) = e_0\).
5. \(G^T_r(\eta, \xi) = G^T_r(\xi, \eta)\), and so \(G^T_r\) is a probabilistic semi-pseudo-metric on \(A\).
6. \(\eta \oplus_T G^T_r(\eta, \xi) \geq \xi + r\) and \(G^T_r(\eta, \xi) \oplus_T \xi \geq \eta + r\).
7. \(G^T_r(\eta, \xi) = e_0\) \iff \(\xi + r \leq \eta\) and \(\eta + r \leq \xi\).
8. \(G^T_r(\eta, \xi)(r + ) = 1\) \iff \(\forall s \in R\): \(\eta(s - r) - r \leq \eta(s)\) and \(\xi(s - r) - r \leq \xi(s)\).
9. \(G^T_1(\eta, \xi) = e_0\).

**Proof.** We have:

1. This holds because \(\xi + r \leq \zeta\) and \(\eta + r \leq \eta\).
2. Because \(p < r \Rightarrow \xi + r \leq \zeta + p\).
3. This follows clearly from 2.
4. This follows from 1, because \(\mathcal{F}_D^T\) is a \(T\)-metric.
5. This follows clearly from (21).
6. Combine (21) with (4).
7. This follows from (21) and 6.
8. We have the equivalences:

\[
G^T_r(\eta, \xi)(r + ) = 1 \iff G^T_r(\eta, \xi) \leq e_r
\]

\[
\iff \eta \oplus_T e_r \geq \xi + r \text{ and } e_r \oplus_T \xi \geq \eta + r
\]

\[
\iff \forall s \in R: \eta(s - r) - r \leq \eta(s) \text{ and } \xi(s - r) - r \leq \xi(s).
\]

9. This holds because \(\zeta + 1 = \eta + 1 = e_{-\infty}\), which is the bottom element of \(A\). \(\square\)
When \( T \) is a continuous t-norm, the Menger \( T \)-semi-pseudo-metrics \( G^T_r \) satisfy a weak form of the \( T \)-triangle inequality, as shown in the next proposition. As the unit interval \( I \) is compact, such \( T \) is uniformly continuous on \( I^2 \). This means that \( \forall r > 0 \exists \delta^T_r > 0 \) such that the following inequalities hold:

\[
\forall p, s \in I: \ (pT(s + \delta_r)) - r \leq pTs \leq (pT(s - \delta_r)) + r, \tag{23}
\]

where \( \delta_r \) is a simplified writing of \( \delta^T_r \), and \( \hat{+} \) and \( \hat{-} \) are the addition and subtraction on \( I \) truncated at 1 and 0, respectively. We shall always take \( \delta_r \leq r \).

**Proposition 5.2.** Let \( T \) be a continuous t-norm. The following three inequalities in \( \Delta^+ \) hold for all positive reals \( p, r \) and all distribution functions \( \eta, \zeta, \rho \):

\[
\rho \oplus_T (\zeta \hat{+} \delta_p) \geq (\rho \oplus_T \zeta) \hat{+} p, \tag{24}
\]

\[
G^T_r(\eta, \zeta) \oplus_T (\zeta \hat{+} \delta_p) \geq \eta \hat{+} (r + p), \tag{25}
\]

\[
G^T_{\delta_p}(\eta, \zeta) \oplus_T G^T_{\delta_p}(\zeta, \rho) \geq G^T_{r+p}(\eta, \rho). \tag{26}
\]

Consequently, the following inequality in \( I \) holds:

\[
G^T_{\delta_p}(\eta, \zeta)(r)T G^T_{\delta_p}(\zeta, \rho)(p) \leq G^T_{r+p}(\eta, \rho)(r + p). \tag{27}
\]

**Proof.** We have \( \forall s > 0 \) by (3),

\[
(r \oplus_T (\zeta \hat{+} \delta_p))(s) = \sup\{\rho(s - t)T(\zeta(s) \hat{+} \delta_p) | 0 < t < s\}
\leq \sup\{\rho(s - t)T\zeta(s) | 0 < t < s\} \hat{+} p \quad \text{(by (23))}
\]

\[
= (\rho \oplus_T \zeta)(s) \hat{+} p,
\]

and we get (24). Consequently, \( G^T_r(\eta, \zeta) \oplus_T (\zeta \hat{+} \delta_p) \geq (G^T_r(\eta, \zeta) \oplus_T \zeta) \hat{+} p \geq (\eta \hat{+} r) \hat{+} p = \eta \hat{+} (r + p) \), by (24) then Proposition 5.1(6). This proves (25).

