Hamming Approximation of NP Witnesses

Daniel Sheldon∗    Neal E. Young†

August 2, 2012

Abstract

Given an instance of an NP-hard problem, how hard is it to compute a (possibly infeasible) solution \(x\), such that \(x\) is guaranteed to agree with some feasible solution \(x^*\) in at least half its bits? Such questions about “structural” approximability are motivated by applications such as Computer Tomography, in which one wants to reconstruct as much of the full structure of the solution as possible. In this spirit, Feige et al. [1] (following Kumar and Sivakumar [4]) show that, for some \(\epsilon > 0\), given an instance \(\Psi\) of 3-SAT, it is NP-hard to compute an assignment \(x\) that agrees with any satisfying assignment \(x^*\) of \(\Psi\) in at least \(n/2 + n^{1-\epsilon}\) of the \(n\) variables. They show similar negative results for other natural NP-complete problems. Guruswami and Rudra [2] strengthen their bounds to \(n/2 + n^{2/3+\epsilon}\) (for all fixed \(\epsilon > 0\)).

The main result in this paper is as follows. For the “universal” NP-complete language \(U\), for any positive \(\epsilon\), it is NP-hard to compute an \(x\) that agrees with a witness \(x^*\) in at least \(n/2 - \epsilon \sqrt{n \log n}\) bits. In contrast to previous results, this is less than half the bits. This result extends to randomized algorithms, for which it is essentially tight.

We also give improved negative results for several natural NP-complete problems, as well as the first positive (algorithmic) results for Vertex Cover, Independent Set, Clique, and \(U\).

1 Introduction

Consider the discrete tomography problem. An instance is specified by numerous two-dimensional x-rays, formed by x-raying a three-dimensional object along various directions. A solution is a discrete function describing the internal structure of the object (specifying the density of matter at each point in the object). Given sufficiently many x-ray images, the internal structure may be determined uniquely, yet computing it exactly is (in general) NP-complete. What form of approximate solution is appropriate in this context? One kind of approximate solution, typical in the study of approximation algorithms, would be an object that, if subject to x-rays from the same perspectives, would yield approximately the same x-ray images (in other words, a solution to the underlying constrained optimization problem that meets the constraints approximately). But such a solution can have a completely different internal structure than any true solution. In this context, where the goal is to discover as much about the internal structure of the object as possible, this form of approximation is unsuitable.

The Hamming distance to a true solution would be a more appropriate metric for approximation. For example, if the discrete tomography problem models an object with just two possible local

∗Department of Computer Science, University of Massachusetts Amherst.
†Department of Computer Science and Engineering, University of California Riverside.
densities \{0,1\}, the problem can be modeled as a system of linear equations over a vector \(x\) of variables taking values in \{0,1\}. A solution is specified by an assignment of values to the variables. A “good” approximate solution \(x\) would be one that agrees with some true solution \(x^*\) in many variables—that is, an \(x\) with small Hamming distance to the set of true solutions. This paper is about the computational complexity of computing such an \(x\), for various NP-complete problems.

**Definitions.** Here we follow [1, 4], with minor variations. Every language \(L\) in NP is characterized by a witness relation \(R_{L}\), such that \(L = \{x : (x, w) \in R_{L} \text{ for some } w\}\) and \((x, w) \in R_{L}\) is decidable in time polynomial in \(|x|\). We call \(w\) such that \((x, w) \in R_{L}\) a witness for \(x \in L\).

An algorithm \(A\) achieves Hamming distance \(h(n)\) (with respect to \(L\) and \(R_{L}\)) if, on any input \((x, n)\) where \(x \in L\) has some witness of size \(n\), the algorithm outputs a string \(A(x)\) of size \(n\) having Hamming distance at most \(h(n)\) to some such witness. If \(A\) is a randomized algorithm, we say that \(A\) achieves Hamming distance \(f(n)\) with probability \(p(n)\) if the algorithm outputs such a string with probability at least \(p(n)\).

Note that small Hamming distance is synonymous with large agreement: \(x\) achieves Hamming distance \(h(n)\) from \(x^*\) iff it agrees with \(x^*\) on at least \(n - h(n)\) bits.

We use \(U\) and \(R_{U}\) to denote a universal NP-complete language and its witness relation, respectively. Specifically, \(R_{U}\) contains encoded tuples \((M, x, b, w)\) such that (1) \(M\) is (the encoding of) a deterministic Turing machine, \(x\), \(b\), and \(w\) are binary strings, and (2) \(M\) accepts input \((x, w)\) within \(|b|^2\) steps (\(b\) is “padding”). Then \(U\) contains encoded tuples \((M, x, b)\) such that \((M, x, b, w) \in R_{U}\) for some witness \(w\).

Here are some basic facts about \(U\).

**Observation 1** \(U\) is NP-complete.

The proof is standard, and we omit it.

Also, \(U\) is as hard to Hamming-approximate as any other NP language:

**Observation 2** Suppose, for \(U\) and \(R_{U}\), that there is a polynomial-time algorithm \(A_{U}\) that achieves Hamming distance \(h(n)\) with probability \(p(n)\).

Let \(L\) be any language in NP, and let \(R_{L}\) be any polynomial-time witness relation for \(L\).

Then (for \(L\) and \(R_{L}\)) there is a polynomial-time algorithm \(A_{L}\) that achieves Hamming distance \(h(n)\) with probability \(p(n)\).

The proof, which is straightforward, is in the Appendix.

