Statistical Estimation Algorithms for Repairs-Time Limit Replacement Scheduling under Earning Rate Criteria

TADASHI DOHI AND AKIRA ASHIOKA
Department of Information Engineering
Graduate School of Engineering
Hiroshima University, 1-4-1 Kagamiyama
Higashi-Hiroshima 739-8527, Japan
dohi@rel.hiroshima-u.ac.jp

NAOTO KAIO
Department of Economic Informatics
Faculty of Economic Sciences
Hiroshima Shudo University
1-1-1 Ozukahigashi, Asaminami-ku
Hiroshima 731-3195, Japan
kaio@shudo-u.ac.jp

SHUNJI OSAKI
Department of Information and Telecommunication Engineering
Faculty of Mathematical Sciences and Information Engineering
Nanzan University, 27 Seirei-Cho
Seto 489-0863, Japan
shunj@it.nanzan-u.ac.jp

Abstract—In this paper, we formulate two repair-time limit replacement problems with imperfect repair under earning rate criteria with and without discounting. First, the optimal repair-time limits which maximize the long-run average profit rate and the expected total discounted profit over an infinite time horizon are analytically derived. Next, we develop the nonparametric algorithms for estimating the optimal repair-time limits, provided that the complete sample data of repair time are given. The basic idea is to apply the Lorenz statistics and to transform the underlying algebraic problems to the graphical ones. Finally, we present some numerical examples to show that the proposed algorithms can be useful to estimate the profit-based repair limit replacement schedule. © 2006 Elsevier Ltd. All rights reserved.

Keywords—Maintenance, Repair-time limit, Imperfect repair, Earning rate, Lorenz statistics, Nonparametric.
1. INTRODUCTION

Since the seminal contribution by Hastings [1], a large number of repair limit replacement problems were considered in literature [2–6]. Nakagawa [2] considered a simple repair-time limit replacement problem under an earning rate criterion. Nguyen and Murthy [4] analyzed a different repair-time limit policy with imperfect repair. In this paper, we focus on a mixed model of Nakagawa [2] and Nguyen and Murthy [4]. Consider a single-unit system where each spare unit is provided only by an order after a lead time, and each failed unit is repairable. When the unit fails, the decision maker (DM) estimates the completion time distribution of repair, which may be a possibly subjective one. If DM estimates that the repair is completed up to a prespecified time-limit, the repair is started immediately, otherwise, the spare unit is ordered with a lead time. Since the repair is imperfect, the unit repaired or even ordered can fail again during a finite time horizon. The problem for DM is to determine the optimal repair-time limit which maximizes any earning rate criterion.

Dohi et al. [7,8] considered the above models with subjective repair time distribution under the expected cost criteria, and developed estimators of the optimal repair-time limits, by applying the Lorenz statistics or the Lorenz curve. Since the knowledge on the repair-time distribution is incomplete in general, such a statistical estimation method for the optimal repair-time limit will be useful in the practical maintenance situation [9]. The Lorenz curve was first introduced by Lorenz [10] to describe income distributions. Since the Lorenz curve is essentially equivalent to the Pareto curve used in the quality control, it will be one of the most important statistics applied in every social sciences. The more general and tractable definition of the Lorenz curve was made by Gastwirth [11]. Goldie [12] proved the strong consistency of the empirical Lorenz curve and discovered its several convergence properties. Chandra and Singpurwalla [13] investigated the relationship between the total time on test statistics [9] and the Lorenz statistics, and derived a few aging and partial ordering properties.

In this paper, we formulate the repair-time limit replacement model with imperfect repair [8] under earning rate criteria with and without discounting. First, after describing the notation and assumptions used here, the optimal repair-time limits which maximize the long-run average profit rate and the expected total discounted profit over an infinite time horizon are analytically derived. Next, we develop the nonparametric algorithms for estimating the optimal repair-time limits, provided that the complete sample data of repair time are given. The basic idea is to apply the Lorenz statistics and to transform the underlying algebraic problems to the graphical ones. Finally, we present some numerical examples to show that the proposed algorithms can be useful to estimate the profit-based repair limit replacement schedule.