Next, we have

\[
(G^T_{\delta_p}(\eta, \zeta) \oplus_T G^T_{\delta_p}(\zeta, \rho)) \oplus_T \rho = G^T_{\delta_p}(\eta, \zeta) \oplus_T (G^T_{\delta_p}(\zeta, \rho) \oplus_T \rho)
\geq G^T_{\delta_p}(\eta, \zeta) \oplus_T (\zeta \hat{+} \delta_p) \quad \text{(by Proposition 5.1(6))}
\geq \eta \hat{+} (\delta_r + p) \quad \text{(by (25))}
\geq \eta \hat{+} (r + p) \quad \text{(because \( \delta_r \leq r \)).}
\]

Similarly, \( (G^T_{\delta_p}(\eta, \zeta) \oplus_T G^T_{\delta_p}(\zeta, \rho)) \oplus_T \eta \geq \rho \hat{+} (r + p) \).

This and (21) yield (26), from which (27) immediately follows. \( \square \)
6. The $T$-metric $\mathcal{M}^T$ on $\Delta$

**Definition 6.1.** For each left continuous t-norm $T$, let $\mathcal{M}^T : \Delta \times \Delta \rightarrow \mathbb{D}[0,1] \subseteq \Delta^+$ be the mapping defined by, \( \forall (\eta, \zeta) \in \Delta \times \Delta \)

$$\mathcal{M}^T(\eta, \zeta)(r) = \sup_{0 < q < r} G^T_q(\eta, \zeta)(q), \quad r > 0. \quad (28)$$

Note that $\mathcal{M}^T(\eta, \zeta) \in \mathbb{D}[0,1]$ because \( \forall r > 1 : \mathcal{M}^T(\eta, \zeta)(r) = G^T(\eta, \zeta)(1) = \varepsilon_0(1) = 1 \), by Proposition 5.1(9).

**Proposition 6.1.** $\mathcal{M}^T \triangleq \mathcal{F}^T_{\mathcal{S}}$.

**Proof.** We have \( \forall (\eta, \zeta, r) \in \Delta \times \Delta \times \mathbb{R}^+ \):

$$\mathcal{M}^T(\eta, \zeta)(r) = \sup_{q < r} G^T_q(\eta, \zeta)(q)$$

$$\geq \sup_{q < r} \mathcal{F}^T_q(\eta, \zeta)(q) \quad (by \ Proposition \ 5.1(1))$$

$$= \mathcal{F}^T_{\mathcal{S}}(\eta, \zeta)(r).$$

This shows that $\mathcal{M}^T \triangleq \mathcal{F}^T_{\mathcal{S}}$. \( \square \)

A (2-place) copula [39] $T$ is a continuous t-norm for which we can take $\forall r > 0 : \delta^T_r = r$ in (23). The basic three continuous t-norms $\text{Min}$, $\prod$ (Product) and $W$ (given by $rW_p = (r + p) - 1$) are copulas [39]. For a copula $T$, inequality (27) becomes:

$$G^T_r(\eta, \zeta)(r)T G^T_p(\zeta, \rho)(p) \leq G^T_{r+p}(\eta, \rho)(r + p). \quad (29)$$

**Theorem 6.1.** Let $T$ be a copula. Then $\mathcal{M}^T$ is a Menger $T$-metric on $\Delta$.

**Proof.** Properties (PM1) and (PM2), in Definition 2.1, follow from assertions 4 and 5 in Proposition 5.1, respectively. We prove (T-PM3) and (PM4):

$$\forall \eta, \zeta, \rho \in \Delta, \forall r, p \in \mathbb{R}^+:$$

$$\mathcal{M}^T(\eta, \rho)(r + p) = \sup_{\varepsilon > 0} G^T_{r+p-2\varepsilon}(\eta, \rho)(r + p - 2\varepsilon)$$

$$\geq \sup_{\varepsilon > 0} [G^T_{r-\varepsilon}(\eta, \zeta)(r - \varepsilon)T G^T_{\rho-\varepsilon}(\zeta, \rho)(p - \varepsilon)] \quad (by \ (29))$$

$$= \left[ \sup_{\varepsilon > 0} G^T_{r-\varepsilon}(\eta, \zeta)(r - \varepsilon) \right] T \left[ \sup_{\varepsilon > 0} G^T_{\rho-\varepsilon}(\zeta, \rho)(p - \varepsilon) \right]$$

$$= \mathcal{M}^T(\eta, \zeta)(r)T \mathcal{M}^T(\zeta, \rho)(p).$$

The net inequality above is a restatement of (T-PM3).
To prove (PM4), let $\eta, \zeta \in \Delta$ be such that $M^T(\eta, \zeta) = \varepsilon_0$. Then $\forall r > 0$:

$$1 = \varepsilon_0(r) = M^T(\eta, \zeta)(r) \leq G^T_r(\eta, \zeta)(r),$$

by virtue of (28) and Proposition 5.1(3). Therefore, $\forall (s, r) \in R \times R^+$ we have by Proposition 5.1(8),

$$\zeta(s - r) - r \leq \eta(s) \quad \text{and} \quad \eta(s - r) - r \leq \zeta(s).$$

