Cellular Networks with \(\alpha\)-Ginibre Configurated Base Stations

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Abstract We consider a cellular network model with base stations configurated according to the \(\alpha\)-Ginibre point process with \(\alpha \in (0, 1]\), which is one of the determinantal point processes. In this model, we focus on the asymptotic behavior of the so-called coverage probability (or link success probability) as the threshold value tends to 0 and \(\infty\), and discuss the Padé approximation of the coverage probability at 0 and the dependence on \(\alpha \in (0, 1]\) of the asymptotic constant at \(\infty\) both numerically and theoretically.

Keywords Cellular network · Ginibre point process · \(\alpha\)-Ginibre point process · Determinantal point process · SINR · Coverage probability · Padé approximation · Stochastic geometry

1 Introduction

Recent studies on cellular networks are based on the theory of stochastic geometry and random geometric graphs by assuming that the base stations of a network are randomly distributed in space (cf. [2, 3, 5, 6]). Cellular networks are usually modeled by the ingredients \((\Phi = \{X_i\}_{i=1}^{\infty}, \ell(r), \{F_i\}_{i=1}^{\infty}, W)\), spatially distributed base stations, a path-loss function, the effect of fading, and noise. A configuration \(\Phi = \sum_{i=1}^{\infty} \delta_{X_i}\) is a simple, stationary point process on \(\mathbb{R}^d\) and it
stands for a realization of spatial distribution of base stations of a cellular network. A decreasing function $\ell: (0, \infty) \to [0, \infty)$ is a path-loss function, which represents the attenuation of signals at distance $r$. What we have in mind is, for example, $\ell(r) = ar^{-d\beta}$ or $\ell(r) = a(r^{-d\beta} \wedge 1)$ with $a > 0$ and $\beta > 1$. A random variable $F_i$ independent of $\Phi$ represents a random effect of fading from the base station $X_i$ to the typical user. Here we assume the so-called Rayleigh fading, i.e., $\{F_i\}_{i=1}^\infty$ are i.i.d. exponential random variables with mean 1. $W$ is a random variable representing thermal noise independent of $\{F_i\}_{i=1}^\infty$ and $\Phi$. Suppose that a typical user is located at the origin and she/he is associated with the nearest base station $X_B$ from the origin, where $B$ is the index corresponding to the nearest base station. SINR (signal-to-interference-plus-noise-ratio) at the origin is defined by

$$\text{SINR}_o = \frac{F_B \ell(|X_B|)}{W + I(B)} = \frac{\text{signal}}{\text{noise}},$$

where $I(B) = \sum_{i \neq B} F_i \ell(|X_i|)$ is the cumulative interference signal from all the base stations other than $B$.

One of the main concerns of the study of wireless networks is to analyze the coverage probability $P(\text{SINR}_o > \theta)$ with given threshold $\theta > 0$ as a functional of $(\Phi, \ell(r), \{F_i\}_{i=1}^\infty, W)$. The coverage probability plays a role of natural metric for measuring the performance of a wireless network. For the first attempt of such an analysis, one often considers a Poisson point process because of its tractability and computability by its spatial independence. The coverage probability for the stationary Poisson point process has been computed explicitly in [1] (see Example 1 below). However, sometimes, it does not seem to be plausible as a real world model since the Poisson points have some clusters due to spatial independence. In our previous paper [9], we treated the cellular network whose base stations are configurated according to the Ginibre point process, which might be more natural as base stations than the Poisson point process since it has repulsion or negative correlation. In the study, we derive an integral representation of the coverage probability by using random infinite products to obtain an asymptotic behavior and perform numerical computation. It is observed in our numerical computation that the coverage probability for the Ginibre point process is larger than that of the Poisson point process in wide range of threshold values $\theta$. In [10], the $\alpha$-Ginibre point process is considered as a one-parameter interpolation between Poisson ($\alpha = 0$) and Ginibre ($\alpha = 1$), and the numerical computation of the coverage probabilities for them is performed.

In this paper, we give some discussions on the asymptotics of the coverage probability of a cellular network model based on stationary point processes. In Sect. 2, we consider the asymptotics as $\theta \to 0$ and propose an idea of the Padé approximation of coverage probability (Fig. 1). In Sect. 3, we focus on the coverage probability for the $\alpha$-Ginibre point processes (Fig. 2) and give an asymptotic behavior as $\theta \to \infty$ (Theorem 1). Also we discuss the asymptotic constant which appears in the limit (Figs. 3, 4 and Proposition 5).
# 2 Cellular Network with Stationary Base Stations

For simplicity, we only consider the interference limited case \( W = 0 \).

**Proposition 1** Suppose that base stations are distributed according to a simple point process \( \Phi = \sum_{i=1}^{\infty} \delta_{X_i} \). Then, the coverage probability is given by the formula

\[
P(\text{SINR}_\theta > \theta) = E \left[ \prod_{j \neq B} \left( 1 + \theta \frac{\ell(|X_j|)}{\ell(|X_B|)} \right)^{-1} \right],
\]

where \( X_B \) is of the least modulus.