2. REPAIR-TIME LIMIT REPLACEMENT MODEL

2.1. Notation

The repair time $X$ for each unit is a nonnegative i.i.d. random variable. The decision maker (DM) has a subjective probability distribution function $\Pr\{X \leq t\} = G(t)$ on the repair time, with density $g(t) (> 0)$ and finite mean $1/\lambda (> 0)$. Suppose that the distribution function $G(t) \in (0, 1)$ is arbitrary, continuous and strictly increasing in $t \in (0, \infty)$, and, in addition, has an inverse function $G^{-1}(\cdot)$. Further, we define

- $t_0 \in [0, \infty)$: repair-time limit (decision variable),
- $F_1(t), f_1(t), 1/\mu_1 (> 0)$: c.d.f., p.d.f., and mean of time to failure for a repaired unit,
- $F_2(t), f_2(t), 1/\mu_2 (> 0)$: c.d.f., p.d.f., and mean of time to failure for a new (spare) unit,
- $k (> 0)$: penalty cost per unit time when the system is in down state,
- $e_0 (> 0)$: earning rate per unit operation time,
- $e_1 (> 0)$: repair cost per unit time,
- $c (> 0)$: fixed cost associated with the ordering of a new unit,
2.2. Model Description

Consider a single-unit repairable system, where each spare is provided only by an order after a lead time $L$ and each failed unit is repairable. When the unit has failed at time $t = 0$, the DM wishes to determine whether he or she should repair it or order a new spare. If DM estimates that the repair is completed within a prespecified time limit $t_0 \in [0, \infty)$, then the repair is started immediately at $t = 0$ and completes at time $t = X$. After the completion of repair, the unit is started to operate again, but can fail again for a finite time span since the repair is imperfect. Then, the mean failure time is $1/\mu_1$.

On the other hand, if DM estimates that the repair time exceeds the time limit $t_0$, then the failed unit is scrapped at time $t = 0$ and a new spare unit is ordered immediately. A new unit is delivered after the lead time $L$. Further, the new unit can also fail for a finite time span and then the mean failure time is $1/\mu_2$. Without any loss of generality, it is assumed that the time required for replacement of a failed unit can be negligible. Under these model setting, we define the interval from the failure time to the following failure time as one cycle. Figure 1 depicts the configuration of the repair-time limit replacement problem with imperfect repair under consideration.

3. ANALYSIS

3.1. Long-Run Average Profit

Define the time interval from a failure point to the next failure point as one cycle. Suppose that the same cycle is repeated again and again over an infinite time horizon. Since the failure point can be regarded as a renewal point, the mean cycle length is given by

$$T_L(t_0) = \int_0^{t_0} t \, dG(t) + \int_0^{\infty} G(t_0) \tilde{F}_1(t) \, dt + \int_0^{\infty} L \, dG(t) + \int_0^{\infty} \tilde{G}(t_0) \tilde{F}_2(t) \, dt$$

$$= \int_0^{t_0} t \, dG(t) + \frac{1}{\mu_1} G(t_0) + (L + \frac{1}{\mu_2}) \tilde{G}(t_0).$$

(1)
Also, the expected total profit during one cycle becomes

\[ V_L(t_0) = e_0 \left\{ \int_0^\infty G(t_0) F_1(t) \, dt + \int_0^\infty G(t_0) F_2(t) \, dt \right\} - (e_1 + k) \int_0^{t_0} t \, dG(t) - k \int_0^\infty L \, dG(t) - c \int_0^\infty dG(t) \]

\[ = \frac{e_0}{\mu_1} G(t_0) + \left\{ \frac{e_0}{\mu_2} - (c + kL) \right\} \dot{G}(t_0) - (e_1 + k) \int_0^{t_0} t \, dG(t). \]

From the familiar renewal reward argument, the long-run average profit rate (the expected total profit per unit time in the steady state) is given by

\[ TP_L(t_0) = \lim_{t \to \infty} \frac{E[\text{total profit on } (0, t)]}{t} = \frac{V_L(t_0)}{T_L(t_0)}, \]

and the problem is to derive the optimal repair-time limit \( t_0 \in [0, \infty) \) satisfying

\[ TP_L(t_0) = \max_{0 \leq t_0 < \infty} TP_L(t_0). \]

Differentiating \( TP_L(t_0) \) with respect to \( t_0 \) yields

\[ \frac{dTP_L(t_0)}{dt_0} = g(t_0) q_L(t_0) / T_L(t_0)^2, \]

where

\[ q_L(t_0) = \left\{ c + kL + e_0 \left( \frac{1}{\mu_1} - \frac{1}{\mu_2} \right) - (e_1 + k) t_0 \right\} T_L(t_0) - \left( t_0 - L + \frac{1}{\mu_1} - \frac{1}{\mu_2} \right) V_L(t_0). \]

Then, we have the following result on the optimal repair-time limit under the assumption \( V_L(t_0) > 0 \) for all \( t_0 \).

