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A combined scalarizing method for multiobjective programming problems

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Abstract

In this paper, a new general scalarization technique for solving multiobjective optimization problems is presented. After studying the properties of this formulation, two problems as special cases of this general formula are considered. It is shown that some well-known methods such as the weighted sum method, the $\epsilon$-constraint method, the Benson method, the hybrid method and the elastic $\epsilon$-constraint method can be subsumed under these two problems. Then, considering approximate solutions, some relationships between $\varepsilon$-(weakly, properly) efficient points of a general (without any convexity assumption) multiobjective optimization problem and $\varepsilon$-optimal solutions of the introduced scalarized problem are achieved.

Keywords: Multiple objective programming, Scalarization method, Approximate solutions, Properly efficient solutions, $\varepsilon$-properly efficient solutions.

1 Introduction

One part of mathematical programming is multiobjective optimization programming when the conflicting objective functions must be minimized over a feasible set of decisions. In many areas in engineering, economics, and science new developments are only possible by the application of multiobjective optimization problems (MOPs) and related methods. There are many recent publications on applications of MOPs,[9, 24, 27, 28, 40, 12] and many others. Various monographs collected many results in theory and methodology, [8, 13, 39], or provided

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a comprehensive review of methods [34]. For solving MOPs, there are a number of methods and algorithms which are classified according to participation of the decision maker in the solution process [25]. The traditional and common approach for solving MOPs is a reformulation as a parameter scalar optimization problem. In other words, they are most commonly solved indirectly by using conventional (single-objective) optimization techniques by the aid of scalarization. In general, scalarization means the replacement of a vector optimization problem by a suitable scalar optimization problem which is an optimization problem with a real valued objective function. Since the scalar optimization theory has been widely developed, scalarization turns out to be of great importance for the vector optimization theory, as it is done in the well known weighted sum method [35, 17], the \( \varepsilon \)-constraint method [36, 5], the hybrid method [20, 26], the Benson method [3], the normal boundary intersection method [6], and so on. For a survey on the scalarizing technique, the reader is referred to [10]. Our focus in this paper is based on the main idea of the elastic \( \varepsilon \)-constraint method introduced by Ehrgott and Ruzika in [11]. Since the \( \varepsilon \)-constraint method has no result about properly efficient solutions, Ehrgott and Ruzika have presented two modifications of the \( \varepsilon \)-constraint method to remedy this weakness. We use their strategy to constitute a general form. We show that the weighted sum method, the \( \varepsilon \)-constraint method, the Benson method, the hybrid method and the elastic \( \varepsilon \)-constraint method can be seen as special cases of our problem. Then, we prove some necessary and sufficient conditions for (weakly, properly) efficient points of a general MOP via optimal solutions of the presented scalarized problem. Researchers have tried to present general formulations for multiobjective optimization problems. For example, Luque et al. [33, 37, 38], introduced a general formulation for several interactive methods. Their general formulation can accommodate some well-known interactive methods. Our formulation in this paper is not for interactive methods and so, is different from the formulation in [33, 38]. It should be mentioned that there exist several publications about properly efficient solutions, [26, 5] and many others, which use terms of stability of the scalarized problem or the K.K.T multipliers. However, our results on proper efficiency are more direct.

On the other hand, the importance of approximation solutions for MOPs in recent decades motivated us to investigate \( \varepsilon \)-efficient solutions. The first notion of approximation was suggested by Kutateladze [29] and extended by Loridan [32]. White [41] investigated six kinds of \( \varepsilon \)-approximate efficient solutions. Many authors studied the properties of this kind of solution. Some necessary and sufficient conditions for \( \varepsilon \)-(weak) efficiency can be found in [7, 21, 22] and others. Engau and Wiecek [14] investigated scalarization approaches to generate \( \varepsilon \)-efficient solutions of MOPs. Since our presented problems are extensions of methods in [14], the results in the current paper are extension of those of special cases in [14].
Also, one of the most important notions in multiobjective optimization theory is proper efficiency introduced by Li and Wang [30]. Liu [31] derived some necessary and sufficient conditions for \( \varepsilon \)-proper efficient solutions of convex MOPs. See also [1, 15, 16]. The methods considered in [14] have no result on \( \varepsilon \)-proper efficiency. So, Ghaznavi and Khorram [18] and Ghaznavi et al.[19], using the elastic \( \varepsilon \)-constraint method, provided some necessary and sufficient conditions for \( \varepsilon \)-(weak,proper) efficiency. Since our problem is a general form and the elastic \( \varepsilon \)-constraint method is a special case of that, the obtained results extend the results obtained in [18, 19, 14]. It is worth mentioning that the obtained results are general and we do not assume any convexity assumption.

The outline of this article is as follows: in Section 2, we provide preliminaries and basic definitions. In Section 3, we present the general formulation and study some properties of this formula. In Sections 4 and 5, two problems are presented which are special cases of the general formula presented in Section 3. Section 6 is devoted to the necessary and sufficient conditions to obtain \( \varepsilon \)-(weakly,properly) efficient solutions in three subsections. The conclusions are derived in Section 7.

2 Preliminaries and basic definitions

In this paper, optimization of the multiple objective problem is studied as follows:

\[
\begin{align*}
\min & \quad f(x) = (f_1(x), f_2(x), \ldots, f_p(x)) \\
\text{s.t.} & \quad g_i(x) \leq 0, \quad i = 1, 2, \ldots, m \\
& \quad h_k(x) = 0, \quad k = 1, 2, \ldots, \hat{m}
\end{align*}
\]

where,

\[
\begin{align*}
f_j, g_i, h_k : \Omega \subset \mathbb{R}^n \to \mathbb{R}, \quad \forall j, i, k,
\end{align*}
\]

and \( \Omega \neq \emptyset \). Here, we show all the feasible points by \( X \). In other words,

\[
X = \{ x \in \Omega | g_i(x) \leq 0, h_k(x) = 0, \forall i, k \}.
\]

Now, the following definitions are presented to determine efficient solutions of the MOP.

**Definition 2.1.** A feasible solution \( x^* \in X \) of the MOP is called (1) efficient optimal solution if there does not exist another \( x \in X \) such that

\[
f_j(x) \leq f_j(x^*) \quad \text{for all} \quad j = 1, 2, \ldots, p \quad \text{and} \quad f(x) \neq f(x^*).
\]
(2) weakly efficient solution if there is no \( x \in X \) such that
\[
f_j(x) < f_j(x^*); \quad j = 1, 2, ..., p.
\]
(3) strictly efficient solution if there does not exist another feasible solution \( x \neq x^* \) such that
\[
f_j(x) \leq f_j(x^*); \quad j = 1, 2, ..., p.
\]

Let \( X_E(X_{wE}, X_{sE}) \) be the set of efficient (weakly, strictly efficient) solutions. If \( x^* \) is an efficient (weakly efficient) solution, \( f(x^*) \) is called a nondominated (weakly nondominated) point. The set of nondominated (weakly nondominated) points is denoted by \( Y_N(Y_{wN}) \). In other words, \( Y_N := f(X_E)(Y_{wN} = f(X_{wE})) \).

We assume throughout this paper that \( Y = f(X) \) is bounded and that \( X_E \) is nonempty. This is guaranteed, e.g. if \( X \) is compact and \( f_i \) are continuous (see [8]).

Throughout this paper, we use the following notations:
- \( R^p_\triangledown := \{ y \in R^p | y_i > 0, i = 1, 2, ..., p \} \).
- \( R^p_\triangledown := \{ y \in R^p | y_i \geq 0, i = 1, 2, ..., p \} \setminus \{0\} \).
- \( R^p_\triangledown := \{ y \in R^p | y_i \geq 0, i = 1, 2, ..., p \} \).

On the other hand, there exists a well-known kind of efficient points which are named properly efficient solutions. Properly efficient points are those efficient solutions that have bounded trade-offs between the objectives. There are some definitions for proper efficiency given by Borwein [4], Hartley [23], Benson [2] and others. Here we use the definition of proper efficiency in the sense of Geoffrion [17].

**Definition 2.2.** A feasible solution \( \hat{x} \in X \) is called properly efficient in Geoffrion’s sense, if it is efficient and if there is a real number \( M > 0 \) such that for all \( i \) and \( x \in X \) satisfying \( f_i(x) < f_i(\hat{x}) \) there exists an index \( j \) such that \( f_j(\hat{x}) < f_j(x) \) and
\[
\frac{f_i(\hat{x}) - f_i(x)}{f_j(x) - f_j(\hat{x})} < M.
\]

The set of properly efficient solutions is denoted by \( X_{pE} \).

**ε(weakly)efficient solutions of MOP (2.1) are defined as follows [32]:**

**Definition 2.3.** Take into consideration MOP (2.1). Let \( \varepsilon \in R^p_\triangledown \). A feasible point \( \hat{x} \in X \) is called:
(1) \( \varepsilon \)-weakly efficient if there is no other \( x \in X \) such that \( f(x) < f(\hat{x}) - \varepsilon \).
(2) \( \varepsilon \)-efficient if there is no other \( x \in X \) such that \( f(x) \leq f(\hat{x}) - \varepsilon \).
Definition 2.4. \([30]\) A feasible point \(\hat{x} \in X\) is called \(\varepsilon\)-properly efficient point of problem (2.1), if it is \(\varepsilon\)-efficient and there is a real positive number \(M > 0\) such that for all \(i \in \{1, 2, \ldots, p\}\) and \(x \in X\) satisfying \(f_i(x) < f_i(\hat{x}) - \varepsilon_i\), there exists an index \(j \in \{1, 2, \ldots, p\}\) such that \(f_j(\hat{x}) - \varepsilon_j < f_j(x)\) and
\[
\frac{f_i(\hat{x}) - f_i(x) - \varepsilon_i}{f_j(x) - f_j(\hat{x}) + \varepsilon_j} < M.
\]

The set of all \(\varepsilon\)-weakly efficient, \(\varepsilon\)-efficient and \(\varepsilon\)-properly efficient solutions of an MOP will be indicated by \(X_{\varepsilon WE}\), \(X_{\varepsilon E}\) and \(X_{\varepsilon PE}\), respectively. Notice that for \(\varepsilon = 0\), \(\varepsilon\)-weak efficiency, \(\varepsilon\)-efficiency and \(\varepsilon\)-properly efficiency collapse in the usual definition of weak efficiency, efficiency, (Definition 2.1) and properly efficiency (Definition 2.2).