**Proof** Straightforward (cf. [9]). \( \square \)

In what follows, we assume that the spatial dimension \( d = 2 \), \( \Phi \) is simple and stationary, and the path-loss function is given by \( \ell(r) = ar^{-2\beta} \) with \( a > 0 \) and \( \beta > 1 \). We denote the coverage probability \( P(\text{SINR}_\theta > \theta) \) by \( p_c(\theta, \beta) \), which does not depend on \( a \) in this case and is given by

\[
p_c(\theta, \beta) = E \left[ \prod_{j \neq B} \left( 1 + \theta \frac{|X_B|}{|X_j|} \beta \right)^{-1} \right].
\]

**Example 1** If \( \Phi \) is the stationary Poisson point process on \( \mathbb{R}^2 \) with intensity \( \pi^{-1}dxdy \), then

\[
p_c^{(\text{Poi})}(\theta, \beta) = \frac{1}{1 + \rho(\theta, \beta)}, \quad \rho(\theta, \beta) = \theta^{1/\beta} \int_{\theta^{-1/\beta}}^{\infty} \frac{du}{1 + u^\beta}.
\]

In particular, as \( \theta \to \infty \),

\[
p_c^{(\text{Poi})}(\theta, \beta) \sim \frac{\sin(\pi/\beta)}{\pi/\beta} \theta^{-1/\beta}.
\]

See [1], for example.

We have another expression for the coverage probability in terms of the number of base stations inside the disk.

**Proposition 2** The base stations are distributed according to a point process \( \Phi = \sum_{i=1}^{\infty} \delta_{X_i} \). Then, the coverage probability is expressed as

\[
p_c(\theta, \beta) = E \left[ \exp \left( -\theta \int_{1}^{\infty} \frac{\tilde{N}_\Phi(s \sqrt{\theta} |X_B|)}{s(\theta + s)} ds \right) \right],
\]

where \( \tilde{N}_\Phi(t) \) is the number of \(|X_i|\)’s less than or equal to \( t \) except \(|X_B|\).
Proof Let \( Z_j = |X_j / X_B|^{2\beta} \). Then, for \( s > 0 \), we have

\[
\{ j \geq 1; j \neq B, Z_j \leq s \} = \{ j \geq 1; j \neq B, |X_j| \leq s^{1/2\beta} |X_B| \} = \tilde{N}_\Phi(s^{1/2\beta} |X_B|).
\]

From (2) and Lemma 1 below, we obtain (4). \(\square\)

Lemma 1 Let \( z_j, j = 1, 2, \ldots \) be an increasing sequence of positive reals. Then, for \( T > 0 \),

\[
\prod_{j=1}^\infty (1 + \frac{T}{z_j}) = \exp \left( T \int_0^\infty \frac{N(s)}{s(T+s)} ds \right),
\]

(5)

where \( N(s) \) is the number of \( z_j \)'s less than or equal to \( s \). Both sides are finite if and only if \( \sum_{j=1}^\infty z_j^{-1} = \int_0^\infty \frac{N(s)}{s} ds < \infty \).

Proof By summation by parts, we see that

\[
\log \prod_{j=1}^\infty (1 + \frac{T}{z_j}) = \sum_{j=1}^\infty \log(1 + \frac{T}{z_j}) = \sum_{j=1}^\infty j \left( \log(1 + \frac{T}{z_j}) - \log(1 + \frac{T}{z_{j+1}}) \right)
\]

\[
= \sum_{j=1}^\infty j \int_{z_j}^{z_{j+1}} \frac{T}{s(T+s)} ds = T \int_0^\infty \frac{N(s)}{s(T+s)} ds.
\]

The last equality follows from \( \sum_{j=1}^\infty z_j^{-1} = \int_0^\infty \frac{dN(s)}{s} \) and integration by parts. \(\square\)

Remark 1 The formula (5) is also valid for a finite sequence of \( z_j, j = 1, 2, \ldots, M \) by replacing \( \prod_{j=1}^\infty \) with \( \prod_{j=1}^M \) on the left-hand side.

By using Proposition 2, we can compute the Taylor expansion at \( \theta = 0 \).