**Previous results.** Hardness of Hamming approximation was previously studied by Feige et al. [1] and Guruswami and Rudra [2]. Feige et al. show, for many natural NP-complete problems, that achieving Hamming distance much less than \(n/2\) (a natural threshold for binary encodings) is NP-hard. Specifically, Feige et al. show (for some \(\delta < 1\), for many standard NP-complete problems, including SAT)) that it is NP-hard to achieve Hamming distance less than \(n/2 - n^\delta\). They extend the result to randomized algorithms: no randomized polynomial-time algorithm achieves Hamming distance less than \(n/2 - n^\delta\) with probability at least \(1/n^c\) (for any fixed \(c\)) unless \(\text{RP} = \text{NP}\). A motivating application was to give evidence that a SAT algorithm by Schoning could not be sped up in a particular way, but Feige et al. were also motivated by related work by Kumar and Ravikumar [4] concerning error-correcting codes.
Guruswami and Rudra [2] use error-correcting codes to strengthen the hardness results of Feige et al. Specifically, they show that, for every NP problem, and every \( \epsilon > 0 \), there is a formulation of the problem for which it is NP-hard to achieve Hamming distance less than \( n/2 - n^{2/3+\epsilon} \), reducing the exponent in the subtracted term from just less than 1 down to about 2/3.

**New results.** We start with some fairly straightforward observations.

- The lower bounds of Feige et al., and of Guruswami and Rudra, can be strengthened, reducing the exponent in the subtracted term to any fixed \( \epsilon > 0 \), for most of the NP-complete problems studied by Feige et al.: no deterministic polynomial-time algorithm achieves Hamming distance less than \( n/2 - n^{\epsilon} \) unless P=NP.

- Likewise, for any positive \( \epsilon \) and \( c \), no randomized polynomial-time algorithm achieves Hamming distance less than \( n/2 - n^{\epsilon} \) with probability \( 1/2 + 1/n^c \) unless RP=NP.

A main point of the above observations is that they don’t rely on error-correcting codes; trivial amplification arguments (closely related to the arguments by Feige et al.) suffice to prove them.

For Vertex Cover, Independent Set, and Clique, we observe the following positive results:

- For unweighted Vertex Cover, unweighted Independent Set, and unweighted Clique, there are polynomial-time algorithms that achieve Hamming distance \( n/2 \).

These observations follow easily from a known combinatorial property of Vertex Cover. These are the first non-trivial positive results that we are aware of.

The threshold of \( n/2 \) is a natural reference point for Hamming approximation, given binary encodings. For example, a random string \( x \) achieves Hamming distance \( n/2 \) from any \( x^* \) in expectation (and close to \( n/2 \) with high probability). Also, for any string \( x \), it is guaranteed that either \( x \) or its complement will have Hamming distance at most \( n/2 \) from any \( x^* \). Given these intuitions, and the hardness results and the algorithmic results above, one might expect a-priori that, for all NP problems, an \( x \) with Hamming distance \( n/2 \) or less from some solution \( x^* \) should be computable in polynomial time. Our main contribution, next, is to prove that this intuition is false:

- For \( U \), for any \( \epsilon > 0 \), no polynomial time algorithm achieves Hamming distance less than \( n/2 + \sqrt{\epsilon n \ln n} \) unless P=NP. (That is, one can’t even get \( n/2 - \sqrt{\epsilon n \ln n} \) of the bits right.)

- Further, no randomized polynomial-time algorithm achieves Hamming distance less than \( n/2 + \sqrt{\epsilon n \ln n} \) with probability at least \( 1 - O((n^{2\epsilon+1}\sqrt{\epsilon \ln n})^{-1}) \) unless RP=NP.

Thus, for \( U \), in contrast to Vertex Cover, Independent Set, and Clique, one cannot achieve Hamming distance \( n/2 \) in polynomial time (unless P=NP).

The intuition behind the proof of the main result is as follows. Given a particular \( x \), of the \( 2^n \) potential witnesses (potential assignments to \( x^* \)), the fraction that lie within Hamming distance \( n/2 + \sqrt{\epsilon n \ln n} \) of \( x \) is \( 1 - n^{-c} \) (for some constant \( c \) depending on \( \epsilon \)). Thus, assuming there is a polynomial-time algorithm to find such an \( x \), with a single call to that algorithm, the number of potential witnesses can be reduced by a factor of \( 1 - n^{-c} \). By iterating, say, \( n^{c+1} \) times (and carefully recoding the set of potential witnesses each time), the number of potential witnesses can be reduced from \( 2^n \) to \( 2^n (1 - n^{-c})^{n^{c+1}} \), which is less than \( \exp(n - n^{-c}n^{c+1}) = O(1) \). Then, each of the \( O(1) \) remaining potential witnesses can be tested using the polynomial-time verifier for \( U \).
For comparison, here are some complementary (and easy) positive results. For any NP language, a trivial deterministic polynomial-time algorithm achieves Hamming distance at most \( n - c \) (for any fixed \( c > 0 \)). (This is a weak upper bound, but it is the best possible for any so-called black-box algorithm.)

Likewise, the naive randomized algorithm (guess a random string) achieves Hamming distance at most \( n/2 + \sqrt{en \ln n} \) with probability \( 1 - O((n^2 \sqrt{\ln n})^{-1}) \). This shows that the randomized hardness result for \( \mathcal{U} \) is essentially tight.

## 2 Improving the hardness results of Feige et al.

This section describes how to improve many of the hardness results of Feige et al., to show, for several of the NP-complete problems they consider, that (for any \( \epsilon > 0 \)) achieving Hamming distance at most \( n/2 - n^\epsilon \) in polynomial time is impossible unless \( P=NP \). The proofs are elementary padding arguments, similar in spirit to the arguments of Feige et al.