Remark 2.5. Obviously, \(X_{\varepsilon PE} \subseteq X_{\varepsilon E} \subseteq X_{\varepsilon WE}\).

The customary approach to solve a given MOP is to formulate a single objective program (SOP) associated with it. Let us consider an SOP as follows:
\[
\min_{x \in X} g(x),
\]
where \(g : X \to \mathbb{R}\). The notation of optimality, \(\varepsilon\)-optimality and strict \(\varepsilon\)-optimality for given SOP are defined as follows:

Definition 2.6. Let \(\varepsilon \geq 0\). For the SOP, a feasible solution \(\hat{x} \in X\) is called:
(1) an optimal solution if \(g(\hat{x}) \leq g(x)\) for all \(x \in X\).
(2) an \(\varepsilon\)-optimal solution if \(g(\hat{x}) \leq g(x) + \varepsilon\) for all \(x \in X\).
(3) a strictly \(\varepsilon\)-optimal solution if \(g(\hat{x}) < g(x) + \varepsilon\) for all \(x \in X\).

3 A general scalarized problem

In this section, using slack and surplus variables we consider the following formulation which is a general scalarizing method for solving MOP (2.1). The objective function equals the positive weighted sum of objectives, the positive weighted sum of surplus variables and the negative weighted sum of slack variables. This general form can be formulated as follows:
\[
\min \quad \sum_{i=1}^{p} \lambda_i f_i(x) - \sum_{i=1}^{p} \gamma_i s_i^+ + \sum_{i=1}^{p} \mu_i s_i^-, \quad (3.1)
\]
\[
f_i(x) + s_i^+ - s_i^- \leq \alpha_i, \quad 1 \leq i \leq p,
\]
\[ x \in X, s^+, s^- \geq 0, \]

where \( \lambda_i, \mu_i \) and \( \gamma_i \), for all \( i \), are nonnegative weights, and \( \alpha_i, (\forall i) \) are given upper bounds.

In SOP (3.1), the slack and surplus variables \( s^+_i \) and \( s^-_i \), for all \( i \), might be changed simultaneously by an amount of \( \beta_i \in R \) without effecting the feasibility of the constraints. This proposition has been discussed completely in [11] and we refer the reader to that for more details.

The first property of the SOP (3.1) is stated as the following lemma.

**Lemma 3.1.** Let \( \gamma \geq 0 \) and the set of optimal solutions of SOP (3.1) be not empty. Then SOP (3.1) has an optimal solution such that all the \( \alpha \)-constraints are active. If \( \gamma > 0 \), then all the \( \alpha \)-constraints are active in every optimal solution of SOP (3.1).

**Proof.** Assume \((\hat{x}, \hat{s}^+, \hat{s}^-)\) is an optimal solution of SOP (3.1), and there is some \( j \) such that \( f_j(\hat{x}) + \hat{s}^+_j - \hat{s}^-_j < \alpha_j \). Define \( I(\hat{x}, \hat{s}^+, \hat{s}^-) = \{ j : f_j(\hat{x}) + \hat{s}^+_j - \hat{s}^-_j < \alpha_j \} \) and \( \delta_j = \alpha_j - f_j(\hat{x}) - \hat{s}^+_j + \hat{s}^-_j > 0 \) \( \forall j \in I(\hat{x}, \hat{s}^+, \hat{s}^-) \). Now, we define \( \hat{s}^+_i = \hat{s}^-_i \), if \( i \notin I(\hat{x}, \hat{s}^+, \hat{s}^-) \) and \( \hat{s}^+_i = \hat{s}^-_i + \delta_i \), if \( i \in I(\hat{x}, \hat{s}^+, \hat{s}^-) \). Clearly, \((\hat{x}, \hat{s}^+, \hat{s}^-)\) is feasible for SOP (3.1) and all the constraints are active. On the other hand, since \( \hat{s}^+ \geq \hat{s}^+ \), we have

\[
\sum_{i=1}^{p} \lambda_i f_i(\hat{x}) - \sum_{i=1}^{p} \gamma_i \hat{s}^+_i + \sum_{i=1}^{p} \mu_i \hat{s}^-_i \leq \sum_{i=1}^{p} \lambda_i f_i(\hat{x}) - \sum_{i=1}^{p} \gamma_i \hat{s}^+_i + \sum_{i=1}^{p} \mu_i \hat{s}^-_i.
\]

This means that \((\hat{x}, \hat{s}^+, \hat{s}^-)\) yields a better objective function value for SOP (3.1) than \((\hat{x}, \hat{s}^+, \hat{s}^-)\) (if \( \gamma_j > 0 \) for some \( j \in I(\hat{x}, \hat{s}^+, \hat{s}^-) \)) or the same as \((\hat{x}, \hat{s}^+, \hat{s}^-)\) (if \( \gamma_j = 0 \) for all \( j \in I(\hat{x}, \hat{s}^+, \hat{s}^-) \)). This contradicts the optimality of \((\hat{x}, \hat{s}^+, \hat{s}^-)\). \( \square \)

Now, we start analyzing the SOP (3.1) theoretically.

**Theorem 3.2.** Let \((\hat{x}, \hat{s}^+, \hat{s}^-)\) be the optimal solution of SOP (3.1). If

(i) \( \lambda + \gamma \geq 0 \) or \( (\lambda + \mu \geq 0 \) and \( \hat{s}^- > 0 \), then \( \hat{x} \) is a weakly efficient solution of the MOP (2.1).

(ii) \( \lambda + \gamma \geq 0 \) or \( (\lambda + \mu \geq 0 \) and \( \hat{s}^- > 0 \) and \( \hat{x} \) is unique, then \( \hat{x} \) is a strictly efficient solution of the MOP (2.1).

(iii) \( \lambda + \gamma > 0 \) or \( (\lambda + \mu > 0 \) and \( \hat{s}^- > 0 \), then \( \hat{x} \) is an efficient solution of the MOP (2.1).

**Proof.** Here, we give the proof of part (iii). The proofs of parts (i) and (ii) are similar and will be omitted.
(iii) Assume \( \hat{x} \) is not an efficient solution. So, there exists a feasible point \( x \in X \) such that for all \( i \) \( f_i(x) \leq f_i(\hat{x}) \) and for some \( j \) the inequality is strict. We have:

\[
f_i(x) + \hat{s}_i^+ - \hat{s}_i^- \leq \alpha_i \quad \forall i \neq j,
\]

\[
f_j(x) + \hat{s}_j^+ - \hat{s}_j^- < \alpha_j.
\]

We consider two cases:

Case 1) Let \( \lambda + \gamma > 0 \). Clearly, there is some \( \nu > 0 \) such that

\[
f_j(x) + \hat{s}_j^+ - \hat{s}_j^- + \nu \leq \alpha_j.
\]

Set \( s_i^+ = \hat{s}_i^+ \) for \( i \neq j \) and \( s_j^+ = \hat{s}_j^+ + \nu \). Obviously, \((x, s^+, \hat{s}^-)\) is feasible for SOP (3.1) and yields a better objective function than \((\hat{x}, \hat{s}^+, \hat{s}^-)\) since \( \lambda_j + \gamma_j > 0 \).

Case 2) If \( \lambda + \mu > 0 \) and \( \hat{s}^- > 0 \). So, there is some \( \nu > 0 \) such that \( \hat{s}_j^- - \nu > 0 \) and

\[
f_j(x) + \hat{s}_j^+ - \hat{s}_j^- + \nu \leq \alpha_j.
\]

In this case define \( s_i^- = \hat{s}_i^- \) for \( i \neq j \) and \( s_j^- = \hat{s}_j^- + \nu \). Therefore, \((x, \hat{s}^+, s^-)\) is feasible for SOP (3.1) and yields a better objective function than \((\hat{x}, \hat{s}^+, \hat{s}^-)\) since \( \lambda_j + \mu_j > 0 \). This contradicts the optimality of \((\hat{x}, \hat{s}^+, \hat{s}^-)\).

Next, we state an easy approach to check the sufficient condition for identifying properly efficient solutions among the solutions of SOP (3.1). For the proof we need a technical lemma relating properly efficient solutions of the MOP with the feasible set of SOP (3.1) and the set \( X \), respectively. This lemma is very similar to the idea mentioned by Ehrgott and Ruzika in [11].

**Lemma 3.3.** Let \( \hat{x} \) be a properly efficient solution of the MOP with feasible set of SOP (3.1). Let there be a partition \( I \cup \bar{I} \) of \( \{1, 2, \ldots, p\} \) such that \( f_i(\hat{x}) < \alpha_i \) for all \( i \in I \) and \( f_i(\hat{x}) > \alpha_i \) for all \( i \in \bar{I} \). Then, \( \hat{x} \) is a properly efficient solution of the MOP with feasible set \( X \).