Proposition 3 For \( t > 0 \), let

\[
\kappa_1(t, \beta) = E[\tilde{N}_\Phi(t^{1/2\beta} |X_B|)],
\]

\[
\kappa_2(t, s, \beta) = E[\tilde{N}_\Phi(t^{1/2\beta} |X_B|) \tilde{N}_\Phi(s^{1/2\beta} |X_B|)],
\]

where \( \tilde{N}_\Phi(s) \) is the number of \( |X_i| \)'s except \( |X_B| \) less than or equal to \( s \). Suppose that \( \kappa_1(t, \beta) \) and \( \kappa_2(t, s, \beta) \) are finite. Then, we have

\[
p_c(\theta, \beta) = 1 - \theta \int_1^\infty \frac{\kappa_1(t, \beta)}{t^2} dt + \frac{\theta^2}{2} \left\{ \int_1^\infty \int_1^\infty \frac{\kappa_2(t, s, \beta)}{t^2 s^2} ds dt + 2 \int_1^\infty \frac{\kappa_1(t, \beta)}{t^3} dt \right\}
\]

\[
+ O(\theta^3) \quad (\theta \to 0).
\]

Proof It follows from the Taylor expansion at \( \theta = 0 \) in (4). \(\square\)
In the case of the stationary Poisson point process on \( \mathbb{R}^2 \) with intensity \( \pi^{-1} dxdy \), it is easy to see that

\[
\kappa_1(t, \beta) = \left( t^{\frac{1}{\beta}} - 1 \right) I_{[1, \infty)}(t), \quad \kappa_2(t, s, \beta) = \kappa_1(t \wedge s, \beta) + 2\kappa_1(t, \beta)\kappa_1(s, \beta)
\]

and that

\[
p_c^{(Poi)}(\theta, \beta) = 1 - \frac{\theta}{\beta - 1} + \frac{\beta^2 \theta^2}{(2\beta - 1)(\beta - 1)^2} + O(\theta^3) \quad (\theta \to 0)
\]

by Proposition 3, although it can also be computed directly from the expression (3) for the Poisson case. In order to apply Proposition 3 to the \( \alpha \)-Ginibre point processes (see Sect. 3.2 for the definition), we need to compute the conditional product moments of random variables \( \tilde{N}_\Phi(u|X_B|), u > 0 \) given \( |X_B| = r \), i.e., the condition that there are no points except \( X_B \) within the disk of radius \( r \). For example, the asymptotic behavior of the conditional first moment (slightly different version) for \( \alpha = 1 \) can be found in [13].

A Padé approximant is a rational function whose power series expansion agrees with a prescribed Taylor series to the highest possible order. Once one knows the second order Taylor expansion of a function, one can compute its \((1, 1)\)-Padé approximant by the formula

\[
\frac{1 + p_1 \theta}{1 + q_1 \theta} = 1 - a_1 \theta + a_2 \theta^2 + O(\theta^3) \quad (\theta \to 0).
\]

Then, if \( p_c(\theta, \beta) = 1 - a_1 \theta + a_2 \theta^2 + O(\theta^3) \) as \( \theta \to 0 \), the \((1, 1)\)-Padé approximant is given by

\[
p_c(\theta, \beta) = \frac{1 + \frac{a_2 - a_1^2}{a_1} \theta}{1 + \frac{a_2}{a_1} \theta} + O(\theta^3).
\]

For example, in the Poisson case above, we have

\[
p_c^{(Poi)}(\theta, \beta) = \frac{1 + \frac{\beta - 1}{2\beta - 1} \theta}{1 + \frac{\beta^2}{(\beta - 1)(2\beta - 1)} \theta} + O(\theta^3).
\]

The \((1, 1)\)-Padé approximant provides a better approximation of the coverage probability (3) in wide range near \( \theta = 0 \) than the second order Taylor expansion. See Fig. 1, in which \( \theta = 0 \) corresponds to \(-\infty \) dB.

More details on the Padé-approximation for coverage probability will be discussed elsewhere.
3 Cellular Network with $\alpha$-Ginibre Configurated Base Stations

We recall the definition of determinantal point processes. The $\alpha$-Ginibre point process is defined as a determinantal point process (cf. [4, 10]). In Sects. 3.3 and 3.4, we discuss the asymptotic behavior of the coverage probability for the $\alpha$-Ginibre point process as $\theta \to \infty$.

3.1 Determinantal Point Processes

Let $\nu$ be a Radon measure on $\mathbb{R}^d$ and $K(\cdot, \cdot): \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}$ a continuous kernel which defines a self-adjoint integral operator $K$ on $L^2(\mathbb{R}^d, \nu)$. Under the assumption that (i) $K$ is of locally trace class, i.e., the restriction of $K$ on a compact set is of trace class, and (ii) the spectrum of $K$ is contained in $[0, 1]$, there exists a unique simple point process $\Phi$ on $\mathbb{R}^d$ such that the $n$-th correlation function with respect to the reference measure $\nu$ is given by

$$
\rho_n(x_1, x_2, \ldots, x_n) = \det(K(x_i, x_j))_{i, j=1}^n.
$$

Here the $n$-th correlation function (if exists) is defined by the formula

$$
E\left[ \sum_{x_1, \ldots, x_n \in \Phi \text{ distinct}} f(x_1, x_2, \ldots, x_n) \right] = \int_{(\mathbb{R}^d)^n} f(x_1, x_2, \ldots, x_n) \rho_n(x_1, x_2, \ldots, x_n) \prod_{i=1}^n \nu(dx_i)
$$
for every symmetric continuous function \( f \) on \((\mathbb{R}^d)^n\) with compact support. Such a point process \( \Phi = \sum \delta_{X_i} \) is called the determinantal point process on \( \mathbb{R}^d \) associated with \( K \) and \( \nu \). Determinantal point processes, a.k.a. fermion point processes, form a nice class of point processes with repulsion and they have been investigated from various points of view over the last 20 years. We refer the reader to \([7, 14, 15]\) for more details on determinantal point processes and their properties.