We will use the following standard observation:

**Observation 3**

(a) If the following problem has a polynomial-time algorithm, then \( P=NP \): Given a 3-SAT formula, find a feasible value for the first variable in the formula, if one exists.

(b) If there is an randomized polynomial-time algorithm that solves any \( n \)-variable instance of the above problem with probability \( 1/2 + 1/n^c \) for any fixed \( c > 0 \), then \( RP=NP \).

If the formula is satisfiable, a feasible value for the variable is one that is consistent with some satisfying assignment. If the formula is not satisfiable, any value can be found.

Part (a) holds by standard arguments. To see why part (b) is true, note that the randomized algorithm could be used to find a satisfying assignment of a given formula with high probability: to determine the likely value of the first variable, run the randomized algorithm, say, \( n^{c+2} \) times, then take the majority value — this standard amplification trick boosts the probability of finding a feasible value to at least \( 1 - 1/n^2 \). Then substitute the likely feasible value for the variable, simplify the formula, then recurse. This would find a full satisfying assignment with probability at least \( 1 - O(1/n) \) in polynomial time, showing that \( RP=NP \).

We start by showing the hardness of Hamming-approximation for 3-SAT:

**Observation 4** Suppose that, for 3-SAT (with the natural witness relation), there exists \( \epsilon > 0 \) such that some polynomial-time algorithm \( A \) achieves Hamming distance \( n/2 - n^\epsilon \). Then \( P=NP \).

**Proof:** The proof is an elementary amplification argument.

Assume the algorithm \( A \) in the statement of Observation 4 exists. Given any 3-SAT formula \( \Psi \), we compute, in polynomial time, a feasible value for the first variable \( V \) as follows.

To \( \Psi \), add \( \lceil n^{1/c} \rceil \) copies of \( V \). Specifically, add new clauses \((V^1 = V) \land (V^2 = V) \land \cdots \land (V^{1/c} = V)\) (where \( a = b \) is shorthand for \((a \lor \overline{b}) \land (a \lor \overline{b})\), and \( V^1, V^2, \ldots, V^{1/c} \) are new variables). This gives a formula \( \Psi' \) with \( n' = n + n^{1/c} \) variables, essentially preserving any satisfying assignments, but forcing the \( n^{1/c} \) added variables to take the same value as \( V \) in any satisfying assignment.

Run \( A \) on \( \Psi' \), and let \( x' \) be the returned value. If \( \Psi' \) is satisfiable, then \( x' \) achieves Hamming distance at most \( n'/2 - (n')^\epsilon \) to an assignment \( x^* \) satisfying \( \Psi' \). That is, \( x' \) agrees with \( x^* \) on at least \( n'/2 + (n')^\epsilon > n^{1/c}/2 + n \) variables. To do so, even if \( x \) agrees with \( x^* \) on all \( n \) of the original variables, it would still have to agree with \( x^* \) on at least half \( (n^{1/c}) \) of the duplicates of \( V \). Thus, the majority value of the duplicate variables in \( x' \) (true or false, whichever \( x' \) assigns to
more duplicates) must be the value that $x^*$ assigns to $V$. This value must also be a feasible value for $V$ in $\Psi$ (if one exists).

Thus, if $A$ exists, then one can compute a feasible value for $V$ in $\Psi$ (if one exists) in polynomial time. By Observation 3 then, $P=NP$. □

**Observation 5** Suppose that, for SAT (with the natural witness relation), there exists $\epsilon > 0$ such that some randomized polynomial-time algorithm $A$ achieves Hamming distance $n/2 - n^\epsilon$ with probability $1/2 + 1/n^\epsilon$ for any fixed $c > 0$. Then $RP=NP$.

**Proof:** Assume the algorithm $A$ exists, and use it just did the preceding proof. Given the formula $\Phi$ with $n$ variables, call $A$ on the formula $\Phi'$ with $O(n')$ variables, where $n' = n^{1/\epsilon}$ ($n$ being the number of variables in $\Phi$). By the properties of $A$ assumed in the observation, the probability that the call succeeds in finding a feasible value for $V$ (if one exists) is $1/2 + 1/(n')^\epsilon = 1/2 + 1/n^{c/\epsilon}$. Thus, by Observation 3(b), RP would equal NP. □

Next we sketch how the same idea applies to other problems.

**Observation 6** Suppose that, for Vertex Cover, Independent Set, or Clique, there exists $\epsilon > 0$ such that some polynomial-time algorithm $A$ achieves Hamming distance $n/2 - n^\epsilon$. Then $P=NP$.

**Proof:** We sketch the proof for Vertex Cover. The rest follow via the standard reductions from Vertex Cover to Independent Set and from Independent Set to Clique, as these reductions preserve Hamming approximation.

Suppose such an algorithm $A$ exists. Given a graph $G = (V,E)$ with $n$ vertices, a non-isolated vertex $w \in V$, and the minimum size, $k$, of any vertex cover in $G$, we will use $A$ to determine in polynomial time either (i) that $G$ has a size-$k$ vertex cover containing $w$, or (ii) that $G$ has a size-$k$ vertex cover not containing $w$. By standard arguments, if this can be done in polynomial time, then $P=NP$.

Determine (i) or (ii) as follows. Construct graph $G'$ from $G$ by adding a copy $w'$ of $w$ (with edges to all neighbors of $w$), and a path $P$ of new vertices connecting $w$ to $w'$, so that $|P|$ (the number of edges in $P$) is even and roughly equals $n^{1/\epsilon}$. Let $n' = n + |P|$ be the number of vertices in $G'$, and let $k' = k + |P|/2$.