**Theorem 1.** There exists a finite and unique optimal repair-time limit \( t_0^* \in (0, \infty) \) satisfying \( q_L(t_0^*) = 0 \), and the corresponding maximum long-run profit rate is given by

\[ TP_L(t_0^*) = \frac{e_0 (1/\mu_1 - 1/\mu_2) + c + kL - (e_1 + k) t_0^*}{T_0^* - L + 1/\mu_1 - 1/\mu_2}. \]

### 3.2. Total Discounted Profit

Next, consider the case where the profit is discounted with the discount rate \( \beta (> 0) \) over an infinite time horizon. Then, the expected total discounted profit during one cycle is given by

\[ V_D(t_0) = \int_0^{t_0} \int_0^t \int_0^\infty e_0 \exp\{-\beta(x + y)\} \, dy \, dF_1(t) \, dG(x) \]

\[ + \int_0^{t_0} \int_0^t \int_0^\infty e_0 \exp\{-\beta(L + y)\} \, dy \, dF_2(t) \, dG(x) - \int_0^{t_0} \int_0^\infty c \exp\{-\beta L\} \, dG(t) \]

\[ - \int_0^{t_0} \int_0^t (e_1 + k) \exp\{-\beta x\} \, dx \, dG(t) - \int_0^{t_0} \int_0^L k \exp\{-\beta x\} \, dx \, dG(t) \]

\[ = \frac{e_0}{\beta} \left( 1 - L \{f_1(\beta)\} \right) \int_0^{t_0} \exp\{-\beta x\} \, dG(x) - \frac{k + e_1}{\beta} \int_0^{t_0} (1 - \exp\{-\beta t\}) \, dG(t) \]

\[ + \frac{e_0 \exp\{-\beta L\}}{\beta} (1 - L \{f_2(\beta)\}) \dot{G}(t_0) \]