Here we just mention one property for later use. For a compact set \( \Lambda \subset \mathbb{R}^d \), let \( K_\Lambda = I_\Lambda K I_\Lambda \) be the restriction of \( K \) onto the compact set \( \Lambda \). By the assumption for \( K \), the operator \( K_\Lambda \) has eigenvalues \( \{\lambda_i(\Lambda), i \in \mathbb{N}\} \) satisfying \( \lambda_i(\Lambda) \in [0, 1] \) for every \( i \). It is known that the number of points inside \( \Lambda \), say \( \Phi(\Lambda) \), is expressed as the sum of independent Bernoulli random variables, that is,

\[
\Phi(\Lambda) \overset{d}{=} \sum_{i \in \mathbb{N}} \xi_i(\Lambda),
\]

where \( \xi_i(\Lambda) \) is a Bernoulli random variable with \( P(\xi_i(\Lambda) = 1) = \lambda_i(\Lambda) \) and \( P(\xi_i(\Lambda) = 0) = 1 - \lambda_i(\Lambda) \). In particular,

\[
E[\Phi(\Lambda)] = \sum_{i \in \mathbb{N}} \lambda_i(\Lambda) = \text{Tr} K_\Lambda, \tag{7}
\]

\[
\text{Var}(\Phi(\Lambda)) = \sum_{i \in \mathbb{N}} \lambda_i(\Lambda)(1 - \lambda_i(\Lambda)) = \text{Tr} K_\Lambda(1 - K_\Lambda). \tag{8}
\]

We note that \( \text{Var}(\Phi(\Lambda)) \leq E[\Phi(\Lambda)] \) and the assumption (i) for \( K \) guarantees that \( E[\Phi(\Lambda)] \) is finite when \( \Lambda \) is compact.

### 3.2 \( \alpha \)-Ginibre Point Processes and Their Properties

We assume that \( \alpha \in [0, 1] \) and \( \alpha = 0 \) is understood to be the limit \( \alpha \to 0 \) as we will see in Remark 3. For \( \alpha \in (0, 1] \), let \( \mu_\alpha \) be the \( \alpha \)-Ginibre point process, i.e., the determinantal point process on \( \mathbb{C}(\cong \mathbb{R}^2) \) associated with

\[
K_\alpha(z, w) = e^{\bar{z}w/\alpha}, \quad g_\alpha(dz) = \frac{\pi^{-1}}{\sqrt{(j - 1)!\alpha^j}} m(dz),
\]

where \( m(dz) \) is the Lebesgue measure on \( \mathbb{C} \). The integral operator \( K_\alpha \) acting on \( L^2(\mathbb{C}, g_\alpha) \) has eigenvalues \( \alpha \) and 0. Indeed, the functions

\[
\phi_j(z) = \frac{z^{j-1}}{\sqrt{(j - 1)!\alpha^j}}, \quad j = 1, 2 \ldots
\]

are the normalized eigenfunctions corresponding to the eigenvalue \( \alpha \), and the functions \( z^n \bar{z}^m, n = 0, 1, 2, \ldots, m = 1, 2, \ldots \) are the eigenfunctions corresponding
to 0. Hence, the spectral decomposition of the kernel $K_\alpha$ is given by

$$K_\alpha(z, w) = \sum_{j=1}^{\infty} \alpha \cdot \phi_j(z) \phi_j(w).$$

**Lemma 2** Let $(K_\alpha)_{D_r}$ be the restriction of $K_\alpha$ on the disk $D_r$ of radius $r$. Then, for each $j \in \mathbb{N}$, $\phi_j(z)$ is also an eigenfunction of $(K_\alpha)_{D_r}$ corresponding to the eigenvalue

$$\lambda_j(r) = \alpha \int_0^{r^2} \frac{s^{j-1}e^{-s}}{(j-1)!} ds = \alpha P(Y_j \leq \frac{r^2}{\alpha}),$$

where $Y_j \sim \text{Gamma}(j, 1)$, i.e., $P(Y_j \leq t) = \int_0^t \frac{s^{j-1}e^{-s}}{(j-1)!} ds$.

**Proof** We can show it by direct computation (cf. [12] for $\alpha = 1$). \qed

This lemma is closely related to the following remark.