Taking the limit as $r \to 0+$, and using the left continuity of $\eta$ and $\zeta$, we find that $\eta$ must coincide with $\zeta$. This yields (PM4), and completes the proof that $M^T$ is a Menger $T$-metric on $\Delta$. \[\square\]

**Theorem 6.2.** Let $T$ be a copula. Then on each bounded closed interval $D[a, b]$ in $\Delta$, the strong uniformity and the strong topology of the $T$-metric $M^T$ coincide with the uniformity and the topology of the modified Lévy metric $d_L$ on $D[a, b]$, respectively. Consequently, the $T$-metric space $(D[a, b], M^T)$ is compact and Hausdorff.

**Proof.** On one hand, the strong uniformity of $(D[a, b], M^T)$ has, by its definition, a basis $U = \{U(r) \subseteq (D[a, b])^2 \mid 0 < r < 1\}$ in which

$$U(r) = \{((\eta, \zeta) \in (D[a, b])^2 \mid M^T(\eta, \zeta)(r) > 1 - r\}, \quad 0 < r < 1.$$ 

On the other hand, the modified Lévy metric $d_L$ on $D[a, b]$ has the uniform basis $V = \{V(r) \mid 0 < r < 1\}$ given in (17). We also have the following entailments for each fixed $0 < r < 1$ and all $(\eta, \zeta) \in (D[a, b])^2$:

$$(\eta, \zeta) \in V(r)
\iff \forall s \in R: \eta(s - r) - r \leq \zeta(s) \leq \eta(s + r) + r \quad \text{and} \quad \zeta(s - r) - r \leq \eta(s) \leq \zeta(s + r) + r$$

$$\Rightarrow \forall s \in R: \eta(s - r) - r \leq \zeta(s) \quad \text{and} \quad \zeta(s - r) - r \leq \eta(s)$$

$$\iff G^T_r(\eta, \zeta)(r+) = 1 \quad (\text{by Proposition 5.1(8)})$$

$$\Rightarrow M^T(\eta, \zeta)(2r+) \geq G^T_r(\eta, \zeta)(r+) = 1$$

$$\Rightarrow M^T(\eta, \zeta)(3r) = 1 > 1 - 3r$$

$$\iff (\eta, \zeta) \in U(3r).$$

This proves that $\forall 0 < r < 1$: $V(r) \subseteq U(3r)$, and so the strong uniformity $[U]$ of $(D[a, b], M^T)$ is coarser than the uniformity $[V]$ of the modified Lévy metric on $D[a, b]$. We also know that the former uniformity is Hausdorff (Theorem 2.1), and the latter uniformity is Hausdorff and compact (Corollary 3.1). Therefore, these two uniformities must coincide ((20)). \[\square\]

**Corollary 6.1.** Let $T$ be a copula. Then the strong uniformity of $(D[a, b], M^T)$ has uniform basis $W = \{W(r) \mid 0 < r < 1\}$ given by

$$W(r) = \{((\eta, \zeta) \in (D[a, b])^2 \mid M^T(\eta, \zeta)(r+) = 1\}$$

$$= \{((\eta, \zeta) \in (D[a, b])^2 \mid M^T(\eta, \zeta) \leq \varepsilon_r\}. \quad (30)$$

$$= \{((\eta, \zeta) \in (D[a, b])^2 \mid M^T(\eta, \zeta) \leq \varepsilon_r\}. \quad (31)$$
Proof. We established within the preceding proof that \( \forall 0 < r < 1: \mathcal{V}(r) \subseteq \mathcal{W}(2r) \subseteq \mathcal{U}(3r) \). The assertion follows from these inclusions, because both \( \{ \mathcal{V}(r) \mid 0 < r < 1 \} \) and \( \{ \mathcal{U}(t) \mid 0 < t < 3 \} \) are bases for the strong uniformity of \((\mathcal{D}[a, b], \mathcal{M}^T)\). \( \Box \)

Corollary 6.2. Let \( T \) be a copula. Then a sequence \( (\eta_n) \) in \((\mathcal{D}[a, b], \mathcal{M}^T)\) converges to \( \eta \in \mathcal{D}[a, b] \) (in the sense of (12)) if and only if it converges weakly to \( \eta \).