Denote the successive vertices on path $P$ as $w = v_0, v_1, v_2, \ldots, v_{|P|} = w'$. Let $P_0$ contain the “even” vertices $v_2, \ldots, v_{|P|} = w'$ (not including $w$) Let $P_1$ contain the “odd” vertices $v_1, v_3, \ldots, v_{|P|} - 1$.

Run $A$ on the instance $\langle G', k' \rangle$ and let $x'$ be the output. Define the Hamming distance of $x'$ to $P_i$ ($i \in \{0, 1\}$) to be the number of vertices $v \in P$ such that $(x_v = 1) \neq (v \in P_i)$. Return “(i) $G$ has a size-$k$ vertex cover containing $w$” if the Hamming distance from $x'$ to $P_0$ is less than the Hamming distance from $x'$ to $P_1$. Otherwise, return “(ii) $G$ has a size-$k$ vertex cover not containing $w$.”

To finish the proof we sketch why this procedure is correct. Let $C'$ be any minimum-size vertex cover of $G'$. By standard arguments, $C'$ has size $k'$ and one of two cases holds:

**Case (1)** $C' = C \cup P_0$ where $C$ is a size-$k$ vertex cover of $G$ and $w \in C$, or

**Case (2)** $C' = C \cup P_1$ where $C$ is a size-$k$ vertex cover of $G$ and $w \not\in C$. 

5
By assumption, $x'$ has Hamming distance at most $n'/2 - (n')^{1/\epsilon}$ to the witness $x^*$ for some such $C'$. That is, $x'$ agrees with $x^*$ on at least $n'/2 + (n')^{1/\epsilon} > n^{1/\epsilon}/2 + n$ vertices. Thus, focusing just on vertices in $P$, $x'$ agrees with $x^*$ on more than half of the vertices in $P$.

If Case (1) above occurs (for the $C'$ that $x^*$ represents), then the Hamming distance from $x'$ to $P_0$ must be less than $|P|/2$, so the Hamming distance from $x'$ to $P_1$ must be more than $|P|/2$, so the algorithm returns “(i) $G$ has a size-$k$ vertex cover containing $w$”. This is correct, given that Case (1) occurs.

By a similar argument, if Case (2) occurs, then the algorithm returns “(ii) $G$ has a size-$k$ vertex cover not containing $w$”, which is correct in this case.

Just as Observation 6 extended to randomized algorithms, the above argument extends to prove the following observation:

\textbf{Observation 7} Suppose that, for Vertex Cover, Independent Set, or Clique, there exists positive $\epsilon$ and $c$ such that some randomized polynomial-time algorithm $A$ achieves Hamming distance $n/2 - n^\epsilon$ with probability $1/2 + 1/n^c$. Then $RP=NP$.

\textbf{Observation 8} Suppose that, for Directed Hamiltonian Cycle (with the witness being a subset of the edges that forms the Hamiltonian cycle) there exists $\epsilon > 0$ such that some polynomial-time algorithm $A$ achieves Hamming distance $n/2 - n^\epsilon$. Then $P=NP$.

\textbf{Proof:} By definition, any polynomial-time reduction from 3-SAT to Directed Hamiltonian Cycle works as follows. Given a 3-SAT formula $\Phi$, the reduction produces, in polynomial time, a directed graph $G = (V, E)$ such that $G$ has a Hamiltonian cycle if and only if $\Phi$ is satisfiable; further, given any Hamiltonian cycle $C$ in $G$, the reduction describes how to compute an assignment $A(C)$ satisfying $G = (V, E)$ in polynomial time.

There exist well known reductions (e.g., see [6]) such that have $\Phi$, $G$, and $A(\cdot)$ have the following further properties. For any variable $V$ in $\Phi$, there are a pair of edges $(u, v)$ and $(v, u)$ such that, for any Hamiltonian cycle $C$, either $C$ contains $(u, v)$ and $A(C)$ assigns $V = true$, or $C$ contains $(v, u)$ and $A(C)$ assigns $V = false$.

Assume the algorithm $A$ from the observation exists. We describe below how to modify any reduction with the above properties so as to solve the following problem in polynomial time: given a 3-SAT formula $\Phi$, determine a feasible value for the first variable in $\Phi$ (if any exists). As in the proof of Observation 6 this is enough to prove $P=NP$.

Given $\Phi$, apply the reduction with the above properties to compute the graph $G = (V, E)$. Then, for the first variable, say, $Z$ in $\Phi$, let $(u, v)$ and $(v, u)$ be the two edges in $G$ for $Z$ as described above. Replace the edges $(u, v)$ and $(v, u)$, respectively, with paths $P_0 = (u = w_0, w_1, w_2, \ldots, w_k, w_{k+1} = v)$ and $P_1 = (v, w_k, w_{k-1}, \ldots, w_1, u)$, where $w_1, w_2, \ldots, w_k$ are new vertices and $k = n^{1/\epsilon}/2$, where $n = |E|$ is the number of edges in $G$. Say that each edge $(w_i, w_{i+1})$ is a duplicate of $(u, v)$, and that each edge $(w_{i+1}, w_i)$ is a duplicate of $(v, u)$. Let $P = P_0 \cup P_1$ be the set of all duplicate edges. Call the resulting graph $G'$, and let $n' = n + n^{1/\epsilon}$ be the number of edges in $G'$.

Run the algorithm $A$ on $G'$, and let $x'$ be the output. Define the Hamming distance from $x'$ to $P_i$ (for $i = 0, 1$) to be the number of edges $e$ in $P$ such that $(x'_e = 1) \neq (e \in P_i)$. If the Hamming distance from $x'$ to $P_0$ is less than the Hamming distance from $x'$ to $P_1$, then return “(i) The value True is feasible for the variable $Z$” and otherwise return “(ii) The value False is feasible for the variable $Z$”.