\[ - \frac{k(1 - \exp\{-\beta L\})}{\beta} + c \exp\{-\beta L\} \dot{G}(t_0). \]
Since the expected present value of the unit profit just after one cycle is given by
\[
\delta(t_o) = \int_0^\infty \int_0^t \exp(-\beta(t + x)) dG(t) dF_1(x) + \int_0^\infty \int_t^\infty \exp(-\beta(L + x)) dG(t) dF_2(x) \tag{8}
\]
the expected total discounted profit over an infinite time horizon becomes
\[
TP_D(t_o) = \sum_{j=0}^\infty V_D(t_o) \delta(t_o)^j = \frac{V_D(t_o)}{\delta(t_o)} \tag{9}
\]
It is evident that
\[
TP_L(t_o) = \lim_{\beta \to 0} \beta \cdot TP_D(t_o). \tag{10}
\]
Of our interest is the derivation of the optimal repair-time limit \(t_0^* \in [0, \infty)\) satisfying
\[
TP_D(t_0^*) = \max_{0 < t_0 < \infty} TP_D(t_0). \tag{11}
\]
Similar to equation (5), it is seen that \(dTP_D(t_0)/dt_0 = g(t_0)q_D(t_0)/\delta(t_0)^2\), where
\[
q_D(t_0) = \left\{ \frac{e_0}{\beta} \left(1 - L\{f_1(\beta)\}\right) \exp(-\beta t_0) - \frac{e_0 \exp(-\beta L)}{\beta} \left(1 - L\{f_2(\beta)\}\right) \right. \\
\left. + c \exp(-\beta L) - \frac{k + e_1}{\beta} \left(1 - \exp(-\beta t_0)\right) + \frac{k(1 - \exp(-\beta L))}{\beta} \right\} \delta(t_0) \tag{12}
\]
\[
TP_D(t_0^*) = \left[ e_0 \left(1 - L\{f_1(\beta)\}\right) \exp(-\beta t_0^*) - e_0 \left(1 - L\{f_2(\beta)\}\right) \exp(-\beta L) \right. \\
\left. - (k + e_1)(1 - \exp(-\beta t_0^*)) + k(1 - \exp(-\beta L)) + c\exp(-\beta L) \right] / \left[ \beta \left(1 - L\{f_2(\beta)\}\right) \exp(-\beta L) - L\{f_1(\beta)\} \exp(-\beta t_0^*) \right] . \tag{13}
\]
\[
(i) \text{ If } q_D(0) > 0 \text{ and } q_D(\infty) < 0, \text{ then there exists a finite and unique optimal repair-time limit } t_0^* (0 < t_0^* < \infty) \text{ satisfying } q_D(t_0^*) = 0, \text{ and the corresponding expected total discounted profit over an infinite time horizon is given by}
\]
\[
TP_D(t_0^*) = \left[ e_0 \left(1 - L\{f_1(\beta)\}\right) \exp(-\beta t_0^*) - e_0 \left(1 - L\{f_2(\beta)\}\right) \exp(-\beta L) \right. \\
\left. - (k + e_1)(1 - \exp(-\beta t_0^*)) + k(1 - \exp(-\beta L)) + c\exp(-\beta L) \right] / \left[ \beta \left(1 - L\{f_2(\beta)\}\right) \exp(-\beta L) - L\{f_1(\beta)\} \exp(-\beta t_0^*) \right] . \tag{13}
\]
\[
(ii) \text{ If } q_D(0) \leq 0, \text{ then the optimal repair-time limit is } t_0^* = 0 \text{ and } TP_D(t_0^*) = TP_D(0).
\]
\[
(iii) \text{ If } q_D(\infty) \geq 0, \text{ then the optimal repair-time limit is } t_0^* \to \infty \text{ and } TP_D(t_0^*) = TP_D(\infty).
\]
In this section, we derived the optimal repair-time limit replacement policies maximizing two kinds of profit functions. It should be noted that the repair time distribution has to be completely known, if one calculates the optimal policies according to Theorem 1 and Theorem 2. However, in general, it is not so easy to identify the repair time distribution for a highly reliable equipment. In the following section, we will develop nonparametric algorithms to estimate the optimal repair-time limits under respective profit functions.

4. GRAPHICAL METHODS

For a continuous repair time c.d.f., \(p = G(t_0)\), define the Lorenz transform,
\[
\phi(p) = \lambda \int_0^{G^{-1}(p)} t \, dG(t), \tag{14}
\]
where
\[ G^{-1}(p) = \inf\{t \geq 0 : G(t) \geq p\}, \quad (15) \]
if the inverse function exists, and \( 1/\lambda = \int_0^{G^{-1}(t)} t \, dG(t) \). Then, the curve \( \mathcal{L} = (p, \phi(p)) \in [0,1] \times [0,1] \) is called the Lorenz curve. From a few algebraic manipulations, we obtain the following useful result to interpret the underlying optimization problem \( \max_{0 \leq p \leq \infty} TP_L(t_0) \) geometrically.

**THEOREM 3.** If \( (1/\mu_1 - 1/\mu_2)(e_0 + e_1 + k) + c - e_1L > 0 \), then obtaining the optimal repair-time limit \( t_0^* \) maximizing the long-run average profit rate \( TP_L(t_0) \) is equivalent to obtaining \( p^* \) \( (0 \leq p^* \leq 1) \) such as
\[
\min_{0 \leq p \leq 1} \frac{\phi(p) + \beta_L}{p + \alpha_L}, \quad (16)
\]
otherwise,
\[
\max_{0 \leq p \leq 1} \frac{\phi(p) + \beta_L}{p + \alpha_L}, \quad (17)
\]
where
\[
\alpha_L = -1 + \frac{(e_0 + e_1 + k)/\mu_1}{(1/\mu_1 - 1/\mu_2)(e_0 + e_1 + k) + c - e_1L}, \quad (18)
\]
\[
\beta_L = \frac{(\lambda/\mu_1)(e_0 + k) + c}{(1/\mu_1 - 1/\mu_2)(e_0 + e_1 + k) + c - e_1L}. \quad (19)
\]

Theorem 3 is the dual of Theorem 1. From this result, it can be seen that the optimal repair-time limit \( t_0^* = G^{-1}(p^*) \) is determined by calculating the optimal point \( p^* \) \( (0 \leq p^* \leq 1) \) minimizing or maximizing the tangent slope from the point \( (-\alpha_L, -\beta_L) \) to the curve \((p, \phi(p)) \in [0,1] \times [0,1] \).