**Remark 2** When $\Phi = \sum_{i \in \mathbb{N}} \delta_{X_i}$ is the original Ginibre ($\alpha = 1$), it is known that $\{\|X_i\|, i \in \mathbb{N}\} \overset{d}{=} \{\sqrt{Y_j}, j \in \mathbb{N}\}$, where $\{Y_j, j \in \mathbb{N}\}$ are independent and $Y_j \sim \text{Gamma}(j, 1)$ ([7, 8]). This fact is useful for computation of the coverage probability since the path-loss function only depends on the distance in our setting.

The $n$-th correlation function with respect to $g_\alpha(dz)$ is given by

$$\rho_n(z_1, \ldots, z_n) = \det(K_\alpha(z_i, z_j))_{i,j=1}^n$$

for each $n \in \mathbb{N}$. For example, the first and second correlation measures are the following.

$$\rho_1(z)g_\alpha(dz) = K_\alpha(z, z)g_\alpha(dz) = \pi^{-1}m(dz),$$

$$\rho_2(z, w)g_\alpha(dz)g_\alpha(dw) = (K_\alpha(z, w)K_\alpha(w, z) - K_\alpha(z, w)K_\alpha(w, z))g_\alpha(dz)g_\alpha(dw) = \pi^{-2}e^{-|z-w|^2/\alpha}m(dz)m(dw).$$

Both measures are motion invariant, i.e., invariant under translation and rotation. Moreover, the $n$-th correlation measure is also motion invariant for every $n \in \mathbb{N}$. Hence, the $\alpha$-Ginibre point process is motion invariant.

**Remark 3** As remarked above, the $\alpha$-Ginibre point process is motion invariant and the intensity is here normalized to be $\pi^{-1}$ for all $\alpha \in (0, 1]$. One can show that $\mu_\alpha$ converges weakly to the Poisson point process with the same intensity as $\alpha \to 0$ so that $\mu_0$ can be regarded as the Poisson point process, which itself is not determinantal.

**Proposition 4** Let $N_r$ be the number of points inside $D_r$. Then, under $\mu_\alpha$, we have

$$E[N_r] = r^2, \quad \text{Var}(N_r) \sim (1 - \alpha)r^2$$

as $r \to \infty$. 


Proof We give a sketch of proof. It follows from (7) that
\[
E[N_r] = \sum_{i=1}^{\infty} \lambda_i(r) = \frac{\alpha \cdot r^2}{\alpha} = r^2.
\]
Note that by Lemma 2 and the law of large numbers, we have
\[
\lambda_n(r) \sim \begin{cases} 
\alpha + o(1), & n \leq (1 - \epsilon)\frac{r^2}{\alpha}, \\
o(1), & n \geq (1 + \epsilon)\frac{r^2}{\alpha}
\end{cases},
\]
for any \( \epsilon > 0 \) and \( o(1) \) is exponentially small by the large deviations result. Hence it follows from (8) that
\[
\text{Var}(N_r) = \sum_{i=1}^{\infty} \lambda_i(r)(1 - \lambda_i(r)) \sim \sum_{i=1}^{\infty} \alpha (1 - \alpha) = (1 - \alpha)r^2.
\]
\[\square\]

Remark 4 When \( \alpha = 1 \), the variance is of \( o(r^2) \) by Proposition 4, which is called small fluctuation property of the (1-)Ginibre point process. Indeed, it is known that \( \text{Var}(N_r) \sim \pi^{-1/2}r \) under \( \mu_1 \) (see [12]).

Remark 5 The \( \alpha \)-Ginibre point process can be constructed from the 1-Ginibre by scaling and independent thinning. We scale the 1-Ginibre point process by factor \( \sqrt{\alpha} \), and for each point in the scaled point process, independently, retain it with probability \( \alpha \) and delete it otherwise. Then the resultant point process is equal in law to the \( \alpha \)-Ginibre point process. The thinning operation makes the variance large for \( \alpha \in (0, 1) \).

### 3.3 Asymptotics of \( p_c^{(\alpha)}(\theta, \beta) \)

The coverage probability for \( \alpha \)-Ginibre point process, denoted by \( p_c^{(\alpha)}(\theta, \beta) \), was discussed in [9] when \( \alpha = 1 \) and in [10] for general \( \alpha \in (0, 1] \). It is given by
\[
p_c^{(\alpha)}(\theta, \beta) = \alpha \sum_{k=1}^{\infty} E \left[ \prod_{j \in \mathbb{N} \setminus \{k\}} q(\alpha, \theta \left( \frac{Y_k}{Y_j} \right)^\beta) 1_{\{Y_j \geq Y_k\}} \right],
\]
where \( Y_j \sim \text{Gamma}(j, 1), \ j = 1, 2, \ldots \) and \( q(\alpha, x) = 1 - \alpha x (1 + x)^{-1} \). Note that \( q(\alpha, x) \) is decreasing in \( x \) and \( \alpha \). The formula (9) can be shown by Proposition 1, Remark 2 and Remark 5.