6
To finish, we prove that this procedure determines a feasible value for $Z$, if there is one.

Assume that there is a feasible value for $Z$ (that is, that $\Psi$ is satisfiable). The output $x'$ of $A$ then has Hamming distance at most $n'/2 - (n')^\epsilon$ to some witness $x^*$ for some Hamiltonian cycle $C'$ in $G'$ (where $x'_e = 1$ iff $e \in C'$). That is, $x'$ agrees with some $C'$ on at least $n'/2 + (n')^\epsilon > n^{1/\epsilon}/2 + n$ edges. Thus, $x'$ agrees with $C'$ on strictly more than $n^{1/\epsilon}/2$ (half) of the edges in $P$. By the properties of the reduction, one of two cases holds:

**Case 1.** $C' \cap P = P_0$, and True is a feasible value for $Z$. In this case, the Hamming distance from $x'$ to $P_0$ must be less than $|P|/2$, so the Hamming distance from $x'$ to $P_1$ must be more than $|P|/2$, so the procedure returns “(i) The value True is feasible for the variable $Z$”, which is correct.

**Case 2.** $C' \cap P = P_1$, and False is a feasible value for $Z$. By similar reasoning, the procedure is correct in this case as well.

\[ \square \]

As do the previous observations, Observation 8 extends to randomized algorithms:

**Observation 9** Suppose that, for Directed Hamiltonian Cycle, there exist positive $\epsilon$ and $c$ such that some randomized polynomial-time algorithm $A$ achieves Hamming distance $n/2 - n^\epsilon$ with probability $1/2 + 1/n^c$. Then $RP=NP$.

## 3 Algorithms for Vertex Cover and related problems

This section presents the positive results for Vertex Cover and related problems.

**Observation 10** There are polynomial-time algorithms achieving Hamming distance $n/2$ for the (unweighted) Vertex Cover, Independent Set, and Clique problems.

The algorithms for Independent Set and Clique work by standard reductions to Vertex Cover. The algorithm for Vertex Cover is based on a classic result of Nemhauser and Trotter.

**Theorem 1** (\cite{5, 3}) Fix any instance $I$ of Weighted Vertex Cover. Let $y$ be any (minimum cost) basic feasible solution to the linear program relaxation of the standard integer linear program for $I$. Then, for each vertex $v$, the variable $y_v$ has value in $\{0, 1/2, 1\}$, and there exists a minimum-weight vertex cover $C^*$ that has the following property. For each vertex $v$, if $y_v = 0$, then $v \not\in C^*$, while if $y_v = 1$, then $v \in C^*$.

The algorithm that achieves Hamming distance $n/2$ for unweighted Vertex Cover is as follows: Given the instance (a graph $G = (V, E)$ and an integer $k$) compute the minimum-cost basic feasible solution $y$ referred to in the theorem, then take the Hamming approximation $C$ to be the set of vertices $v$ such that $y_v > 0$.

Since the basic feasible solution $y$ can be computed in polynomial time, the algorithm clearly runs in polynomial time. Next we argue that this $C$ has Hamming distance at most $n/2$ from the minimum-size vertex cover $C^*$ referred to in the theorem. (Note that, as long as the instance $(G, k)$ is feasible, this $C^*$ will be a witness.)

Since there is a feasible solution of cost $|C^*|$ to the linear program, while $y$ is a minimum-cost solution, it follows that $|C^*| \geq \sum_i y_i$. The choice of $C$ implies $\sum_i y_i \geq |C|/2$. Thus, $|C^*| \geq |C|/2$. 

7
The choice of $C$ also implies that $C^* \subseteq C$. This and $|C^*| \geq |C|/2$ imply that $C$ is within Hamming distance $|C|/2$ (which is at most $n/2$) from $C^*$.

This proves the correctness of the algorithm for Vertex Cover.

The algorithm for Independent Set is as follows: *Given the instance* (a graph $G = (V, E)$ and an integer $k$), run the above algorithm for Vertex Cover to compute a Hamming approximation $C$ for the Vertex Cover instance $(G, n - k)$. Return the complement of $C$, i.e., $\overline{C} = V - C$.

To see why the algorithm is correct, recall that a vertex set $S \subseteq V$ is an independent set in a graph $G = (V, E)$ if and only if its complement $\overline{S} = V - S$ is a vertex cover of $G$. Thus, $\overline{C}$ has Hamming distance $n/2$ from some independent set of size $k$ if and only if $C$ has Hamming distance $n/2$ from some vertex cover of size $n - k$.

The algorithm for Clique is as follows: *Given the instance* (a graph $G = (V, E)$ and an integer $k$), run the above algorithm for Independent Set to compute a Hamming approximation $P$ for the Independent Set instance $(\overline{G}, k)$, where $\overline{G} = (E, V)$ is the graph whose edge set is the complement of $E$. Return $C$.

The algorithm is correct simply because a vertex set $S$ is a clique in $G$ if and only if $S$ is an independent set in $\overline{G}$.

This completes the proof of Observation 10.

4 Hardness of the universal language

This section presents the main result: the hardness of achieving Hamming distance less than $n/2 + \sqrt{en\log n}$ for $U$. First we introduce the utility functions $H$ and $P$ and state a relation between them. Define $H(n, \epsilon) = \sqrt{en\ln n}$. Define $P(n, \epsilon) = (n^\epsilon \sqrt{\epsilon \ln n})^{-1}$.