**Proof.** The proof is similar to the Lemma 3.2 in [11] and will be omitted here. \( \square \)

**Theorem 3.4.** Let \( \lambda + \mu > 0 \) and \( \lambda + \gamma > 0 \). If \((\hat{x}, \hat{s}^+, \hat{s}^-)\) is an optimal solution of SOP (3.1) and also there is a partition \( I \cup \bar{I} \) of \( \{1, 2, \ldots, p\} \) such that

\[
\hat{s}_i^+ = 0, \hat{s}_i^- > 0 \quad \text{for } i \in I \quad \text{and} \quad \hat{s}_i^- = 0, \hat{s}_i^+ > 0 \quad \text{for } i \in \bar{I},
\]

then \( \hat{x} \) is a properly efficient solution of the MOP.

**Proof.** Helping part (iii) of Theorem 3.2, \( \hat{x} \) is efficient. On the other hand, we
have:

\[ \sum_{i=1}^{p} \lambda_i f_i(\hat{x}) - \sum_{i=1}^{p} \gamma_i \hat{s}_i^+ + \sum_{i \in \bar{I}} \mu_i \hat{s}_i^- = \sum_{i=1}^{p} \lambda_i f_i(\hat{x}) - \sum_{i \in \bar{I}} \gamma_i \hat{s}_i^+ + \sum_{i \in I} \mu_i \hat{s}_i^- \]

and since \((\hat{x}, \hat{s}^+, \hat{s}^-)\) is an optimal solution of SOP (3.1) without loss of generality, we can write:

\[ = \sum_{i=1}^{p} \lambda_i f_i(\hat{x}) - \sum_{i \in \bar{I}} \gamma_i (\alpha_i - f_i(\hat{x})) + \sum_{i \in I} \mu_i (f_i(\hat{x}) - \alpha_i). \]

Therefore \(\hat{x}\) is an optimal solution of the weighted sum problem

\[ \min \{\sum_{i \in \bar{I}} (\lambda_i + \gamma_i) f_i(\hat{x}) + \sum_{i \in I} (\lambda_i + \mu_i) f_i(\hat{x}) : f_i(\hat{x}) < \alpha_i, i \in \bar{I}, f_i(\hat{x}) > \alpha_i, i \in I\}. \]

By Geoffrion’s theorem [17], \(\hat{x}\) is a properly efficient solution of the MOP with additional constraints. Using Lemma 3.3, \(\hat{x}\) is properly efficient for the MOP with feasible set \(X\) and the proof is completed.

\[ \square \]

**Remark 3.5.** If \(\lambda = e_k, \gamma_k = 0\) and \(\mu_k = 0\), Theorem 3.4 reduces to Theorem 5.2 in [11].

In the following theorem, we present a necessary condition for properly efficient solutions of the MOP. We will show how properly efficient solutions can be obtained by appropriate choices of parameters.

**Theorem 3.6.** Let \(\hat{x}\) be properly efficient for the MOP. Then, there exist \(\lambda, \mu, \gamma \geq 0, \alpha < \infty\) and \(\hat{s}^+, \hat{s}^-\), such that \((\hat{x}, \hat{s}^+, \hat{s}^-)\) is an optimal solution of SOP (3.1).

**Proof.** Set \(\gamma = 0, \hat{s}^+ = 0\). We set \(\alpha_i := f_i(\hat{x}), i = 1, 2, \ldots, p\). So, we can choose \(\hat{s}^- = 0\). Also, let \(\lambda \geq 0\) such that \(\sum_{i=1}^{p} \lambda_i = 1\). Since \(\hat{x}\) is properly efficient, there is \(M > 0\) such that, for all \(x \in X\) and for all \(i\) with \(f_i(x) < f_i(\hat{x})\), there exists \(j \neq i\) such that \(f_j(\hat{x}) < f_j(x)\) and

\[ (f_i(\hat{x}) - f_i(x))/(f_j(x) - f_j(\hat{x})) < M. \]

Now, define \(\hat{\mu}_i = M, \forall i\).

Let \((x, s^+, s^-)\) be a feasible point for SOP (3.1). Since \(\gamma = 0\), we can put \(s^+ = 0\) and \(s^-\) as follows:

\[ s^- = \max \{0, f_i(x) - \alpha_i\} = \max \{0, f_i(x) - f_i(\hat{x})\}, \]
that is the smallest possible value it can take. We need to show that
\[ \sum_{i=1}^{p} \lambda_i f_i(x) + \sum_{i=1}^{p} \mu_i s_i \geq \sum_{i=1}^{p} \lambda_i f_i(\hat{x}) + \sum_{i=1}^{p} \mu_i s_i = \sum_{i=1}^{p} \lambda_i f_i(\hat{x}). \]

Let \( k \in \{1, 2, ..., p\} \). We have two possible cases:

(case 1) If \( f_k(x) \geq f_k(\hat{x}) \), then
\[ f_k(x) + \sum_{i=1}^{p} \hat{\mu}_i s_i \geq f_k(\hat{x}). \]  \( \text{(a)} \)

(case 2) If \( f_k(x) < f_k(\hat{x}) \), set \( I^* = \{ i : f_i(x) > f_i(\hat{x}) \} \). The set \( I^* \neq \emptyset \) because \( \hat{x} \in X_pE \). We can write:
\[ f_k(x) + \sum_{i=1}^{p} \hat{\mu}_i s_i = f_k(x) + \sum_{i=1}^{p} \hat{\mu}_i \max\{0, f_i(x) - f_i(\hat{x})\} \]
\[ = f_k(x) + \sum_{i \in I^*} \hat{\mu}_i (f_i(x) - f_i(\hat{x})). \]

Since \( \hat{x} \in X_pE \) and \( f_k(x) < f_k(\hat{x}) \), there is \( k^* \in I^* \) such that \( f_k(x) - f_k(\hat{x}) < M \).

So, we have
\[ f_k(x) + \sum_{i \in I^*} \hat{\mu}_i (f_i(x) - f_i(\hat{x})) \geq f_k(x) + \mu_{k^*} (f_{k^*}(x) - f_{k^*}(\hat{x})) \]
\[ > f_k(x) + f_k(\hat{x}) - f_{k^*}(\hat{x}) (f_{k^*}(x) - f_{k^*}(\hat{x})) = f_k(\hat{x}). \]  \( \text{(b)} \)

Applying (a) and (b) we have:
\[ \sum_{i=1}^{p} \lambda_i f_i(x) + \sum_{i=1}^{p} \lambda_i (\sum_{i=1}^{p} \hat{\mu}_i s_i) \geq \sum_{i=1}^{p} \lambda_i f_i(\hat{x}), \]
and since \( \sum_{i=1}^{p} \lambda_i = 1 \), we have
\[ \sum_{i=1}^{p} \lambda_i f_i(x) + \sum_{i=1}^{p} \hat{\mu}_i s_i \geq \sum_{i=1}^{p} \lambda_i f_i(\hat{x}). \]

The above inequality is true for all \( \mu \geq \hat{\mu} \) and the proof is complete. \( \Box \)
In Theorem 3.6, the boundedness of \( f(X) \) cannot be omitted. Examples 3.2 and 4.2 in [11] shows that if \( f(X) \) is unbounded the result is no longer true. Additionally, we can obtain a necessary condition for efficient solutions as follows:

**Theorem 3.7.** Let \( \hat{x} \) be efficient for the MOP. Then, there exist \( \lambda, \mu, \gamma \geq 0 \), \( \alpha < \infty \) and \( \hat{s}^+, \hat{s}^- \), such that \((\hat{x}, \hat{s}^+, \hat{s}^-)\) is an optimal solution of SOP (3.1).

**Proof.** It is sufficient to set \( \alpha_i = f_i(\hat{x}) \) for all \( i \), \( \gamma = 0 \), \( \mu = 0 \), \( \hat{s}^+ = 0 \) and \( \hat{s}^- = 0 \). Therefore, by Theorem 4.7 in [8] there exists \( \lambda > 0 \) such that \( \hat{x} \) is an optimal solution of SOP (3.1).

In Section 4 and 5, we investigate two special cases of the SOP (3.1). We study these two problems with more details. Also, we show that some well-known scalarizing methods for solving the MOP can be seen as special cases of our problems.

### 4 A problem with slack variables

In this section, we consider problem (3.1) only with slack variables for solving the MOP. The objective function equals the positive weighted sum of objectives and the negative weighted sum of the slack variables. In other words, in SOP (3.1) we put \( \mu = 0 \) and \( s^- = 0 \). So, we have:

\[
\min \sum_{i=1}^{p} \lambda_i f_i(x) - \sum_{i=1}^{p} \gamma_i s_i,
\]

\[
f_i(x) + s_i \leq \alpha_i, \quad 1 \leq i \leq p,
\]

\[
x \in X, s \geq 0,
\]

where \( \lambda_i \) and \( \gamma_i \), for all \( i \), are nonnegative weights, and \( \alpha_i, (\forall i) \) are given upper bounds. We will suppose that in SOP (4.1), \( \alpha_i, (\forall i) \) are selected such that the mentioned problem remains feasible. The SOP (4.1) is the extended form of some of the well known scalarizing methods. Table 1 shows these relations.