Proof. Combine Theorem 6.2 with Theorem 3.2. \( \Box \)

7. Some properties of the \( T \)-metric \( \mathcal{M}^T \)

Some of our proofs in this section will need the following elementary properties of the residuation implication \( \mathcal{I}_T \) of a left continuous t-norm \( T \) (see [1,13,17]) \( \forall p, q, r \in I \):

\[
\mathcal{I}_T(p, q) = 1 \quad \text{iff} \quad p \leq q, \\
\mathcal{I}_T(1, q) = q, \\
pT \mathcal{I}_T(p, q) \leq q, \\
r \leq \mathcal{I}_T(p, q) \quad \text{iff} \quad pTr \leq q, \\
\mathcal{I}_T(rTp, rTq) \geq \mathcal{I}_T(p, q), \\
\mathcal{I}_T \left( \sup_j p_j, \sup_j q_j \right) \geq \inf_j \mathcal{I}_T(p_j, q_j),
\]

for all subfamilies \((p_j)_{j \in J}\) and \((q_j)_{j \in J}\) of \( I \) that have a common index set \( J \).

Proposition 7.1. Let \( T \) be a left continuous t-norm. Then \( \forall \eta, \zeta \in \Delta \), \( \mathcal{G}_T^T(\eta, \zeta) \in \Delta^+ \) is given by: \( \forall s > 0 \)

\[
\mathcal{G}_T^T(\eta, \zeta)(s) = \sup_{0 < i < s} \inf_{p \in R} \min \{ \mathcal{I}_T(\zeta(p), \eta(p + t) \hat{+} r), \mathcal{I}_T(\eta(p), \zeta(p + t) \hat{+} r) \}. 
\]

Proof. The right-hand side of (38) obviously defines a ddf. Denote this ddf by \( \rho \). Then on one hand, \( \forall s > 0 \)

\[
(\rho \oplus_T \zeta)(s) = \sup_{0 < b} \zeta(s - b)Tp(b) \\
\leq \sup_{0 < b} \zeta(s - b)T \left( \sup_{0 < t < b} \inf_{p \in R} \mathcal{I}_T(\zeta(p), \eta(p + t) \hat{+} r) \right) \\
= \sup_{0 < b} \sup_{0 < t < b} \zeta(s - b)T \left( \inf_{p \in R} \mathcal{I}_T(\zeta(p), \eta(p + t) \hat{+} r) \right) \\
\quad \text{(by the left continuity of } T) 
\]
\[ \sup \sup_{0 < b \leq 0 < t \leq b} \zeta(s - b) T I_T(\zeta(s - b), \eta(s - (b - t)) \hat{\oplus} r) \]
\[ \sup \sup_{0 < b \leq 0 < t \leq b} \eta(s - (b - t)) \hat{\oplus} r \quad \text{(by (34))} \]
\[ = \eta(s) \hat{\oplus} r \quad \text{(by the left continuity of } \eta). \]

This shows that \( \eta \hat{\oplus} r \ll \rho \hat{\oplus}_T \zeta \). Similarly, \( \zeta \hat{\oplus} r \ll \eta \hat{\oplus}_T \rho \). Hence by the definition of \( G_T(\eta, \zeta) \), we get
\[ G_T(\eta, \zeta) \ll \rho. \]

On the other hand, denoting \( G_T(\eta, \zeta) \) simply by \( \xi \), we get from Proposition 5.1(6),
\[ \eta \hat{\oplus} r \ll \xi \hat{\oplus}_T \xi \quad \text{and} \quad \zeta \hat{\oplus} r \ll \eta \hat{\oplus}_T \zeta. \]

Consequently, \( \forall p \in R \), \( \forall t \in R^+ \), we find that
\[ \eta(p + t) \hat{\oplus} r \geq (\xi \hat{\oplus}_T \zeta)(p + t) \geq \xi(p) T \zeta(t). \]

From this inequality and (35) we deduce that
\[ \zeta(t) \leq T(\xi(p), \eta(p + t) \hat{\oplus} r). \]

Similarly, \( \zeta(t) \leq T(\xi(p), \eta(p) \hat{\oplus} \xi)(p + t) \hat{\oplus} r). \)

These two inequalities yield, \( \forall s > 0 \)
\[ \zeta(s) = \sup_{0 < t < s} \zeta(t) \]
\[ \leq \sup_{0 < t < s} \inf_{p \in R} \{ T(\xi(p), \eta(p + t) \hat{\oplus} r), T(\xi(p), \xi(p + t) \hat{\oplus} r) \} = \rho(s). \]

This shows that \( G_T(\eta, \zeta) = \xi \gg \rho \), which completes the proof of (38).

We are now in a position to derive a direct computational formula for \( \mathcal{M}_T \).