**Observation 11** Fix any $\epsilon \in (0, 1/2)$. The probability that a random $n$-bit string has more than $n/2 + H(n, \epsilon)$ ones is $\Theta(P(n, \epsilon))$. The probability that a random $(n - 1)$-bit string has more than $n/2 + H(n, \epsilon)$ ones is also $\Theta(P(n, \epsilon))$.

This proof, which is standard, is in the Appendix.

Here is the main result for deterministic algorithms:

**Theorem 2** Fix any $\epsilon > 0$. Suppose, for $U$ and $R_U$, that there is a deterministic polynomial-time algorithm $A_U$ that achieves Hamming distance $n/2 + H(n, \epsilon)$. Then $P=NP$.

**Proof:** Assume that there exists $\epsilon$ and $A_U$ as in the theorem. We will describe a polynomial-time algorithm for $U$. Since $U$ is NP-complete, the theorem follows.

The algorithm, given a tuple $(M, x, b, n)$, calls the subroutine shown in Fig. 1 with $u = 2^n$. If necessary, $M$ is first modified so that the precondition holds. The algorithm returns an $n$-bit witness $w$ such that $(M, x, b, w) \in R_U$ (if such a witness exists, and “none” otherwise).

The intuition is the following. The possible witnesses for machine $M$ are the elements of $[u]$. The algorithm calls the oracle $A_U$ to get some string $v$ within small Hamming distance from some actual witness in $[u]$ (if there is one). It then filters the set of possible witnesses to just those in $[u]$ that are close enough in Hamming distance to $v$ (this filtered set must include an actual witness, if there is one). It then computes the size $u'$ of the filtered set, and constructs $M'$ so that its possible witnesses are the elements of $[u']$, and so that $M'(x, u')$ accepts if $M(x, \phi(u'))$ accepts, where $\phi$ is a bijection between $[u']$ and the filtered set. Since $u' < u$, the algorithm eventually terminates.
\textbf{Witness}_e(M, x, b, u) \rightarrow \text{find } w \text{ such that } (M, x, b, w) \in R_d

preconditions: (1) If there is a witness \( w \), then there is one such that \( w \in [u] \), where \([u]\) denotes the set of \( u \) binary strings \( s \) such that \( |s| = |u| = \lfloor \log_2 u \rfloor \) and \( s < u \) (lexicographically). (2) \( M \) halts within \(|b|^2\) steps.

1. (base case) If \( u \leq |(M, x, b)|^c \) then do the following. For each \( w \in [u] \): simulate \( M(x, w) \) and return \( w \) if \( M(x, w) \) accepts. If no \( w \) causes \( M \) to accept, return “none”.

2. Let \( v = A_d(M, x, n) \) and let \( N(v) \) denote the \( n \)-bit binary strings within Hamming distance \( n/2 + H(n, \epsilon) \) from \( v \), where \( n = |u| \).

3. Compute \( u' = [N(v) \cap [u]] \). Let \( \phi : [u'] \rightarrow (N(v) \cap [u]) \) be the bijection mapping the \( i \)-th string in \([u']\) to the \( i \)-th string in \( N(v) \cap [u] \) (both sets ordered lexicographically).

4. Construct Turing machine \( M' \) that does the following on input \((x, w')\):
   1. If \( w' \in [u'] \), then return \( M(x, \phi(w')) \)
   2. else reject.

5. Let \( w'' = \text{Witness}_e(M', x, b', u') \), where \( b' \) is chosen so \( M' \) halts within \(|b'|^2\) steps.

6. Return \( \phi(w'') \), or “none” if \( w' = “none” \).

Figure 1: Assuming \( A_d(M, x, n) \) computes a string \( v \) that achieves Hamming distance \( n/2 + H(n, \epsilon) \) to a witness, \( \text{Witness}_e(M, x, b, u) \) finds such a witness (if one exists). \( \text{Witness}_e \) excludes from the set of possible witnesses \([u]\) those not in the Hamming-neighborhood of \( v \), then recurses on the filtered set. \( H(n, \epsilon) = \sqrt{\epsilon n \ln n} \).

Correctness. Correctness follows from a straightforward inductive proof, based on the fact that \((M, x, b) \in U\) has a witness in \([u]\) if and only if \((M', x, b') \in U\) has a witness in \([u']\).

Running time. To prove that the algorithm terminates in polynomial time, we first prove that \( u' \leq u - \Omega(2^n P(n, \epsilon)) \). (Recall \( n = |u| = \lfloor \log_2 u \rfloor \).) This implies that the algorithm recurses \( O(1/P(n, \epsilon)) \) times before \( u \) decreases by a factor of 2, and \( n \) decreases by at least 1. This implies that the algorithm recurses \( O(n/P(n, \epsilon)) \) times total. To finish, we then argue that each step can be implemented in polynomial time.

Here are the details. Consider the set \([2^{n-1}] \subset [u]\). If we choose a random string \( r \) in \([2^{n-1}]\), the probability that \( r \) has Hamming distance more than \( n/2 + H(n, \epsilon) \) to \( v \) is at least the probability that the last \( n - 1 \) bits of \( r \) have Hamming distance more than \( n/2 + H(n, \epsilon) \) to the last \( n - 1 \) bits of \( v \). This is the same as the probability that \( n - 1 \) random bits have at least \( n/2 + H(n, \epsilon) \) ones. By Observation 11, this probability is \( \Theta(P(n, \epsilon)) \). Thus, the number of integers in \([2^{n-1}]\) with Hamming distance more than \( n/2 + H(n, \epsilon) \) to \( v \) is at least \( 2^{n-1} \Theta(P(n, \epsilon)) \). Thus, the number of elements of \([u]\) within Hamming distance \( n/2 + H(n, \epsilon) \) to \( v \) is at most \( u - 2^{n-1} \Theta(P(n, \epsilon)) \).