For \( k \in \mathbb{N} \) and \( \beta > 1 \), let
Fig. 2  The coverage probability for the $\alpha$-Ginibre point process for $\beta = 2$ [10]

\[ A_k(\alpha) = A_{k,\beta}(\alpha) := \alpha \int_0^\infty \frac{v^{k-1}}{(k-1)!} \prod_{j \in \mathbb{N}\setminus\{k\}} E \left[ q(\alpha, \left(\frac{v}{Y_j}\right)^\beta) \right] dv, \]  

\[ p_c^{(\alpha)}(\theta, \beta) \sim A_1(\alpha)\theta^{-1/\beta} \]  

where $Y_j$’s and $q(\alpha, x)$ are defined as above. Then, we can show the following.

**Theorem 1** Fix $\beta > 1$. Then, for $\alpha \in (0, 1]$,

\[ p_c^{(\alpha)}(\theta, \beta) \sim A_1(\alpha)\theta^{-1/\beta} \]  

as $\theta \to \infty$.

We note that the decay rate $\theta^{-1/\beta}$ is the same as that of the Poisson point process as in Example 1. As we will see in Lemma 4 and Proposition 5, the asymptotic constant $A_1(\alpha)$ is finite (see Fig. 3), and $A_1(\alpha)$ converges to that of the Poisson point process as $\alpha \to 0+$, as was naturally expected from Remark 3.

We also observe that $A_1(\alpha)$ is not increasing in $\alpha$ for $\beta = 1.25$ (near $\beta = 1$) in Fig. 3, and it seems that $A_{1,\beta}(\alpha)$ is increasing in $\beta$ and the asymptotic constant is bounded below by $A_1(0+)$ in Fig. 4.

**Lemma 3** Let $k \in \mathbb{N}$ be fixed. Then,

\[ \lim_{\theta \to \infty} \theta^{k/\beta} \cdot \alpha E \left[ \prod_{j \in \mathbb{N}\setminus\{k\}} q(\alpha, \theta \left(\frac{Y_k}{Y_j}\right)^\beta) \mathbf{1}_{\{Y_j \geq Y_k\}} \right] = A_k(\alpha). \]  

**Proof** By a change of variables and the monotone convergence theorem, we see that
Fig. 3 The asymptotic constant $A_1(\alpha)$

Fig. 4 The asymptotic constant $A_1(\alpha)$ as a function of $\beta > 1$

$$\theta^{k/\beta} \cdot \alpha E\left[ \prod_{j \in \mathbb{N}\setminus\{k\}} \left\{ q(\alpha, \theta \left( \frac{Y_k}{Y_j} \right)^\beta )^\beta 1_{\{Y_j \geq Y_k\}} \right\} \right]$$

$$= \theta^{k/\beta} \cdot \alpha \int_0^{\infty} \frac{u^{k-1}e^{-u}}{(k-1)!} \prod_{j \in \mathbb{N}\setminus\{k\}} E\left[ q(\alpha, \theta \left( \frac{u}{Y_j} \right)^\beta )^\beta 1_{\{Y_j \geq u\}} \right] du$$
\[
= \alpha \int_0^\infty \frac{v^{k-1}e^{-\theta^{-1/\beta}v}}{(k-1)!} \prod_{j \in \mathbb{N} \setminus \{k\}} E \left[ q(\alpha, \left(\frac{v}{Y_j}\right)^\beta) 1_{\{Y_j \geq \theta^{-1/\beta}v\}} \right] dv
\]

\[
/ \alpha \int_0^\infty \frac{v^{k-1}}{(k-1)!} \prod_{j \in \mathbb{N} \setminus \{k\}} E \left[ q(\alpha, \left(\frac{v}{Y_j}\right)^\beta) \right] dv = A_k(\alpha)
\]

as \( \theta \to \infty \). \qed

**Lemma 4** Let \( k \geq 1 \). Then \( A_k(\alpha) < \infty \). Moreover, for every \( N \geq 1 \),

\[
\lim_{\theta \to \infty} \theta^{1/\beta} \cdot \alpha \sum_{k=1}^N \left[ \prod_{j \in \mathbb{N} \setminus \{k\}} \left\{ q(\alpha, \theta \left(\frac{Y_k}{Y_j}\right)^\beta) \right\} \right] = A_1(\alpha). \quad (13)
\]