**Theorem 7.1.** Let \( T \) be a left continuous \( t \)-norm. Then \( \forall \eta, \zeta \in \Delta \), \( \mathcal{M}_T(\eta, \zeta) \in \mathcal{D}[0, 1] \) is given by: \( \forall r > 0 \)
\[ \mathcal{M}_T(\eta, \zeta)(r) = \sup_{0 < q < r} \inf_{p \in R} \{ T(\xi(p), \eta(p + q) \hat{\oplus} q), T(\xi(p), \xi(p + q) \hat{\oplus} q) \}. \]  

**Proof.** By substituting for \( G_T(\eta, \zeta) \) from (38) into (28), we get
\[ \mathcal{M}_T(\eta, \zeta)(r) = \sup_{0 < q < r} G_T(\eta, \zeta)(q) \]
\[ = \sup_{0 < q < r} \sup_{0 < t < q} \inf_{p \in R} \{ T(\xi(p), \eta(p + t) \hat{\oplus} q), T(\xi(p), \xi(p + t) \hat{\oplus} q) \}. \]
Hence, on one hand,
\[
\mathcal{M}^T(\eta, \zeta)(r) \geq \sup_{0 < q < r} \sup_{0 < t < q} \inf_{p \in \mathbb{R}} \min \{ \mathcal{I}_T(\zeta(p), \eta(p + t) \hat{+} t), \mathcal{I}_T(\eta(p), \zeta(p + t) \hat{+} t) \},
\]
and on the other hand,
\[
\mathcal{M}^T(\eta, \zeta)(r) \leq \sup_{0 < q < r} \sup_{0 < t < q} \inf_{p \in \mathbb{R}} \min \{ \mathcal{I}_T(\zeta(p), \eta(p + q) \hat{+} q), \mathcal{I}_T(\eta(p), \zeta(p + q) \hat{+} q) \}.
\]

The two right-hand sides of the last two inequalities, above, obviously coincide with the right-hand side of (39). So, (39) follows. \( \Box \)

We end by listing some basic properties of \( \mathcal{M}^T \), in the next eight propositions.

**Proposition 7.2.** Let \( T \) be a left continuous t-norm. Then \( \forall a, b \in \mathbb{R}, \mathcal{M}^T(\varepsilon_a, \varepsilon_b) \in \mathcal{D}[0, 1] \) is given by: \( \forall r \geq 0 \)
\[
\mathcal{M}^T(\varepsilon_a, \varepsilon_b)(r) = \begin{cases} r, & r \leq \min \{1, |b - a|\}, \\ 1, & r > \min \{1, |b - a|\}. \end{cases}
\]
(40)

**Proof.** Direct computation in (39); using (32) and (33). \( \Box \)

**Proposition 7.3.** Let \( T \) be a left continuous t-norm. Then \( \forall \eta \in \mathcal{A}^+ \):
\[
\mathcal{M}^T(\eta, \varepsilon_0)(r) = \eta(r) \hat{+} r, \quad r > 0.
\]
(41)

Consequently,
\[
\mathcal{M}^T(\eta, \varepsilon_0)(0+) = \eta(0+).
\]
(42)

**Proof.** Direct computation in (39); using (32) and (33). \( \Box \)

**Proposition 7.4.** Let \( T \) be a left continuous t-norm, and let \( \iota: \mathbb{R} \rightarrow I \) be the function given by
\[
\iota(r) = \begin{cases} 0, & r \leq 0 \\ r, & 0 < r \leq 1 \\ 1, & 1 < r \end{cases}.
\]
Then \( \forall \eta \in \mathcal{A}^+, \forall \zeta \in \mathcal{A} \):
\[
\eta \oplus_T \mathcal{M}^T(\eta, \zeta) \geq \zeta \hat{+} \iota.
\]
(43)

**Proof.** For \( r > 0 \) we have from (3) and (39)
\[
(\eta \oplus_T \mathcal{M}^T(\eta, \zeta))(r) = \sup_{0 < t < r} \eta(t)T \mathcal{M}^T(\eta, \zeta)(r - t)
\leq \sup_{0 < t < r} \sup_{0 < q < r - t} \eta(t)T \mathcal{I}_T(\eta(t), \zeta(t + q) \hat{+} q)
\]

\[
\leq \sup_{0 < t < r} \sup_{0 < q < r - t} \zeta(t + q) \hat{+} q \quad \text{(by (34))}
\]
\[
= \zeta(r) \hat{+} r \quad \text{(by the left continuity of } \zeta) \]
\[
= (\zeta \hat{+} 1)(r),
\]
and for \( r \leq 0 \) we have
\[
(\eta \oplus_T \mathcal{M}^T(\eta, \zeta))(r) = 0 \leq (\zeta \hat{+} 1)(r).
\]
This proves (43). \( \square \)