To finish, we argue that each step can be implemented in polynomial time. There are two non-trivial issues:

i. The set \( N(v) \cap [u] \) has exponential size, so we need to say how to implement the operations involving this set. In particular, the algorithm needs to be able to compute the size \( |u'| \) of the set and to compute the bijection \( \phi : u' \rightarrow N(v) \cap [u] \).

Let \( N(s) \) denote the number of \( n \)-bit binary strings having string \( s \) as a prefix and having Hamming distance at most \( k = \lfloor n/2 + H(n, \epsilon) \rfloor \) to \( v \). The first \(|s| \) bits of such a string agree
with \( s \); the remaining \( n - |s| \) bits of such a string differ in at most \( k - \ell \) places from \( v \), where \( \ell \) is the Hamming distance between \( s \) and the first \( |s| \) bits of \( v \). Thus, \( N(s) = \sum_{j=0}^{k-\ell} \binom{n-\ell}{j} \).

Thus, given \( s \), \( N(s) \) can be computed in polynomial time.

In step 3, the algorithm needs to compute \( u' = |N(v) \cap [u]| \). This is \( \sum_{\ell=0}^{n-1} N(b_1 b_2 \cdots b_{n-1} 0) \), where \( b_1 \cdots b_n \) is the \( n \)-bit binary representation of \( u \). So \( u' \) can be computed in polynomial time.

In steps 4 and 6, the Turing machine and the algorithm need to be able to compute \( \phi(w') \) given \( w' \). Given \( w' \), it is easy to compute its rank \( i \) in \([u]\). Then \( \phi(w') \) is the largest \( n \)-bit string \( w \) such that \( |N(v) \cap [w]| \leq i \). Since \( |N(v) \cap [w]| \) can be computed as described in the previous paragraph, \( w \) can be found in polynomial time using binary search.

ii. We must check that the padding string \( b' \) constructed in each call to \textsc{Witness}_e (including the recursive calls) has size polynomial in the size of the original input \((M, x, b, u)\). This follows from the observation that only polynomially many recursive calls are made, and with each one, \(|b'| = |b| + n^{O(1)}\).

\[
\square
\]

**Theorem 3** Suppose that, for \( \mathcal{U} \) and \( \mathcal{R}_\mathcal{U} \), some randomized polynomial-time algorithm achieves Hamming distance \( n/2 + H(n, \epsilon) \) with probability \( 1 - O(P(n, \epsilon)/n) \). Then \( \text{RP} = \text{NP} \).

**Proof:** Consider the algorithm \textsc{Witness}_e described in the proof of Theorem 2 with input \((M, x, b, u)\). \textsc{Witness}_e calls the algorithm \( A_{\mathcal{U}}(M', x, \ell) \) with \( \ell \in \{\lceil c \log n \rceil, \ldots, n\} \). For each value of \( \ell \), \textsc{Witness}_e makes \( O(1/P(\ell, \epsilon)) \) calls to \( A_{\mathcal{U}} \).

Suppose \( A_{\mathcal{U}} \) were randomized and had probability \( O(P(\ell, \epsilon)/\ell) \) of failure on any input \((M', x, \ell)\). Then the probability that none of the calls to \( A_{\mathcal{U}} \) would fail would be at least

\[
\prod_{\ell=\lceil c \log n \rceil}^{n} (1 - O(P(\ell, \epsilon)/\ell))^{O(1/P(\ell, \epsilon))} = \exp(-O(\sum_{\ell=\lceil c \log n \rceil}^{n} 1/\ell)) = 1/n^{O(1)}.
\]

Thus, the algorithm \textsc{Witness}_e, described in that proof would have probability \( 1/n^{O(1)} \) of producing a witness in polynomial time. This shows that \( \mathcal{U} \in \text{RP} \). Since \( \mathcal{U} \) is NP-complete, the result follows.

\[
\square
\]

5 **Discussion**

Is it likely that the hardness results for \( \mathcal{U} \) are tight?

Among polynomial-time algorithms, the best randomized one that we know of is the trivial algorithm: guess \( n \) random bits. By Observation 11, this algorithm achieves Hamming distance \( n/2 + H(n, \epsilon) \) with probability \( \Theta(P(n, \epsilon)) \). Thus, the hardness result for randomized algorithms for \( \mathcal{U} \) is essentially tight, at least for this particular trade-off of approximation parameter and the probability of success.

The best deterministic polynomial-time algorithm we know of is a naive algorithm. It achieves Hamming distance \( n - c \) for any fixed \( c \). The algorithm is as follows: Test each \( n \)-bit string with \( c \) or fewer 1’s. If one is a witness, return it, otherwise return \( 0^n \). Note that \( 0^n \) is within Hamming distance \( n - c \).
distance \( n - c \) of all untested strings, and therefore within Hamming distance \( n - c \) of any witness (if the algorithm returns \( 0^n \)).

A simple adversary argument shows that this algorithm is essentially optimal among deterministic algorithms that use the verifier \( M \) as a black box. (Such an algorithm, given an instance \( I \), determines information about \( I \) only by querying the NP verifier \( M(I,x^*) \) with various potential witnesses \( x^* \).) The adversary behaves as follows. Whenever the algorithm queries \( M(I,x^*) \) with a given choice of \( x^* \), the adversary has \( M \) return “no”. Suppose the algorithm runs in \( o(n^{c-1}) \) time for some constant \( c \), before it returns its answer \( x \). There are at least \( \binom{n}{n-c+1} = \Omega(n^{c-1}) \) strings whose Hamming distance to \( x \) is \( n - c + 1 \). At least one of these strings \( x^* \) was not queried by the algorithm. The adversary can take \( x^* \) to be the true witness (taking \( M(I,x^*) = 1 \) only for this \( x^* \)). This means the algorithm’s answer, \( x \), does not achieve Hamming distance \( n - c \). In sum, any “black-box” deterministic algorithm that achieves Hamming distance \( n - c \) must take time \( \Omega(n^{c-1}) \) in the worst case.