For the discounted case, define the modified Lorenz transform,
\[
\phi_D(p) = \int_0^{G^{-1}(p)} \exp\{-\beta t\} \, dG(t) \int_0^{G^{-1}(t)} \exp\{-\beta t\} \, dG(t), \quad (20)
\]

**THEOREM 4.** If \( (e_0 + e_1 + k) \exp\{-\beta L\} \mathcal{L}\{f_2(\beta)\} - (e_0 + e_1 - c\beta) \exp\{-\beta L\} \mathcal{L}\{f_1(\beta)\} > 0\), then obtaining the optimal repair-time limit \( t_0^* \) maximizing the expected total discounted profit over an infinite time horizon \( TP_D(t_0) \) is equivalent to obtaining \( p^* \) \( (0 \leq p^* \leq 1) \) such as
\[
\max_{0 \leq p \leq 1} \frac{\phi_D(p) + \beta_D}{p + \alpha_D}, \quad (21)
\]
otherwise
\[
\min_{0 \leq p \leq 1} \frac{\phi_D(p) + \beta_D}{p + \alpha_D}, \quad (22)
\]
where
\[
\alpha_D = (\mathcal{L}\{f(\beta)\} - 1)(k + e_0 + e_1)/(e_1 + (e_1 + k - c\beta) \exp\{-\beta L\}) \mathcal{L}\{f_1(\beta)\} - (e_0 + e_1 + k) \exp\{-\beta L\} \mathcal{L}\{f_2(\beta)\} - 1, \quad (23)
\]
\[
\beta_D = [(e_0 + e_1 + k) \exp\{-\beta L\} \mathcal{L}\{f_2(\beta)\} - (k + e_0 - c\beta) \exp\{-\beta L\} - e_1]/[\mathcal{L}\{g(\beta)\}(e_1 + (e_0 + e_1 + k - c\beta) \exp\{-\beta L\}) \mathcal{L}\{f_1(\beta)\} - (e_0 + e_1 + k) \exp\{-\beta L\} \mathcal{L}\{f_2(\beta)\}] \quad (24)
\]
Next, suppose that the optimal repair-time limit has to be estimated from $n$ ordered complete observations: $0 = x_0 \leq x_1 \leq x_2 \leq \cdots \leq x_k$ of the repair times from a continuous c.d.f. $G$, which is unknown. Then, the empirical distribution for this sample, is given by

$$G_{in}(x) \equiv \begin{cases} i/n & \text{for } x_i \leq x < x_{i+1}, \\ 1, & \text{for } x_n \leq x, \end{cases}$$

(25)

where $i = 0, 1, 2, \ldots, n - 1$. Then, the Lorenz statistics [12,13] can be defined by

$$\phi_{in} = \sum_{i=1}^{[n]} x_i / \sum_{i=1}^{n} x_i,$$

(26)

where $[a]$ denotes the least integer not smaller than $a$ (ceiling function). From equations (25) and (26), plotting $(i/n, \phi_{in})$, ($i = 0, 1, 2, \ldots, n$) on $\mathbb{R}^2$ and connecting them by line segment yield the sample Lorenz curve $L_{in} = (i/n, \phi_{in}) \in [0, 1] \times [0, 1]$. From the analogy of Theorem 3, we obtain an estimator of the optimal repair-time limit which maximizes the long-run average profit rate as follows.