**Proposition 7.5.** Let \( T \) be a copula, and let \( \iota \) be the ddf defined in the preceding proposition. Then
\[
\mathcal{M}^T \leq \mathcal{F}^T_{\hat{+} \iota}.
\]

**Proof.** For all \( t, s, q \in R^+ \) we have since \( T \) is a copula, \( tT(I_T(t,s) \hat{+} q) \leq tT(I_T(t,s) \hat{+} q) \leq s \hat{+} q \), by (34).
Consequently by (32),
\[
I_T(t,s) \hat{+} q \leq I_T(t,s \hat{+} q).
\]

For all \( \eta, \zeta \in \Delta \) and all \( r > 0 \), we now find by means of formula (20) for \( \mathcal{F}^T_{\hat{+} \iota} \) and formula (39) for \( \mathcal{M}^T \) that:
\[
(\mathcal{F}^T_{\hat{+} \iota} (\eta, \zeta))(r) = \mathcal{F}^T_{\hat{+} \iota} (\eta, \zeta)(r) \hat{+} r
\]
\[
= \sup_{q < r} \inf_{p \in R} \min \{ I_T(\zeta(p), \eta(p + q)), I_T(\eta(p), \zeta(p + q)) \} \hat{+} q \sup_{q < r} q
\]
\[
= \sup_{q < r} \inf_{p \in R} \min \{ I_T(\zeta(p), \eta(p + q)) \hat{+} q, I_T(\eta(p), \zeta(p + q) \hat{+} q) \}
\]
\[
\quad \text{(because all terms are monotone in } q \}
\]
\[
\leq \sup_{q < r} \inf_{p \in R} \min \{ I_T(\zeta(p), \eta(p + q) \hat{+} q), I_T(\eta(p), \zeta(p + q) \hat{+} q) \}
\]
\[
= \mathcal{M}^T(\eta, \zeta)(r).
\]
This proves (44). \( \square \)

It is easy to verify that the right-hand side of (44), as well as its left-hand side, defines a Menger \( T \)-metric whenever \( T \) is a copula. The two \( T \)-metrics \( \mathcal{F}^T_{\hat{+} \iota} \) and \( \mathcal{F}^T_{\hat{+} \iota} \) have the same strong uniformity. Therefore, on one hand, equality must fail in (44) when \( T \) is Min, because \( (\mathcal{D}[0,1], \mathcal{F}^\text{Min}) \) is not compact. On the other hand, the residuation implication of the copula \( W \) is the well known \textbf{Łukasiewicz implication}, given by
\[
I_W(t,s) = s \hat{+} (1 - t), \quad t, s \in I.
\]
This satisfies \( \forall t,s,q \in I: \mathcal{I}_W(t,s) \hat{\oplus} q = \mathcal{I}_W(t,s+q) \). In consequence, the proof of Proposition 7.5 clearly yields an identity in place of the inequality (44) when \( T = W \). We then conclude that all bounded closed intervals \((D[a,b], \mathcal{I}_W)\) are compact.

**Proposition 7.6.** Let \( T \) be a left continuous \( t \)-norm, let \((X, \mathcal{F})\) be a Menger \( T \)-pseudo-metric space, and let \( S \) be a nonempty subset of \( X \). Define a function \( F_S : X \to \Delta^+ \) by \( F_S(x) = \mathcal{F}(x,S) = \cap \{\mathcal{F}(x,z) | z \in S\}, x \in X \).

Then \( \forall x,y \in X \):

\[
\mathcal{M}^T(F_S(x), F_S(y)) \leq \mathcal{F}(x, y),
\]

and for all \( x, y, z, w \in X \):

\[
\mathcal{M}^T(\mathcal{F}(x,z), \mathcal{F}(y,w)) \leq \mathcal{F}(x,y) \oplus_T \mathcal{F}(z,w).
\]

**Proof.** These follow trivially from Lemma 4.2 and Lemma 4.1, respectively, because \( \mathcal{M}^T \leq \mathcal{F}_I^T \).

**Proposition 7.7.** Let \( T \) be a copula. Let \( \star \) be a right nonexpansive (see Section 4) and right monotonic binary operation on \( R \) that satisfies the following condition:

\[
\forall u, v \in R, \forall 1 \geq q > 0 \exists w \in R: u \star w = u \star v + q,
\]

and let \( \otimes_T \) be the sup-\( T \) extension of \( \star \) over \( I^R \). Then \( \otimes_T \) is right nonexpansive in \( \Delta \) with respect to the \( T \)-metric \( \mathcal{M}^T \); that is, for all \( \eta, \zeta, \xi \in \Delta \) we have

\[
\mathcal{M}^T(\xi \otimes_T \eta, \xi \otimes_T \zeta) \leq \mathcal{M}^T(\eta, \zeta).
\]

In particular,

\[
\mathcal{M}^T(\xi \oplus_T \eta, \xi \oplus_T \zeta) \leq \mathcal{M}^T(\eta, \zeta).
\]

Similar conclusion holds for continuous, left nonexpansive, left monotonic binary operations.