<table>
<thead>
<tr>
<th>Conditions on parameters</th>
<th>The achieved scalarized method</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \gamma = 0, \alpha = \infty )</td>
<td>Weighted-Sum method</td>
</tr>
<tr>
<td>( \gamma = 0, \lambda_k = 1, \lambda_{i \neq k} = 0, \alpha_k = \infty )</td>
<td>( \epsilon )-Constraint method</td>
</tr>
<tr>
<td>( \gamma = 0, \alpha = f(x^0) ) where ( x^0 ) is an arbitrary feasible point</td>
<td>Hybrid method</td>
</tr>
<tr>
<td>( \lambda = 0, \gamma = 1, \alpha = f(x^0) ) where ( x^0 ) is an arbitrary feasible point</td>
<td>Benson method</td>
</tr>
<tr>
<td>( \lambda_k = 1, \lambda_{i \neq k} = 0, \alpha_k = \infty )</td>
<td>Elastic Constraint method</td>
</tr>
</tbody>
</table>
SOP (4.1) has some properties. The first one is that similar to the SOP (3.1) there are always some optimal solutions such that the additional constraints are active at these points. In other words, SPO (4.1) has the properties presented in Lemma (3.1).

Depending on the choice of the weight vectors, different results can be derived for SOP (4.1). The next theorem shows some results.

**Theorem 4.1.** (1) Let \((\hat{x}, \hat{s})\) be an optimal solution of SOP (4.1) with \(\lambda + \gamma > 0\). Then, \(\hat{x}\) is an efficient solution of MOP (2.1).

(2) Let \((\hat{x}, \hat{s})\) be an optimal solution of SOP (4.1) with \(\lambda + \gamma \geq 0\). Then, \(\hat{x}\) is a weakly efficient solution of MOP (2.1).

(3) Let \((\hat{x}, \hat{s})\) be an optimal solution of SOP (4.1) with \(\lambda + \gamma \geq 0\). If \(\hat{x}\) is unique, then \(\hat{x}\) is a strictly efficient solution of MOP (2.1).

**Proof.** Putting \(\gamma = 0\) and \(s^- = 0\), this theorem is a special case of Theorem 3.2 and the proof is obvious.

The results obtained by Theorem 4.1 are true for the special cases presented in Table 1. In other words, the properties of the weighted sum method, the \(\epsilon\)-constraint method, the Benson method, the hybrid method and the elastic \(\epsilon\)-constraint method can be considered as special cases of Theorem 4.2 and achieved by that.

The next theorem states an easy to check sufficient condition for identifying properly efficient solutions of the MOP among the solutions of SOP (4.1).

**Theorem 4.2.** If \((\hat{x}, \hat{s})\) is an optimal solution of SOP (4.1) with \(\lambda + \gamma > 0\) and \(\hat{s} > 0\), then \(\hat{x}\) is a properly efficient solution of MOP (2.1).

**Proof.** Since in SOP (4.1) \(s^- = 0\), we can assume that \(\mu > 0\). Hence, the proof is achieved by Theorem 3.4.

**Remark 4.3.** Putting \(\lambda_k = 1, \lambda_i \neq k = 0, \gamma_k = 0\), Theorem 3.2 in [11] can be seen as a special case of Theorem 4.2.

Similar to Theorem 3.6, any efficient solution can be considered as an optimal solution of SOP (4.1) with positive weights. So, we have:

**Theorem 4.4.** Let \(\hat{x}\) be an efficient solution of the MOP. Then there exist \(\alpha < \infty, \hat{s}, \lambda\) and \(\gamma\), such that \((\hat{x}, \hat{s})\) is an optimal solution of SOP (4.1).

Notice that in Theorem 4.4 the parameters \(\alpha_i, \forall i\) are finite. So, for some scalarizing technique with infinite values for some \(\alpha_i\)(see Table 1), we cannot use
this theorem. For this kind of scalarized problem more assumptions are needed, i.e. convexity for the weighted sum method or proper efficiency for the elastic \( \varepsilon \)-constraint method and so on.

In the following section, we allow the added constraints to be violated and then penalize these violations in the objective function.

5 A problem with flexible constraints

In this section, we study the problem (3.1) in a special case when \( \gamma = 0 \) and \( s^+ = 0 \). So, consider the following problem:

\[
\min \sum_{i=1}^{p} \lambda_i f_i(x) + \sum_{i=1}^{p} \mu_i s_i, \tag{5.1}
\]

\[
f_i(x) - s_i \leq \alpha_i, \quad 1 \leq i \leq p,
\]

\[
x \in X, s \geq 0,
\]

where \( \lambda_i \) and \( \mu_i \), for all \( i \), are nonnegative weights, and \( \alpha_i, (\forall i) \) are given upper bounds. We will suppose that in SOP (5.1), \( \alpha_i, (\forall i) \) are selected such that the mentioned problem remains feasible. Note that if \((\hat{x}, \hat{s})\) is an optimal solution, then we may assume without loss of generality that \( \hat{s}_i = \max\{0, f_i(\hat{x}) - \alpha_i\} \).

It should be mentioned that although the SOP (5.1) is a special case of the SOP (3.1), since \( s^+ = 0 \), it does not have the property of Lemma 3.1. In other words, the \( \alpha \)-constraints are not always active in optimality. Hence, some properties of the problem (3.1) and also (4.1) are not true for the SOP (5.1).

Remark 5.1. If \( \lambda_k = 1, \lambda_{i \neq k} = 0 \) and \( \mu_k = 0 \), the SOP (5.1) reduces to the second modification of the \( \varepsilon \)-constraint method introduced by Ehrgott and Ruzika [11].

The following results obtained by Theorem 3.2 and Theorem 3.4, are extensions of the results in [11].

Theorem 5.2. Let \((\hat{x}, \hat{s})\) be the optimal solution of SOP (5.1). If

(1) \( \lambda \neq 0 \) or \( (\lambda + \mu \geq 0 \) and \( \hat{s} > 0 \)), then \( \hat{x} \) is a weakly efficient solution.

(2) \( \lambda \neq 0 \) or \( (\lambda + \mu \geq 0 \) and \( \hat{s} > 0 \)) and \( \hat{x} \) is a unique solution, then \( \hat{x} \) is a strictly efficient solution.

(3):

(a) \( \lambda > 0 \), then \( \hat{x} \) is an efficient solution.

(b) \( \lambda + \mu > 0 \) and \( \hat{s} > 0 \), then \( \hat{x} \) is an efficient solution.
Theorem 5.3. If $(\hat{x}, \hat{s})$ is an optimal solution of SOP (5.1) with $\lambda + \mu > 0$ and $\hat{s} > 0$, then $\hat{x}$ is a properly efficient solution of the MOP.

Remark 5.4. Theorem 5.3 extends the result obtained by Theorem 4.1 in [11].

Using Theorem 3.6, we now turn to the problem of showing that properly efficient solutions of the MOP are optimal solutions of SOP (5.1) for appropriate choices of $\alpha$ and $\mu$.

Theorem 5.5. Let $\hat{x}$ be a properly efficient solution of the MOP. Then, there are $\alpha < \infty, \hat{s}, \lambda, \hat{\mu}$ with $\hat{\mu}_i < \infty$ for all $i$, such that $(\hat{x}, \hat{s})$ is an optimal solution of SOP (5.1) for all $\mu \in \mathbb{R}^p, \mu \geq \hat{\mu}$.

In SOP (4.1), we use slack variables. Insertion of these variables results in obtaining some information about proper efficiency. The negative sign of the weight coefficients of the slack variables $s$ in objective functions allows $s$ to be as large as possible. On the other hand, the constraints limit the magnitude of slack variables. Besides the additional constraints in SOP (4.1) are inflexible. Thus, we decide to address the inflexibility of the constraints. In SOP (5.1), the constraints are allowed to be violated using the variable $s$ and these violations are penalized in the objective function with positive weight coefficients $\mu_i$ for each violation. The idea of flexible constraints is that, in some problems, a small deviation in constraints may result in the attainment of a better solution. The SOP (3.1) has the advantages of two SOPs (4.1) and (5.1).

In the next section we investigated approximate solutions and obtained some necessary and sufficient conditions for $\varepsilon$-(weakly, properly)efficient solutions via approximate solutions of SOPs (3.1), (4.1) and (5.1).

6 $\varepsilon$-(weakly, properly)efficient solutions

In this section, we are going to characterize the approximate (weakly, properly) efficient solutions of the general multi-objective optimization problem (2.1) using the SOPs (3.1), (4.1) and (5.1).

6.1 Some necessary/sufficient conditions for problem (3.1)

In this subsection, we consider SOP (3.1) and provide some necessary/sufficient conditions for characterizing (weakly, properly)efficient solutions of the MOP through SOP (3.1).

The following theorem provides sufficient conditions for $\varepsilon$-efficiency.
Theorem 6.1. Assume \( \varepsilon \in \mathbb{R}_\geq^p \).

i) Let \( \varepsilon \leq \sum_{i=1}^{p} (\lambda_i + \gamma_i) \varepsilon_i, \) and \( \lambda + \gamma > 0 \). If \((\hat{x}, \hat{s}^+, \hat{s}^-)\) is an \( \varepsilon \)-optimal solution of SOP (3.1), then \( \hat{x} \) is an \( \varepsilon \)-efficient solution of the MOP.

ii) If \((\hat{x}, \hat{s}^+, \hat{s}^-)\) is an \( \varepsilon \)-optimal solution of SOP (3.1) where \( 0 \leq \varepsilon < \hat{s}^-, \lambda + \mu > 0 \) and \( \varepsilon \leq \sum_{i=1}^{p} (\lambda_i + \mu_i) \varepsilon_i, \) then \( \hat{x} \) is an \( \varepsilon \)-efficient solution of the MOP.