**Proof** Since \( q(\alpha, x) \leq \exp(-\alpha \frac{x}{1+x}) \), we have

\[
A_k(\alpha) \leq \alpha \int_0^\infty \frac{v^{k-1}}{(k-1)!} E \left[ \prod_{\substack{j=1^v \atop j \neq k}}^{c_2^v} \exp \left( -\alpha \frac{(\frac{v}{Y_j})^\beta}{1 + (\frac{v}{Y_j})^\beta} \right) \right] dv
\]

\[
= \alpha \int_0^\infty \frac{v^{k-1}}{(k-1)!} E \left[ \exp \left( -\alpha \sum_{\substack{j=1^v \atop j \neq k}}^{c_2^v} \frac{v^\beta}{Y_j^\beta + v^\beta} \right) \right] dv
\]

for \( 0 < c_1 < c_2 \). For \( \epsilon > 0 \), let

\[
B_v := \bigcap_{c_1 v \leq j \leq c_2 v} \{ Y_j \leq (1 + \epsilon) j \}.
\]

Now we estimate

\[
E \left[ \exp \left( -\alpha \sum_{\substack{j=1^v \atop j \neq k}}^{c_2^v} \frac{v^\beta}{Y_j^\beta + v^\beta} \right) \right]
\]

\[
= E \left[ \exp \left( -\alpha \sum_{\substack{j=1^v \atop j \neq k}}^{c_2^v} \frac{v^\beta}{Y_j^\beta + v^\beta} \right); B_v \right] + E \left[ \exp \left( -\alpha \sum_{\substack{j=1^v \atop j \neq k}}^{c_2^v} \frac{v^\beta}{Y_j^\beta + v^\beta} \right); B_v^c \right]
\]

\[
= (I) + (II).
\]

For \( (I) \), we have

\[
(I) \leq E \left[ \exp \left( -\alpha \sum_{\substack{j=1^v \atop j \neq k}}^{c_2^v} \frac{v^\beta}{((1 + \epsilon) j)^\beta + v^\beta} \right); B_v \right] \leq \exp \left( -\alpha \frac{(c_2 - c_1) v}{((1 + \epsilon)c_2)^\beta + 1} \right).
\]
For (II), by large deviations result, we see that

\[(II) \leq P(B'_v) = P\left( \bigcup_{c_1 v \leq j \leq c_2 v} [Y_j > (1 + \epsilon) j] \right) \leq \sum_{c_1 v \leq j \leq c_2 v} P(Y_j > (1 + \epsilon) j) \]

\[\leq \sum_{c_1 v \leq j \leq c_2 v} \exp\left( -I(1 + \epsilon) j \right) \]

\[\leq (c_2 - c_1) v \exp\left( -c_1 I(1 + \epsilon) v \right),\]

where \(I(x) = x - 1 - \log x (x > 0)\) is the rate function with which the large deviation principle holds for sum of i.i.d. exponential random variables with mean 1. Hence, for some \(c, C > 0\) independent of \(v\), we have

\[(I) + (II) \leq c(1 + v)e^{-Cv}.\]

Therefore,

\[A_k(\alpha) \leq \alpha \int_0^\infty \frac{u^{k-1}}{(k-1)!} \cdot c(1 + v)e^{-Cv} dv < \infty.\]

The second part of the assertion immediately follows from the above and Lemma 3.

\[\square\]

**Proof of Theorem 1** We observe that

\[\alpha \sum_{k=N+1}^\infty E\left[ \prod_{j \in \mathbb{N}\setminus[k]} \left( q(\alpha, \theta \left( \frac{Y_k}{Y_j} \right) )^{\beta} 1_{\{Y_j \geq Y_k\}} \right) \right] \]

\[\leq \alpha q(\alpha, \theta)^{-1} \sum_{k=N+1}^\infty \int_0^\infty \frac{u^{k-1}e^{-u}}{(k-1)!} \prod_{j \in \mathbb{N}} E\left[ q(\alpha, \theta \left( \frac{u}{Y_j} \right) )^{\beta} \right] du \]

\[\leq \alpha q(\alpha, \theta)^{-1} \int_0^\infty \frac{u^N}{N!} \prod_{j \in \mathbb{N}} E\left[ q(\alpha, \theta \left( \frac{u}{Y_j} \right) )^{\beta} \right] du \]

\[\leq q(\alpha, \theta)^{-1} A_{N+1}(\alpha) \theta^{-\frac{N+1}{p}} \]

\[= \begin{cases} O(\theta^{-\frac{N+1}{p}}) & 0 < \alpha < 1, \\ O(\theta^{1-\frac{N+1}{p}}) & \alpha = 1. \end{cases} \]

The last equality follows from Lemma 3 since \(q(\alpha, \theta) \geq 1 - \alpha\) when \(0 < \alpha < 1\) and \(q(\alpha, \theta) = (1 + \theta)^{-1}\) when \(\alpha = 1\). Therefore, letting \(N = \lfloor \beta \rfloor + 1\) when \(\alpha = 1\), we have the asymptotic formula

\[\lim_{\theta \to \infty} \theta^{1/\beta} p_c^{(\alpha)}(\theta, \beta) = A_1(\alpha). \quad (14)\]

This together with (13) in Lemma 4 completes the proof. \(\square\)
3.4 A Remark on the Asymptotic Constant $A_1(\alpha)$

In this subsection, we give a probabilistic representation of $A_1(\alpha)$ and its asymptotic behavior as $\alpha \to 0^+$. 