**Proof.** For all \( p \in R \) and all \( 1 \geq q > 0 \) we have

\[
\inf_{p \in R} \mathcal{I}_T((\xi \otimes_T \zeta)(p), (\xi \otimes_T \eta)(p + q)) \\
= \inf_{p \in R} \mathcal{I}_T(\sup_{v \in R} \{\zeta(v) | v \in R, u \star v = p\} | u \in R),
\]

\[
\sup\{(\zeta(u)T \sup_{w \in R} \{\eta(w) | w \in R, u \star w = p + q\}) \hat{+} q | u \in R\})
\geq \inf_{p \in R} \inf_{u \in R} \mathcal{I}_T(\zeta(u)T \sup_{v \in R} \{\zeta(v) | v \in R, u \star v = p\},
\]

\[
(\zeta(u)T \sup_{w \in R} \{\eta(w) | w \in R, u \star w = p + q\}) \hat{+} q)
\]

(by (37))

\[
\geq \inf_{p \in R} \inf_{u \in R} \mathcal{I}_T(\zeta(u)T \sup_{v \in R} \{\zeta(v) | v \in R, u \star v = p\},
\]
\[ \zeta(u)T(\sup\{\eta(w) \mid w \in R, u \star w = p + q\}) \]

(because $T$ is a copula)

\[ \geq \inf_{p \in R} \inf_{u \in R} I_T(\sup\{\zeta(v) \mid v \in R, u \star v = p\}, \sup\{\eta(w) \mid w \in R, u \star w = p + q\}) \]

(by (36))

\[ = \inf_{u \in R} I_T(\sup\{\zeta(v) \mid v \in R\}, \sup\{\sup\{\eta(w) \mid w \in R, u \star w = u \star v + q\} \mid v \in R\}) \]

\[ \geq \inf_{u \in R} \inf_{v \in R} I_T(\zeta(v), \sup\{\eta(w) \mid w \in R, u \star w = u \star v + q\}) \]

(by (37) again).

Condition (47) on $\star$ ensures that the last supremum above is not an empty supremum. We now use the nonexpansion and monotonicity of $\star$ in the right argument. These two properties yield the following entailment:

\[ (u \star w = u \star v + q) \Rightarrow (w \geq v + q). \]

Therefore,

\[ \inf_{p \in R} I_T((\zeta \oplus_T \zeta)(p), (\zeta \oplus_T \eta)(p + q) + q) \]

\[ \geq \inf_{v \in R} I_T(\zeta(v), \eta(v + q) + q) \]

Consequently, for every $1 \geq r > 0$ we get

\[ M_T(\zeta \oplus_T \eta, \zeta \oplus_T \zeta)(r) \]

\[ = \sup_{0 < q < r} \inf_{p \in R} \min\{I_T((\zeta \oplus_T \zeta)(p), (\zeta \oplus_T \eta)(p + q) + q), I_T((\zeta \oplus_T \eta)(p), (\zeta \oplus_T \zeta)(p + q) + q)\} \]

\[ \geq \sup_{0 < q < r} \inf_{v \in R} \min\{I_T(\zeta(v), \eta(v + q) + q), I_T(\eta(v), \zeta(v + q) + q)\} \]

\[ = M_T(\eta, \zeta)(r). \]

This establishes (48). \[ \Box \]

If, in the preceding proposition, $\Delta$ is narrowed down to some subset of distribution functions, then in condition (47) we can replace $R$ by appropriate subsets of real numbers. Ordinary multiplication
on $R$ satisfies the properties listed in the preceding proposition for ★, provided that the left multiple is in $(0, 1]$. By retracing its proof, we obtain

**Proposition 7.8.** Let $T$ be a copula, and let $\odot_T$ be the sup-$T$ extension over $I^R$ of the multiplication on $R$. Then for all $\eta, \zeta \in \Lambda$ and all $\xi \in \mathcal{D}[0, 1]$ we have

$$ \mathcal{M}^T(\xi \odot_T \eta, \xi \odot_T \zeta) \leq \mathcal{M}^T(\eta, \zeta). \quad (50) $$

**Proposition 7.9.** Let $T$ be a left continuous t-norm, and $b \in R^+$. Then $\forall \eta, \zeta \in \Lambda^+$:

If $1 < b$, then $\mathcal{M}^T(\xi_b \odot_T \eta, \xi_b \odot_T \zeta) \leq \xi_b \odot_T \mathcal{M}^T(\eta, \zeta)$. \hspace{1cm} (51)

If $0 < b \leq 1$, then $\xi_b \odot_T \mathcal{M}^T(\eta, \zeta) \leq \mathcal{M}^T(\xi_b \odot_T \eta, \xi_b \odot_T \zeta) \leq \mathcal{M}^T(\eta, \zeta)$. \hspace{1cm} (52)

**Proof.** It is direct to verify that $\forall \xi \in \Lambda^+$, $\forall t, b > 0 : (\xi_b \odot_T \xi)(t) = \xi(\frac{t}{b})$.