**Theorem 5.** Define the estimator of the optimal repair-time limit by $t^*_0 = x^*_1$. If $(1/\mu_1 - 1/\mu_2)(e_0 + e_1 + k) + c - e_1L > 0,$

$$\begin{cases} x^*_1 \min_{0 \leq i \leq n} \phi_{in} + \beta L \\ \max_{0 \leq i \leq n} \phi_{in} + \beta L \end{cases}$$

(27)

otherwise,

$$\begin{cases} x^*_1 \min_{0 \leq i \leq n} \phi_{in} + \beta L \\ \max_{0 \leq i \leq n} \phi_{in} + \beta L \end{cases}$$

(28)

where $1/\lambda$ in $\beta L$ (equation (19)) is replaced by the sample mean,

$$\sum_{i=1}^{n} x_i / n.$$

In fact, Goldie [12] proves the strong consistency of the Lorenz statistics given in equation (26), that is, $\phi_{in} \to \phi(p)$ as $n \to \infty$, a.s. This fact means that the estimator of the optimal repair-time limit $t^*_0$ may be also consistent. The graphical procedure proposed here has an educational value for better understanding of the optimization problem and it is convenient for performing sensitivity analysis of the optimal repair-time limit when different values are assigned to the model parameters. The special interest is, of course, to estimate the optimal repair-time limit without specifying the repair time distribution. Although some typical theoretical distribution functions such as the lognormal distribution are often assumed for the repair time distribution, our nonparametric estimation algorithm can estimate the optimal repair-time limit based on the on-line knowledge about the observed repair times.

Next, define the statistics of the modified Lorenz transform by

$$\phi_{in}^\beta = \left[ 1 - \left( 1 - \frac{i}{n} \right) \exp\{-\beta x_i\} - \beta \sum_{j=1}^{i} \left( 1 - \frac{j-1}{n} \right) (x_j - x_{j-1}) \exp\{-\beta x_j\} \right]$$

$$\sqrt{1 - \beta \sum_{j=1}^{n} \left( 1 - \frac{j-1}{n} \right) (x_j - x_{j-1}) \exp\{-\beta x_j\} \left[ \sum_{j=1}^{n} \left( 1 - \frac{j-1}{n} \right) (x_j - x_{j-1}) \exp\{-\beta x_j\} \right].$$

(29)

By plotting $(i/n, \phi_{in}^\beta)$ and connecting them by line segment, we obtain the modified sample Lorenz curve.
THEOREM 6. Define the estimator of the optimal repair-time limit by $\hat{t}_n^* = x_{\hat{t}_n}^*$. If

$$(e_0 + e_1 + k) \exp \{-\beta L\} \mathcal{L}\{f_2(\beta)\} - [e_1 + (e_0 + k - c\beta) \exp \{-\beta L\}] \mathcal{L}\{f_1(\beta)\} > 0,$$

then

$$
\begin{align*}
\hat{t}_n^* &= \max_{0 \leq i \leq n} \frac{\phi_{\frac{\alpha}{n}, \frac{\beta}{n}}^D}{t/n + \alpha_D} \\
\hat{t}_n^* &= \min_{0 \leq i \leq n} \frac{\phi_{\frac{\alpha}{n}, \frac{\beta}{n}}^D}{t/n + \alpha_D}
\end{align*}
$$

In this section, we developed estimation algorithms for the optimal repair-time limits under different earning rate criteria. As mentioned before, it can be easily understood that the estimator $\hat{t}_n^*$ of the optimal repair-time limit maximizing the long-run average profit rate has strongly consistent, i.e., $\hat{t}_n^* \rightarrow t^*$ as $n \rightarrow \infty$. However, it is an open question whether the estimate in the discounted case can converge to the real optimum as $n \rightarrow \infty$. For the practical purpose, we will examine the convergence properties of nonparametric estimators for the optimal repair-time limits through a simulation study in the following section.

5. ILLUSTRATIVE EXAMPLES

Suppose that the repair time distribution is the following Weibull distribution,

$$G(t) = 1 - e^{-\left(t/\theta\right)^{m}},$$

where the scale parameter $\theta = 1.2$ and the shape parameter $m = 2.2$. The other parameters are $1/\mu_1 = 0.4300$, $1/\mu_2 = 0.4500$, $e_0 = 8.0000$, $e_1 = 0.4000$, $k = 0.8000$, $c = 2.5000$, and $L = 0.2500$. We generated 20 samples from the Weibull distribution above and regard them as the repair time data. Figure 2a illustrates an estimation of the optimal repair-time limit when

$$(e_0 + e_1 + k) \exp \{-\beta L\} \mathcal{L}\{f_2(\beta)\} - [e_1 + (e_0 + k - c\beta) \exp \{-\beta L\}] \mathcal{L}\{f_1(\beta)\} > 0.$$ 