Proof. i) Let \( \hat{x} \notin X_{\varepsilon E} \). Then, there exists \( x \in X \) such that \( f(x) \leq f(\hat{x}) - \varepsilon \). In other words, \( f_i(x) \leq f_i(\hat{x}) - \varepsilon_i \) for all \( i \) and for some index, namely \( j \), the inequality is strict. So,

\[
f_i(x) + \varepsilon_i + \hat{s}_i^+ - \hat{s}_i^- \leq f_i(\hat{x}) + \hat{s}_i^+ - \hat{s}_i^- \leq \alpha_i, \quad i \neq j.
\]

Also, we have

\[
f_j(x) + \varepsilon_j + \hat{s}_j^+ - \hat{s}_j^- + \nu \leq f_j(\hat{x}) + \hat{s}_j^+ - \hat{s}_j^- \leq \alpha_j,
\]

for some \( \nu > 0 \). Therefore, if we define \( s_i^+ = \hat{s}_i^+ + \varepsilon_i, \forall i \neq j \) and \( s_j^+ = \hat{s}_j^+ + \varepsilon_j + \nu \), then \((x, s^+, s^-)\) is feasible for SOP (3.1). So, we have:

\[
\sum_{i=1}^{p} \lambda_i f_i(x) - \sum_{i=1}^{p} \gamma_i \hat{s}_i^+ + \sum_{i=1}^{p} \mu_i \hat{s}_i^- = \sum_{i=1}^{p} \lambda_i f_i(x) - \sum_{i=1}^{p} \gamma_i (\hat{s}_i^+ + \varepsilon_i) - \gamma_j \nu + \sum_{i=1}^{p} \mu_i \hat{s}_i^- \\
= \sum_{i=1}^{p} \lambda_i f_i(x) - \sum_{i=1}^{p} \gamma_i \hat{s}_i^+ + \sum_{i=1}^{p} \mu_i \hat{s}_i^- - \sum_{i=1}^{p} \gamma_i \varepsilon_i - \gamma_j \nu
\]

and since \( \lambda_j + \gamma_j > 0 \),

\[
\leq \sum_{i=1}^{p} \lambda_i f_i(\hat{x}) - \varepsilon_i - \sum_{i=1}^{p} \gamma_i \hat{s}_i^+ + \sum_{i=1}^{p} \mu_i \hat{s}_i^- - \sum_{i=1}^{p} \gamma_i \varepsilon_i \\
= \sum_{i=1}^{p} \lambda_i f_i(\hat{x}) - \sum_{i=1}^{p} \gamma_i \hat{s}_i^+ + \sum_{i=1}^{p} \mu_i \hat{s}_i^- - \sum_{i=1}^{p} (\lambda_i + \gamma_i) \varepsilon_i \\
\leq \sum_{i=1}^{p} \lambda_i f_i(\hat{x}) - \sum_{i=1}^{p} \gamma_i \hat{s}_i^+ + \sum_{i=1}^{p} \mu_i \hat{s}_i^- - \varepsilon,
\]

which is a contradiction.
Let $\hat{x} \notin X_{E}$. So,

$$f_i(x) + \varepsilon_i + s_i^+ - s_i^- \leq \alpha_i, \quad \forall i \neq j.$$ 

It is easy to show that there is some $\nu > 0$ such that

$$f_j(x) + \varepsilon_j + s_j^+ - s_j^- + \nu \leq \alpha_j,$$

and

$$s_j^- - \varepsilon_j - \nu \geq 0.$$

Now, define $s_i^- = s_i^- - \varepsilon_i, \forall i \neq j$ and $s_j^- = s_j^- - \varepsilon_j - \nu$. So, $(\hat{x}, \hat{s}^+, \hat{s}^-)$ is feasible for SOP (3.1). With a similar calculation like part (i) we will have:

$$\sum_{i=1}^{p} \lambda_i f_i(x) - \sum_{i=1}^{p} \gamma_i s_i^+ + \sum_{i=1}^{p} \mu_i s_i^- < \sum_{i=1}^{p} \lambda_i f_i(\hat{x}) - \sum_{i=1}^{p} \gamma_i \hat{s}_i^+ + \sum_{i=1}^{p} \mu_i \hat{s}_i^- - \epsilon,$$

which is a contradiction.

The following theorem provides some sufficient conditions for $\varepsilon$-(weakly) efficient solutions.

**Theorem 6.2.**

1) Let $\varepsilon \leq \sum_{i=1}^{p} (\lambda_i + \gamma_i) \varepsilon_i$, and $\lambda + \gamma \geq 0$. If $(\hat{x}, \hat{s}^+, \hat{s}^-)$ is an $\varepsilon$-optimal solution of SOP (3.1), then $\hat{x}$ is a weakly $\varepsilon$-efficient solution of the MOP.

2) If $(\hat{x}, \hat{s}^+, \hat{s}^-)$ is an (strict) $\varepsilon$-optimal solution of SOP (3.1) where $0 \leq \varepsilon < \hat{s}^-$, $\lambda + \mu \geq 0$ and $\varepsilon \leq \sum_{i=1}^{p} (\lambda_i + \mu_i) \varepsilon_i$, then $\hat{x}$ is a ($\varepsilon$-efficient) weakly $\varepsilon$-efficient solution of the MOP.

3) Let $\varepsilon < \sum_{i=1}^{p} (\lambda_i + \gamma_i) \varepsilon_i$, and $\lambda + \gamma \geq 0$. If $(\hat{x}, \hat{s}^+, \hat{s}^-)$ is an $\varepsilon$-optimal solution of SOP (3.1), then $\hat{x}$ is an $\varepsilon$-efficient solution of the MOP.

4) If $(\hat{x}, \hat{s}^+, \hat{s}^-)$ is an $\varepsilon$-optimal solution of SOP (3.1) where $0 \leq \varepsilon < \hat{s}^-$, $\lambda + \mu \geq 0$ and $\varepsilon < \sum_{i=1}^{p} (\lambda_i + \mu_i) \varepsilon_i$, then $\hat{x}$ is an $\varepsilon$-efficient solution of the MOP.

5) Let $\varepsilon \leq \sum_{i=1}^{p} (\lambda_i + \gamma_i) \varepsilon_i$, and $\lambda + \gamma \geq 0$. If $(\hat{x}, \hat{s}^+, \hat{s}^-)$ is a strict $\varepsilon$-optimal solution of SOP (3.1), then $\hat{x}$ is an $\varepsilon$-efficient solution of the MOP.

**Proof.** The proofs of all parts are similar to Theorem 6.1 and will be omitted here.
Now, we utilize SOP (3.1) to provide a sufficient condition for \( \varepsilon \)-proper efficiency.

**Theorem 6.3.** Let \( \varepsilon \in \mathbb{R}_+^p \) and suppose \( 0 \leq \varepsilon \leq \sum_{i=1}^p (\lambda_i + \gamma_i) \varepsilon_i, \gamma > 0 \) and \( \sum_{i=1}^p \lambda_i = 1 \). If \((\hat{x}, \hat{s}^+, \hat{s}^-)\) is an \( \varepsilon \)-optimal solution of SOP (3.1) with \( f_i(\hat{x}) + \hat{s}_i^+ - \hat{s}_i^- < \alpha_i \) for all \( i \), then \( \hat{x} \) is an \( \varepsilon \)-properly efficient solution to the MOP.

**Proof.** From Theorem 6.1 it follows that \( \hat{x} \in X_{\varepsilon E} \). Now, we prove that \( \hat{x} \) is an \( \varepsilon \)-properly efficient solution. By contradiction, assume \( \hat{x} \notin X_{\varepsilon PE} \). Then there exists sequence \( \{M_\beta\} \) of positive scalars such that \( \lim_{\beta \to \infty} M_\beta = \infty \) and for each \( M_\beta \) there is an \( x_\beta \in X \) and an index \( i \in \{1, 2, \ldots, p\} \) with \( f_i(x_\beta) < f_i(\hat{x}) - \varepsilon_i \) and

\[
\frac{f_i(\hat{x}) - f_i(x_\beta) - \varepsilon_i}{f_j(x_\beta) - f_j(\hat{x}) + \varepsilon_j} > M_\beta, \tag{6.1}
\]

for each \( j \neq i \) with \( f_j(\hat{x}) - \varepsilon_j < f_j(x_\beta) \). Without loss of generality, we can consider an unbounded subsequence of \( \{M_\beta\} \) such that index \( i \) and the set \( Q = \{j : f_j(x_\beta) > f_j(\hat{x}) - \varepsilon_j\} \) is constant for each \( \beta \). Now choose \( j \in \{1, 2, \ldots, p\} \). We have two possible cases:

**Case 1:** If \( j \notin Q \), then,

\[
f_j(x_\beta) \leq f_j(\hat{x}) - \varepsilon_j < \alpha_j - \hat{s}_j^+ + \hat{s}_j^- - \varepsilon_j,
\]

\[
\Rightarrow f_j(x_\beta) + \hat{s}_j^+ - \hat{s}_j^- + \varepsilon_j < \alpha_j.
\]

So, there is some \( \nu_{j\beta} > 0 \) such that

\[
f_j(x_\beta) + \hat{s}_j^+ - \hat{s}_j^- + \varepsilon_j + \nu_{j\beta} \leq \alpha_j.
\]

**Case 2:** If \( j \in Q \), then \( f_j(x_\beta) > f_j(\hat{x}) - \varepsilon_j \). Since \( f(X) \) is bounded, by inequality (6.1), we have:

\[
\lim_{\beta \to \infty} f_j(x_\beta) = f_j(\hat{x}) - \varepsilon_j < \alpha_j - \hat{s}_j^+ + \hat{s}_j^- - \varepsilon_j.
\]