Given the “universality” of \( U \), it may be difficult to design an algorithm for \( U \) other than a black-box algorithm as described above. If so, by the simple argument above, it will be difficult to find a deterministic polynomial-time algorithm that improves on the naive upper bound of \( n - O(1) \).

On the other hand, there is also a barrier to improving the corresponding lower bound (of \( n/2 + O(\sqrt{n \log n}) \) in the hardness result for deterministic algorithms for \( U \)). The barrier is essentially that the proof technique would have to distinguish a deterministic algorithm from an algorithm that, although randomized, has only exponentially small likelihood of failure. More specifically, note that the proof in this paper is essentially a reduction from \( U \) to the problem of achieving Hamming distance \( n/2 + O(\sqrt{n \log n}) \) for \( U \). The proof describes how, given an instance \( I \) of \( U \), to reduce \( I \) to a sequence of instances of the easier problem. For any such reduction, it would work just as well to solve the sequence of easier instances with high probability. And, for any bound much above \( n/2 + O(\sqrt{n \log n}) \), the naive randomized algorithm will do just that, and so, combined with the reduction, would give a randomized polynomial-time algorithm for \( U \), showing that RP=NP. Thus, it is unlikely that any proof that uses \( A_U \) as a black box (that is, a proof by reduction) can establish a lower bound above \( n/2 + O(\sqrt{n \log n}) \), even for deterministic algorithms.

It might be interesting to explore this question in the context of non-uniform complexity. For example, it seems likely that standard derandomization methods might yield a deterministic P/POLY algorithm for achieving Hamming distance \( n/2 + O(\sqrt{n \log n}) \) for \( U \).

Achieving Hamming distance \( n/2 \) for \( U \) is NP-hard. Achieving Hamming distance \( n/2 \) for Vertex Cover (and Clique and Independent Set) is in P. This raises a natural question: how hard is it to achieve Hamming distance \( n/2 \) for other natural problems? Is there a natural problem in NP (other than \( U \)) for which it is NP-hard to achieve Hamming distance \( n/2 \)?

References


Appendix

Here is the proof of Observation 2.

Proof: Let $M_L$ be a Turing machine that decides the witness relation $(x, w) \in R_L$ in polynomial time $t(|x|)$. Let $b$ be a string of $t(|x|)$ 1’s.

By the definition of the universal language, the size-$n$ witnesses for $x \in L$ are exactly the size-$n$ witnesses for $(M_L, x, b) \in U$.

Given $(x, n)$, the algorithm $A_L$ does the following: run $A_U$ on $(M_L, x, p, n)$ and return the result. This takes polynomial time because $A_U$ takes polynomial time and $|p| \leq t(|x|)$.

By the definition of the universal language, the size-$n$ witnesses for $x \in L$ are exactly the size-$n$ witnesses for $(M_L, x, b) \in U$. Thus, whatever string $A_U$ returns, the string is a size-$n$ witness for $x \in L$ if and only if it is a size-$n$ witness for $x \in L$.

Thus, if $x \in L$ has a size-$n$ witness, then $A_L(x)$ is within Hamming distance $H(|w|)$ to some such witness with probability at least $P(|w|)$.

Here is the proof of Observation 11.

Proof: Assume for simplicity of notation that $n$ is even. (The case when $n$ is odd is similar.) Let $H = \lceil H(n, \epsilon) \rceil$. Let $p_i = \binom{n}{n/2+1}/2^n$. The probability that $n$ random bits have at least $H$ ones is $\sum_{i=H}^{n/2} p_i$. Note that

$$p_{i+1} = \frac{1 - 2i/n}{1 + 2i/n}.$$  

From this it follows that the sum $\sum_{i=H}^{n/2} p_i$ is proportional to the sum of its first $\Theta(n/H)$ or so terms, and that each of the first $\Theta(n/H)$ terms is proportional to the first term. Thus, the entire sum is $\Theta((n/H)p_H)$. A calculation shows that $p_H = \Theta(n^{-2\epsilon+1/2})$. Thus, the entire sum is $\Theta(n^{-2\epsilon+1/2}/H)$. Plugging in the definition of $H$ gives the first claim.

The second claim follows easily because the probability in question is at least $\sum_{i=H+2}^{n/2} p_i$, which by the above considerations is also $\Theta((n/H)p_H)$.

For the thorough reader, here are the intermediate steps in the calculation that $p_H = \Theta(n^{-2\epsilon+1/2})$. We use Stirling’s approximation $i! = \Theta((i/e)^i/\sqrt{i})$, then we use $(1+a)^b = \exp(b \ln(1+a)) = \Theta(b^n)$.
\( \exp(b(a + O(a^2))) = \Theta(\exp(ab)) \) when \( a^2b = O(1) \) and \( |a| < 1. \)

\[
\frac{2^n}{\binom{n}{n/2+H} \sqrt{n}} = \Theta \left( (1 - 2H/n)^{n/2-H} (1 + 2H/n)^{n/2+H} \right)
\]

\[
= \Theta \left( \exp(4H^2/n) \right)
\]

\[
= \Theta \left( n^{2\epsilon} \right).
\]