In this case, since $(-\alpha_L, -\beta_L) = (-0.7852, -0.8617)$, it is observed that $\hat{t}_0^* = 0.9382$ and $TP_L(t_0^*) = 2.1165$. On the other hand, Figure 2b depicts an example for the case of

$$(e_0 + e_1 + k) \exp \{-\beta L\} \mathcal{L}\{f_2(\beta)\} - [e_1 + (e_0 + k - c\beta) \exp \{-\beta L\}] \mathcal{L}\{f_1(\beta)\} \leq 0.$$ 

The model parameters are $\theta = 2.2$, $m = 2.0$, $1/\mu_1 = 0.3500$, $1/\mu_2 = 0.7500$, $e_0 = 3.5000$, $e_1 = 0.4500$, $k = 0.2000$, $c = 1.5000$, and $L = 3.0000$. For 20 sample from the Weibull distribution, we get $(-\alpha_D, -\beta_D) = (1.9619, 1.6782)$, $\hat{t}_0^* = x_{\hat{t}_0}^* = 1.6052$, and $TP_D(t_0^*) = 0.1913$.

Next, consider the discounted case. Figure 3a is an estimation result for the case of

$$(e_0 + e_1 + k) \exp \{-\beta L\} \mathcal{L}\{f_2(\beta)\} - [e_1 + (e_0 + k - c\beta) \exp \{-\beta L\}] \mathcal{L}\{f_1(\beta)\} > 0,$$

where $\theta = 15$, $m = 1.2$, $\mathcal{L}\{f_1(\beta)\} = 0.9550$, $\mathcal{L}\{f_2(\beta)\} = 0.9500$, $\beta = 0.0330$, $e_0 = 6.5000$, $e_1 = 0.3500$, $k = 0.3500$, $c = 9.8000$, and $L = 5.5000$. In this case, we generated 50 random numbers from $G(t)$ as the repair time data. From $(-\alpha_D, -\beta_D) = (-0.8828, -0.7426)$, we obtain $\hat{t}_0^* = x_{\hat{t}_0}^* = 10.8862$ and $TP_D(t_0^*) = 12.0033$. In Figure 3b, we present an example for the case of

$$(e_0 + e_1 + k) \exp \{-\beta L\} \mathcal{L}\{f_2(\beta)\} - [e_1 + (e_0 + k - c\beta) \exp \{-\beta L\}] \mathcal{L}\{f_1(\beta)\} \leq 0,$$

where $\theta = 15$, $m = 1.2$, $\mathcal{L}\{f_1(\beta)\} = 0.9800$, $\mathcal{L}\{f_2(\beta)\} = 0.9750$, $\beta = 0.0500$, $e_0 = 5.0000$, $e_1 = 0.5500$, $k = 0.1200$, $c = 0.1250$, and $L = 10.0000$. For 50 repair time data from $G(t)$, we estimate $\hat{t}_0^* = x_{\hat{t}_0}^* = 2.4629$ and $TP_D(t_0^*) = 1.0195$ with $(-\alpha_D, -\beta_D) = (1.5027, 2.2648)$.

Finally, we investigate the convergence properties of the estimator, $\hat{t}_n^* = x_{\hat{t}_n}^*$ for the discounted case. In Figure 4, the asymptotic behavior of the estimate of optimal repair-time limit and its associated expected total discounted profit over an infinite time horizon is illustrated, where $\theta = 15$, $m = 1.2$, $\mathcal{L}\{f_1(\beta)\} = 0.9550$, $\mathcal{L}\{f_2(\beta)\} = 0.9500$, $\beta = 0.0330$, $e_0 = 6.5000$, $e_1 = 0.3500$, $k = 0.3500$, $c = 9.8000$, and $L = 5.5000$. In the figures, the horizontal lines denote the real optimal repair-time limit and the corresponding maximum expected total discounted profit over an infinite time horizon. From these results, it is seen that the optimal repair-time limit can be estimated with higher accuracy when more than 50 repair time data are available.
Figure 2. Estimation of the optimal repair-time limit maximizing the long-run average profit.
Figure 3. Estimation of the optimal repair-time limit maximizing the expected total discounted profit over an infinite time horizon.
Figure 4. Asymptotic behavior of the estimate of optimal repair-time limit and its associated expected total discounted profit.

REFERENCES