Hence, there exists \( \beta_0 > 0 \) such that

\[
f_j(x_\beta) + \hat{s}_j^+ - \hat{s}_j^- + \varepsilon_j < \alpha_j \quad \forall \beta \geq \beta_0.
\]

Also, for all \( \beta \geq \beta_0 \) there is \( \nu_{j\beta} > 0 \) such that

\[
f_j(x_\beta) + \hat{s}_j^+ - \hat{s}_j^- + \varepsilon_j + \nu_{j\beta} \leq \alpha_j.
\]
Now, define \( s^+_{\beta_j} = \hat{s}_j^+ + \varepsilon_j + \nu_{\beta j} \) for all \( 1 \leq j \leq p \) and \( \beta \geq \beta_0 \). So, 
\( (x_\beta, s^+_{\beta_j}, \hat{s}^-) \) is feasible for SOP (3.1) for all \( \beta \geq \beta_0 \). On the other hand, when \( j \in Q \) and \( \beta \geq \beta_0 \), we have
\[
\lim_{\beta \to \infty} f_j(x_\beta) = f_j(\hat{x}) - \varepsilon_j,
\]
\[
\Rightarrow \lim_{\beta \to \infty} (-f_j(x_\beta) + f_j(\hat{x}) - \varepsilon_j + \gamma_1 \nu_{\beta j}) = \gamma_1 \nu_{\beta j} > 0.
\]
So, there exists some \( \beta_0 > \beta_0 \) such that for all \( \beta \geq \beta_0 \)
\[
f_j(x_\beta) < f_j(\hat{x}) - \varepsilon_j + \gamma_1 \nu_{\beta j}. \quad (a)
\]
If \( j \notin Q \), we also have:
\[
f_j(x_\beta) \leq f_j(\hat{x}) - \varepsilon_j. \quad (b)
\]
Now, select some \( \beta^* > \beta_0 \) and put \( (\vec{x}, \vec{s}^+, \vec{s}^-) = (x_\beta^*, s_{\beta^*}, \hat{s}^-) \). By applying (a) and (b), we can write:
\[
\sum_{i=1}^{p} \lambda_i f_i(\vec{x}) - \sum_{i=1}^{p} \gamma_i \hat{s}_i^+ + \sum_{i=1}^{p} \mu_i \hat{s}_i^- = \sum_{i=1}^{p} \lambda_i f_i(\vec{x}) - \sum_{i=1}^{p} \gamma_i \hat{s}_i^+ - \sum_{i=1}^{p} \mu_i \hat{s}_i^- - \sum_{i \in Q} \lambda_i f_i(\vec{x}) - \sum_{i \in Q} \gamma_i \hat{s}_i^+ + \sum_{i \in Q} \mu_i \hat{s}_i^- - \sum_{i \notin Q} \lambda_i f_i(\vec{x}) - \sum_{i \notin Q} \gamma_i \hat{s}_i^+ - \sum_{i \notin Q} \mu_i \hat{s}_i^-
\]
\[
\leq \sum_{i \in Q} \lambda_i (f_i(\vec{x}) - \varepsilon_i) + \sum_{i \in Q} \lambda_i (f_i(\vec{x}) - \varepsilon_i + \gamma_1 \nu_{\beta j}) - \sum_{i \in Q} \gamma_i \hat{s}_i^+ + \sum_{i \in Q} \mu_i \hat{s}_i^- - \sum_{i \in Q} \gamma_i \hat{s}_i^+ - \sum_{i \in Q} \gamma_i \hat{s}_i^- + \sum_{i \in Q} \mu_i \hat{s}_i^- + \sum_{i \in Q} \lambda_i (\lambda_i + \gamma_i) \varepsilon_i + \gamma_1 \nu_{\beta j} \sum_{i \in Q} \lambda_i - \sum_{i \in Q} \gamma_i \nu_{\beta j} \lambda_i,
\]
and since \( \sum_{i \in Q} \lambda_i \leq 1 \),
\[
< \sum_{i=1}^{p} \lambda_i f_i(\vec{x}) - \sum_{i=1}^{p} \gamma_i \hat{s}_i^+ + \sum_{i=1}^{p} \mu_i \hat{s}_i^- - \sum_{i=1}^{p} (\lambda_i + \gamma_i) \varepsilon_i
\]
<p>\[ \sum_{i=1}^{p} \lambda_i f_i(\hat{x}) - \sum_{i=1}^{p} \gamma_i \hat{s}_i^+ + \sum_{i=1}^{p} \mu_i \hat{s}_i^- - \epsilon \]

which is a contradiction and the proof is completed. \qed</p>

The next subsection deals with SOP (4.1) to provide some conditions for approximate solutions.

### 6.2 Some necessary/sufficient conditions for problem (4.1)

Consider SOP (4.1). Most of the results in this subsection are obtained from Subsection 6.1, indirectly. It is easy to show that part (i) of Theorem 6.1 and parts 1, 3 and 5 of Theorem 6.2 is true for SOP (4.1).

**Remark 6.4.** Considering SOP (4.1) and putting \( \gamma = 0, \alpha = \infty \), Theorem 6.1 reduces to Proposition 3.1 (i) in [14]. If \( \gamma = 0, \lambda_k = 1, \lambda_i \neq k = 0, \alpha_k = \infty \) then, Proposition 3.2 (i) in [14], if \( \gamma = 0, \alpha = f(x^0) \) where \( x^0 \) is an arbitrary feasible point, then Proposition 3.3 (i) in [14], and finally, if \( \lambda_k = 1, \lambda_i \neq k = 0, \alpha_k = \infty \), then Theorem 4.4 in [19] will be obtained.

**Remark 6.5.** Similar to Remark 6.4, we can obtain the conditions of \( \varepsilon \)-weakly efficient of the techniques presented in Table 1 as special cases of SOP (4.1) with Theorem 6.6.

As a special case of Theorem 6.3, the following theorem utilizes SOP (4.1) to provide a sufficient condition for \( \varepsilon \)-proper efficiency.

**Theorem 6.6.** Let \( \varepsilon \in \mathbb{R}^p_{\geq} \) and suppose \( 0 \leq \epsilon \leq \sum_{i=1}^{p} (\lambda_i + \gamma_i) \varepsilon_i, \gamma_i > 0, \forall i \) and \( \sum_{i=1}^{p} \lambda_i = 1 \). If \( (\hat{x}, \hat{s}) \) is an \( \epsilon \)-optimal solution of SOP (4.1) with \( f_i(\hat{x}) + \hat{s}_i < \alpha_i \) for all \( i \), then \( \hat{x} \) is an \( \epsilon \)-properly efficient solution to the MOP.

In the next subsection, the SOP (5.1) will be investigated.

### 6.3 Some necessary/sufficient conditions for problem (5.1)

In this subsection, we restrict our attention to SOP (5.1). We provide some necessary/sufficient conditions for characterizing \( \varepsilon \)-(weakly, properly) efficient points of MOP via SOP (5.1).

The following theorem provides a sufficient condition for \( \varepsilon \)-weak efficiency.
Theorem 6.7. Suppose \( \varepsilon \in \mathbb{R}_+^p \) and \( \varepsilon \leq \sum_{i=1}^p \lambda_i \varepsilon_i \). If \((\hat{x}, \hat{s})\) is an \( \varepsilon \)-optimal solution of SOP (5.1), then, \( \hat{x} \in X_{\varepsilon \text{WE}} \).

Proof. The proof is easily obtained and will be omitted here. \( \square \)

Remark 6.8. Theorem 6.7 extends Theorem 3.2 in [18], which is obtained by \( \lambda = e_k \) and \( \mu_k = 0 \).

SOP (5.1) satisfies in part (ii) of Theorem 6.1 and parts 2 and 4 of Theorem 6.2. Additionally, in the following theorem, we present more necessary conditions for \( \varepsilon \)-efficiency of the MOP.

Theorem 6.9. Given \( \varepsilon \in \mathbb{R}_+^p \).

1) If \((\hat{x}, \hat{s})\) is a strictly \( \varepsilon \)-optimal solution of SOP (5.1) and \( \varepsilon \leq \sum_{i=1}^p \lambda_i \varepsilon_i \), then \( \hat{x} \in X_{\varepsilon \text{E}} \).

2) If \((\hat{x}, \hat{s})\) is an \( \varepsilon \)-optimal solution of SOP (5.1) and \( \varepsilon < \sum_{i=1}^p \lambda_i \varepsilon_i \), then \( \hat{x} \in X_{\varepsilon \text{E}} \).

Proof. 1) Suppose \( \hat{x} \) is not \( \varepsilon \)-efficient. Then, there exists some \( x \in X \) with \( f_i(x) \leq f_i(\hat{x}) - \varepsilon_i \). It is a simple matter to show that \((x, \hat{s})\) is feasible for SOP (5.1). So,

\[
\sum_{i=1}^p \lambda_i f_i(x) + \sum_{i=1}^p \mu_i \hat{s}_i + \sum_{i=1}^p \lambda_i \varepsilon_i \leq \sum_{i=1}^p \lambda_i f_i(\hat{x}) + \sum_{i=1}^p \mu_i \hat{s}_i + \varepsilon_i \leq \sum_{i=1}^p \lambda_i f_i(\hat{x}) + \sum_{i=1}^p \mu_i \hat{s}_i,
\]

a contradiction.

2) Similar to Part (1). \( \square \)

Remark 6.10. By letting \( \lambda = e_k \) and \( \mu_k = 0 \), Theorem 6.9 reduces to Theorem 3.7 in [18].