Fix $\beta > 1$ and let $f_j(v) = E\left[\frac{v^\beta}{\sum_{j} v^\beta + v^\beta}\right]$. Then, it is easy to see that

$$A_1(\alpha) = \alpha \int_0^\infty \prod_{j \geq 2} (1 - \alpha f_j(v)) dv,$$  

and

$$A_1'(\alpha) = \int_0^\infty \left\{ \prod_{j \geq 2} (1 - \alpha f_j(v)) - \sum_{k \geq 2} \alpha f_k(v) \prod_{j \geq 2, j \neq k} (1 - \alpha f_j(v)) \right\} dv.$$  

Note that $f_j(v) \in [0, 1]$ is increasing in $v$ with $f_j(0) = 0$ and $f_j(\infty) = 1$ for every $j$. We consider independent Bernoulli random variables $\{\xi_{j, \alpha}(v), j \geq 2\}$ such that $P(\xi_{j, \alpha}(v) = 1) = \alpha f_j(v)$ and $P(\xi_{j, \alpha}(v) = 0) = 1 - \alpha f_j(v)$, and set $X_{\alpha}(v) = \sum_{j \geq 2} \xi_{j, \alpha}(v)$. Then,

$$E[X_{\alpha}(v)] = \alpha \int_0^\infty \frac{v^\beta}{f^\beta + v^\beta} (1 - e^{-tv}) dt = \alpha v \int_0^\infty \frac{1}{s^\beta + 1} (1 - e^{-vs}) ds. \quad (16)$$

It follows from (16) that as $v \to \infty$

$$E[X_{\alpha}(v)] \sim \alpha v \int_0^\infty \frac{ds}{s^\beta + 1} = \alpha \frac{\pi/\beta}{\sin(\pi/\beta)} v \quad (\beta > 1)$$

and as $v \to 0$

$$E[X_{\alpha}(v)] \sim \alpha \begin{cases} -\Gamma(1 - \beta)v^\beta & 1 < \beta < 2, \\ -v^2 \log v & \beta = 2, \\ \frac{\pi/\beta}{\sin(2\pi/\beta)} v^2 & \beta > 2. \end{cases}$$

In terms of $X_{\alpha}(v)$, the quantities $A_1(\alpha)$ and $A_1'(\alpha)$ can be rewritten as

$$A_1(\alpha) = \alpha \int_0^\infty P(X_{\alpha}(v) = 0) dv,$$

$$A_1'(\alpha) = \int_0^\infty \{P(X_{\alpha}(v) = 0) - P(X_{\alpha}(v) = 1)\} dv.$$  

Although the convergence of the numerical computation is too slow to observe the values near $\alpha = 0$ in Fig. 3, we can show the following limiting behavior as $\alpha \to 0^+$.

**Proposition 5** For every $\beta > 1$, it holds that
\[
\lim_{\alpha \to 0^+} A_1(\alpha) = \frac{\sin(\pi/\beta)}{\pi/\beta}.
\]

The right-hand side is the asymptotic constant for the Poisson case as in Example 1.

**Proof** For every \( \delta > 0 \) there exists \( 0 < x_\delta < 1 \) such that
\[
e^{-(1+\delta)x} \leq 1 - x \leq e^{-x} \quad (0 \leq x \leq x_\delta).
\]
Fix \( \delta > 0 \). From (15) and (17), since \( f_j(v) \in [0, 1] \) for all \( j \) and \( v \), we see that for any sufficiently small \( \alpha > 0 \)
\[
\alpha \int_0^\infty e^{-(1+\delta)E[X_\alpha(v)]} dv \leq A_1(\alpha) \leq \alpha \int_0^\infty e^{-E[X_\alpha(v)]} dv,
\]
and hence we have
\[
\int_{\epsilon}^\infty e^{-(1+\delta)E[X_\alpha(u/\alpha)]} du \leq A_1(\alpha) \leq \epsilon + \int_{\epsilon}^\infty e^{-E[X_\alpha(u/\alpha)]} du
\]
for any \( \epsilon > 0 \). From (16) we see that \( E[X_\alpha(u/\alpha)] \geq (1 - e^{-\epsilon})u \int_1^\infty \frac{ds}{s^{\beta+1}} \) uniformly in \( u \in [\epsilon, \infty) \) and that \( E[X_\alpha(u/\alpha)] \nearrow \frac{\pi/\beta}{\sin(\pi/\beta)} u \) as \( \alpha \searrow 0 \). Therefore, we obtain the assertion by the monotone convergence theorem since \( \epsilon \) and \( \delta \) are arbitrary. \( \square \)

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**References**