From this and (39), we get $\forall r > 0$:

$$ \mathcal{M}^T(\xi_b \odot_T \eta, \xi_b \odot_T \zeta)(r) $$

$$ = \sup_{0 < q < r} \inf_{p \in R} \min \left\{ \mathcal{I}_T \left( \zeta \left( \frac{p}{b} \right), \eta \left( \frac{p}{b} + \frac{q}{b} \right) \right), \mathcal{I}_T \left( \eta \left( \frac{p}{b} \right), \zeta \left( \frac{p}{b} + \frac{q}{b} \right) \right) \right\} $$

$$ = \sup_{0 < q < r} \inf_{u \in R} \min \left\{ \mathcal{I}_T \left( \zeta(u), \eta \left( u + \frac{q}{b} \right) \right), \mathcal{I}_T(\eta(u), \zeta \left( u + \frac{q}{b} \right) \right) \right\}. $$

When $1 < b$, $q > q/b$, and so we get by writing $s = q/b$:

$$ \mathcal{M}^T(\xi_b \odot_T \eta, \xi_b \odot_T \zeta)(r) $$

$$ \geq \sup_{0 < s < r/b} \inf_{u \in R} \min \left\{ \mathcal{I}_T(\zeta(u), \eta(u + s) \hat{+} s), \mathcal{I}_T(\eta(u), \zeta(u + s) \hat{+} s) \right\} $$

$$ = \mathcal{M}^T(\eta, \zeta)(r) = (\xi_b \odot_T \mathcal{M}^T(\eta, \zeta))(r), $$

which yields (51).

When $0 < b \leq 1$, $q \leq q/b$, and so the opposite of the above inequality holds, and yields $\xi_b \odot_T \mathcal{M}^T(\eta, \zeta) \leq \mathcal{M}^T(\xi_b \odot_T \eta, \xi_b \odot_T \zeta)$. We also obtain in this case

$$ \mathcal{M}^T(\xi_b \odot_T \eta, \xi_b \odot_T \zeta)(r) $$

$$ \geq \sup_{0 < q < r} \inf_{u \in R} \min \left\{ \mathcal{I}_T(\zeta(u), \eta(u + q) \hat{+} q), \mathcal{I}_T(\eta(u), \zeta(u + q) \hat{+} q) \right\} $$

$$ = \mathcal{M}^T(\eta, \zeta)(r); $$

that is, $\mathcal{M}^T(\xi_b \odot_T \eta, \xi_b \odot_T \zeta) \leq \mathcal{M}^T(\eta, \zeta)$, and the proof of (52) is complete. \hfill \Box

We leave to the reader the verification that the sequence $(\eta_n)$, described at the end of Section 4, converges in $(\mathcal{D}[0, 1], \mathcal{M}^T)$ to $\xi_1$, for any copula $T$. 
8. Conclusion

The outcome of this work is the family of Menger $T$-metrics $\mathcal{M}^T$, one for each copula $T$, on the set $\Lambda$ of distribution functions. On one hand, the strong uniformity of $\mathcal{M}^T$ is compact on every bounded closed interval $\mathcal{D}[a,b]$ in $\Lambda$, because it coincides there with the uniformity of the modified Lévy metric $d_L$. On the other hand, $\mathcal{M}^T$ interacts well with the two binary operations $\oplus_T$ and $\odot_T$ on $\Lambda$. As $\mathcal{M}^T$ is given two alternative, but equivalent, descriptions (28) and (39), it may prove to be amenable to practical computations.

For a continuous $t$-norm $T$ which is not a copula, readers can easily verify that while $\mathcal{M}^T$ need not satisfy the $T$-triangle inequality ($T$-PM3), the nested family $\{\mathcal{U}(r) = \{(\eta, \zeta) \mid \mathcal{M}^T(\eta, \zeta)(r) > 1 - r\} : 0 < r < 1\}$ remains a basis for the compact uniformity of the metric $d_L$, on each $\mathcal{D}[a,b]$ in $\Lambda$ (the proof of Theorem 6.2 remains valid).

References