The following theorem utilizes SOP (5.1) to provide a sufficient condition for \( \varepsilon \)-proper efficiency.

Theorem 6.11. Suppose \( \varepsilon \in \mathbb{R}_+^p \) and \( \varepsilon \leq \sum_{i=1}^p \lambda_i \varepsilon_i \). If \((\hat{x}, \hat{s})\) is an \( \varepsilon \)-optimal point of SOP (5.1) with \( \lambda + \mu > 0 \) and \( \hat{s} > 0 \), then \( \hat{x} \in X_{\varepsilon \text{PE}} \).

Proof. Let us first prove that \( \hat{x} \) is \( \varepsilon \)-efficient. Without loss of generality, assume \( \hat{s}_i = \max\{0, f_i(\hat{x}) - \alpha_i\} \). Since \( \hat{s} > 0 \), it follows that \( \hat{s}_i = f_i(\hat{x}) - \alpha_i > 0, \forall i \). Let \( \hat{x} \) not be an \( \varepsilon \)-efficient solution. Therefore, there exists \( x \in X \) such that \( f_i(x) \leq f_i(\hat{x}) - \varepsilon_i, \forall i \) and for at least one index...
j, \ f_j(x) < f_j(\hat{x}) - \varepsilon_j. \text{ Define } s_i = \max\{0, f_i(x) - \alpha_i\} \text{ for all } i \neq j. \text{ Since } f_j(x) - \hat{s}_j < \alpha_j, \text{ there is some } \nu > 0 \text{ such that } f_j(x) - \hat{s}_j + \nu \leq \alpha_j \text{ and } s_j = \hat{s}_j - \nu > 0. \text{ Put } s_i = \hat{s}_i, \forall i \neq j \text{ and } s_j = \hat{s}_j + \nu. \text{ Obviously, } (x, s) \text{ is feasible for SOP (5.1) and } s \leq \hat{s}, \text{ specially, } s_j < \hat{s}_j. \text{ Because } \lambda_j + \mu_j > 0,

\lambda_j f_j(x) + \lambda_j \varepsilon_j + \mu_j s_j < \lambda_j f_j(\hat{x}) + \mu_j \hat{s}_j,

and for all \ i \neq j,

\lambda_i f_i(x) + \lambda_i \varepsilon_i + \mu_i s_i \leq \lambda_i f_i(\hat{x}) + \mu_i \hat{s}_i.

So,

\sum_{i=1}^{p} \lambda_i f_i(x) + \sum_{i=1}^{p} \mu_i s_i + \sum_{i=1}^{p} \lambda_i \varepsilon_i < \sum_{i=1}^{p} \lambda_i f_i(\hat{x}) + \sum_{i=1}^{p} \mu_i \hat{s}_i,

\Rightarrow \sum_{i=1}^{p} \lambda_i f_i(x) + \sum_{i=1}^{p} \mu_i s_i + \epsilon < \sum_{i=1}^{p} \lambda_i f_i(\hat{x}) + \sum_{i=1}^{p} \mu_i \hat{s}_i,

a contradiction to \epsilon-optimality of (\hat{x}, \hat{s}).

To show that \hat{x} is an \epsilon-properly efficient point, by our assumption \hat{x} is an \epsilon-optimal point of the following problem:

\begin{align*}
\min & \sum_{i=1}^{p} \lambda_i f_i(x) + \sum_{i=1}^{p} \mu_i (f_i(x) - \alpha_i) \\
& \quad \text{subject to } f_i(x) > \alpha_i, \forall i \in X
\end{align*}

This problem can be rewritten as follows:

\begin{align*}
\min & \sum_{i=1}^{p} (\lambda_i + \mu_i) f_i(x) - \sum_{i=1}^{p} \mu_i \alpha_i \\
& \quad \text{subject to } f_i(x) > \alpha_i, \forall i \in X
\end{align*}

Now, since \lambda + \mu > 0, employing Theorem 2 in [31], yields \hat{x} as an \epsilon-properly efficient point with added constraints \ f_i(x) > \alpha_i, \forall i. \text{ Then from a result similar to Lemma 3.3, we conclude that } \hat{x} \in X_{\varepsilon PE}. \quad \square

**Remark 6.12.** Letting \lambda = e_k \text{ and } \mu_k = 0, Theorem 6.11 reduces to Theorem 3.14 in [18].

In the following theorem, a necessary condition for \epsilon-properly efficient solutions of MOP (2.1) is obtained. This theorem extends Theorem 3.21 in [18].

**Theorem 6.13.** Suppose \hat{x} \in X_{\varepsilon PE} \text{ and } \sum_{i=1}^{p} \lambda_i = 1. \text{ Then, there are
\[ \alpha < \infty, \hat{s}, \hat{\mu} \] with \( \hat{\mu}_i < \infty \), such that \((\hat{x}, \hat{s})\) is an \( \epsilon \)-optimal point of SOP (5.1) with \( \epsilon = \sum_{i=1}^{p}(\lambda_i + \mu_i)\epsilon_i \) for all \( \mu \geq \hat{\mu} \).

**Proof.** Let \( \alpha_i = f_i(\hat{x}) - \epsilon_i \) and \( \hat{s}_i = \epsilon_i \) for all \( i \). Since \( \hat{x} \in X_{\epsilon PE} \), there is \( M > 0 \) such that, for all \( x \in X \) and for all \( i \) with \( f_i(x) < f_i(\hat{x}) - \epsilon_i \), there exists \( j \neq i \) such that \( f_j(\hat{x}) - \epsilon_j < f_j(x) \) and

\[ (f_i(\hat{x}) - \epsilon_i - f_i(x)/(f_j(x) - f_j(\hat{x}) + \epsilon_j)) < M. \]

We define \( \hat{\mu}_i = M, \forall i \).

Let \( x \in X \) and \( s \) be such that:

\[ s_i = \max\{0, f_i(x) - \alpha_i\} = \max\{0, f_i(x) - f_i(\hat{x}) + \epsilon_i\}, \]

the smallest possible value it can take. We need to show that

\[ \sum_{i=1}^{p} \lambda_i f_i(x) + \sum_{i=1}^{p} \mu_i s_i \geq \sum_{i=1}^{p} \lambda_i f_i(\hat{x}) + \sum_{i=1}^{p} \mu_i \hat{s}_i - \epsilon. \]

Let \( k \in \{1, 2, \ldots, p\} \). We have two possible cases:

**Case 1)** If \( f_k(x) \geq f_k(\hat{x}) - \epsilon_k \), then

\[ f_k(x) + \sum_{i=1}^{p} \mu_i s_i \geq f_k(\hat{x}) - \epsilon_k. \quad (a) \]

**Case 2)** If \( f_k(x) < f_k(\hat{x}) - \epsilon_k \), set \( I^* = \{ i : f_i(x) > f_i(\hat{x}) - \epsilon_i \} \). The set \( I^* \neq \emptyset \) because \( \hat{x} \in X_{\epsilon PE} \). We can write:

\[
\begin{align*}
  f_k(x) + \sum_{i=1}^{p} \mu_i s_i &= f_k(x) + \sum_{i=1}^{p} \hat{\mu}_i \max\{0, f_i(x) - f_i(\hat{x}) + \epsilon_i\} \\
  &= f_k(x) + \sum_{i \in I^*} \hat{\mu}_i (f_i(x) - f_i(\hat{x}) + \epsilon_i).
\end{align*}
\]

Since \( \hat{x} \in X_{\epsilon PE} \) and \( f_k(x) < f_k(\hat{x}) - \epsilon_k \), there is \( k^* \in I^* \) such that \( f_k(\hat{x}) - f_k(x) - \epsilon_k < f_{k^*}(\hat{x}) - f_{k^*}(x) + \epsilon_{k^*} < M \). So, we have

\[
\begin{align*}
  f_k(x) + \sum_{i \in I^*} \hat{\mu}_i (f_i(x) - f_i(\hat{x}) + \epsilon_i) &\geq f_k(x) + \hat{\mu}_{k^*} (f_{k^*}(x) - f_{k^*}(\hat{x}) + \epsilon_{k^*}) \\
  &> f_k(x) + \frac{f_k(\hat{x}) - f_k(x) - \epsilon_k}{f_{k^*}(\hat{x}) - f_{k^*}(x) + \epsilon_{k^*}} (f_{k^*}(x) - f_{k^*}(\hat{x}) + \epsilon_{k^*}) = f_k(\hat{x}) - \epsilon_k. \quad (b)
\end{align*}
\]
Applying (a) and (b) we have:

\[
\sum_{i=1}^{p} \lambda_i f_i(x) + \sum_{i=1}^{p} \lambda_i (\hat{\mu}_i s_i) \geq \sum_{i=1}^{p} \lambda_i f_i(\hat{x}) - \sum_{i=1}^{p} \lambda_i \epsilon_i,
\]

and since \(\sum_{i=1}^{p} \lambda_i = 1\), we have

\[
\sum_{i=1}^{p} \lambda_i f_i(x) + \sum_{i=1}^{p} \hat{\mu}_i s_i \geq \sum_{i=1}^{p} \lambda_i f_i(\hat{x}) + \sum_{i=1}^{p} \mu_i \epsilon_i - \epsilon
\]

\[
= \sum_{i=1}^{p} \lambda_i f_i(\hat{x}) + \sum_{i=1}^{p} \mu_i \hat{s}_i - \epsilon.
\]

The above inequality is true for all \(\mu \geq \hat{\mu}\) and the proof is completed.

\[\square\]

7 Conclusion

In this paper, a general form of scalarization technique for solving multiobjective optimization has been proposed. It is shown that some well-known methods such as the weighted sum method, the \(\epsilon\)-constraint method, the Benson method, the hybrid method and the elastic \(\epsilon\)-constraint method can be seen as special cases and can be subsumed under this general problem. With this problem, we are able to prove some results on (weakly, properly) efficiency of optimal solutions. Additionally, we relax the constraints and obtain some other results. On the other hand, by considering the relationships between \(\epsilon \in \mathbb{R}^p_{\geq}\) and \(\epsilon \in \mathbb{R}_{\geq}\) for the multiobjective and presented problems, we derived necessary and/or sufficient conditions for \(\epsilon\)-(weakly, properly) efficient solutions of the MOP (2.1). Our results extend some results obtained by Ehrgott and Ruzika [11], Engau and Wiecek [14], Ghaznavi and Khorram [18] and Ghaznavi et al. [19]. In Tables 2 and 3, we summarize some of the results obtained for these problems.
Theorem 5.2 (1)

Theorem 5.2 (2)

Theorem 5.2 (3)

Theorem 5.2 (3,a)

Theorem 5.3

Table 2: Summary of results obtained for SOPs (3.1), (4.1) and (5.1)

<table>
<thead>
<tr>
<th>Scalarization method</th>
<th>Parameters</th>
<th>Implication for $\hat{x}$</th>
<th>Reference</th>
</tr>
</thead>
<tbody>
<tr>
<td>SOP (3.1)</td>
<td>$\lambda + \gamma \geq 0 \text{ or } (\lambda + \mu \geq 0, \delta &gt; 0)$</td>
<td>$\hat{x} \in X_{wE}$</td>
<td>Theorem 3.2 (i)</td>
</tr>
<tr>
<td>SOP (3.1)</td>
<td>$\lambda + \gamma \geq 0 \text{ or } (\lambda + \mu \geq 0, \delta &gt; 0)$ and $\hat{x}$ is a unique solution</td>
<td>$\hat{x} \in X_{wE}$</td>
<td>Theorem 3.2 (ii)</td>
</tr>
<tr>
<td>SOP (3.1)</td>
<td>$\lambda + \gamma \geq 0 \text{ or } (\lambda + \mu \geq 0, \delta &gt; 0)$ and $\hat{x}$ is a unique solution</td>
<td>$\hat{x} \in X_{wE}$</td>
<td>Theorem 3.2 (iii)</td>
</tr>
<tr>
<td>SOP (4.1)</td>
<td>$\lambda + \gamma &gt; 0$</td>
<td>$\hat{x} \in X_{E}$</td>
<td>Theorem 4.2 (1)</td>
</tr>
<tr>
<td>SOP (4.1)</td>
<td>$\lambda + \gamma \geq 0$</td>
<td>$\hat{x} \in X_{wE}$</td>
<td>Theorem 4.2 (2)</td>
</tr>
<tr>
<td>SOP (4.1)</td>
<td>$\lambda + \gamma \geq 0 \text{ and } \hat{x}$ is a unique solution</td>
<td>$\hat{x} \in X_{E}$</td>
<td>Theorem 4.2 (3)</td>
</tr>
<tr>
<td>SOP (5.1)</td>
<td>$\lambda \neq 0$ or ($\lambda + \mu \geq 0$ and $\delta &gt; 0$)</td>
<td>$\hat{x} \in X_{wE}$</td>
<td>Theorem 5.2 (1)</td>
</tr>
<tr>
<td>SOP (5.1)</td>
<td>$\lambda \neq 0$ or ($\lambda + \mu \geq 0$ and $\delta &gt; 0$) and $\hat{x}$ is a unique solution</td>
<td>$\hat{x} \in X_{wE}$</td>
<td>Theorem 5.2 (2)</td>
</tr>
<tr>
<td>SOP (5.1)</td>
<td>$\lambda &gt; 0$</td>
<td>$\hat{x} \in X_{E}$</td>
<td>Theorem 5.2 (3,a)</td>
</tr>
<tr>
<td>SOP (5.1)</td>
<td>$\lambda + \mu &gt; 0, \delta &gt; 0$</td>
<td>$\hat{x} \in X_{wE}$</td>
<td>Theorem 5.3</td>
</tr>
</tbody>
</table>

Table 3: Sufficient conditions for generating $\varepsilon$-(weakly,properly) efficient points of SOPs (3.1), (4.1) and (5.1)

<table>
<thead>
<tr>
<th>Scalarization method</th>
<th>Parameters</th>
<th>Results</th>
<th>Reference</th>
</tr>
</thead>
<tbody>
<tr>
<td>SOP (3.1)</td>
<td>$\lambda + \gamma &gt; 0, \varepsilon \leq \sum_{i=1}^{p} (\lambda_{i} + \gamma_{i}) \varepsilon_{i}$</td>
<td>$\varepsilon$-opt. $\rightarrow$ $\varepsilon$-eff.</td>
<td>Theorem 6.1 (i)</td>
</tr>
<tr>
<td>SOP (3.1)</td>
<td>$\lambda + \mu &gt; 0, \varepsilon \leq \sum_{i=1}^{p} (\lambda_{i} + \gamma_{i}) \varepsilon_{i}, \varepsilon &lt; \delta -$</td>
<td>$\varepsilon$-opt. $\rightarrow$ $\varepsilon$-eff.</td>
<td>Theorem 6.1 (ii)</td>
</tr>
<tr>
<td>SOP (3.1)</td>
<td>$\lambda + \gamma &gt; 0, \varepsilon \leq \sum_{i=1}^{p} (\lambda_{i} + \gamma_{i}) \varepsilon_{i}$</td>
<td>$\varepsilon$-opt. $\rightarrow$ $\varepsilon$-weak eff.</td>
<td>Theorem 6.2 (1)</td>
</tr>
<tr>
<td>SOP (3.1)</td>
<td>$\lambda + \mu &gt; 0, \varepsilon \leq \sum_{i=1}^{p} (\lambda_{i} + \gamma_{i}) \varepsilon_{i}, \varepsilon &lt; \delta -$</td>
<td>$\varepsilon$-opt. $\rightarrow$ $\varepsilon$-weak eff.</td>
<td>Theorem 6.2 (2)</td>
</tr>
<tr>
<td>SOP (3.1)</td>
<td>$\lambda + \gamma &gt; 0, \varepsilon \leq \sum_{i=1}^{p} (\lambda_{i} + \gamma_{i}) \varepsilon_{i}$</td>
<td>$\varepsilon$-opt. $\rightarrow$ $\varepsilon$-eff.</td>
<td>Theorem 6.2 (3)</td>
</tr>
<tr>
<td>SOP (3.1)</td>
<td>$\lambda + \mu &gt; 0, \varepsilon \leq \sum_{i=1}^{p} (\lambda_{i} + \gamma_{i}) \varepsilon_{i}, \varepsilon &lt; \delta -$</td>
<td>$\varepsilon$-opt. $\rightarrow$ $\varepsilon$-eff.</td>
<td>Theorem 6.2 (4)</td>
</tr>
<tr>
<td>SOP (3.1)</td>
<td>$\gamma &gt; 0, \varepsilon \leq \sum_{i=1}^{p} (\lambda_{i} + \gamma_{i}) \varepsilon_{i}, \sum_{i=1}^{p} \lambda_{i} = 1$</td>
<td>$\varepsilon$-opt. $\rightarrow$ $\varepsilon$-pro. eff.</td>
<td>Theorem 6.3</td>
</tr>
<tr>
<td>SOP (4.1)</td>
<td>$\gamma &gt; 0, \varepsilon \leq \sum_{i=1}^{p} (\lambda_{i} + \gamma_{i}) \varepsilon_{i}, \gamma \geq 0$</td>
<td>$\varepsilon$-opt. $\rightarrow$ $\varepsilon$-eff.</td>
<td>Theorem 6.6</td>
</tr>
<tr>
<td>SOP (5.1)</td>
<td>$\varepsilon \leq \sum_{i=1}^{p} \lambda_{i} \varepsilon_{i}, \lambda + \mu &gt; 0, \delta &gt; 0$</td>
<td>$\varepsilon$-opt. $\rightarrow$ $\varepsilon$-eff.</td>
<td>Theorem 6.7</td>
</tr>
<tr>
<td>SOP (5.1)</td>
<td>$\varepsilon \leq \sum_{i=1}^{p} \lambda_{i} \varepsilon_{i}$</td>
<td>$\varepsilon$-opt. $\rightarrow$ $\varepsilon$-eff.</td>
<td>Theorem 6.9 (1)</td>
</tr>
<tr>
<td>SOP (5.1)</td>
<td>$\varepsilon \leq \sum_{i=1}^{p} \lambda_{i} \varepsilon_{i}$</td>
<td>$\varepsilon$-opt. $\rightarrow$ $\varepsilon$-eff.</td>
<td>Theorem 6.9 (2)</td>
</tr>
<tr>
<td>SOP (5.1)</td>
<td>$\varepsilon \leq \sum_{i=1}^{p} \lambda_{i} \varepsilon_{i}, \lambda + \mu &gt; 0, \delta &gt; 0$</td>
<td>$\varepsilon$-opt. $\rightarrow$ $\varepsilon$-eff.</td>
<td>Theorem 6.11</td>
</tr>
</tbody>
</table>

References


Research highlights

- A new scalarization technique for multiobjective programming is presented.
- It is shown that some well-known scalarization methods can be seen as special case of that.
- We prove some results on (weakly,properly) efficient solutions.
- We deal with approximate solutions and derive some necessary/sufficient conditions.
- We summarize the obtained results in two tables.