Feedback Stabilization of Discrete-Time Networked Systems over Fading Channels

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Abstract—This paper is concerned with the mean square stabilization problem for discrete-time networked control systems over fading channels. We show that there exists a requirement on the network over which an unstable plant can be stabilized. For the case of state feedback, necessary and sufficient conditions on the network for mean square stabilizability are derived. Under a parallel transmission strategy and the assumption that the overall mean square capacity of the network is fixed and can be assigned among parallel input channels, a tight lower bound on the overall mean square capacity for mean square stabilizability is presented in terms of the Mahler measure of the plant. The minimal overall capacity for stabilizability is also provided under a serial transmission strategy. For the case of dynamic output feedback, a tight lower bound on the capacity requirement for stabilization of SISO plants is given in terms of the antistable poles, nonminimum phase zeros and relative degree of the plant. Sufficient and necessary conditions are further derived for triangularly decoupled MIMO plants. The effect of pre- and post-channel processing and channel feedback is also discussed, where the channel feedback is identified as a key component in eliminating the limitation on stabilization induced by the nonminimum phase zeros and high relative degree of the plant. Finally, the extension to the case with output fading channels and the application of the results to vehicle platooning are presented.

Index Terms—Fading channel, feedback stabilization, mean square capacity, networked control system, vehicle platooning.

I. INTRODUCTION

In the past few years, networked control systems (NCSs) have attracted recurring interests from the control community due to their wide applications in advanced manufacturing, industrial automation, intelligent transportation, and national defense. Such applications are propelled by the use of sensor networks and distributed systems. See the survey paper [1]. NCSs have numerous advantages over classical feedback control systems with wired point-to-point connections, including low cost, high flexibility, easy installation, and simple maintenance. However, networked systems require new tools for analysis and control synthesis since, in executing estimation and control operations, we cannot ignore the limited communication resources such as channel capacity and bandwidth as well as uncertainties and unreliability of sensing/communication networks, especially wireless networks. On the one hand, depending on applications, different measures such as signal-to-noise ratio, delay spread and Rician factor have been proposed by communication engineers to characterize the channel quality [2], [3]. On the other hand, the effect of channel capacity and uncertainties on the stability and performance of networked systems has been widely studied by control researchers; see, e.g., [4]–[7] on packet loss, [8], [9] on quantization, [10] on time delay, [11], [12] on fading, [13] on limited data rate, [14] on limited capacity, etc. Since the network is an integral part of an NCS, there is a trend toward a joint design of communication and control; see, e.g., [15]–[17].

It is now well known that the Mahler measure [18] or the instability degree [19] of the plant to be controlled poses a fundamental limitation on the NCS. For instance, Nair and Evans [13] presented a tight lower bound on the data rate required to stabilize an unstable plant over an error-free digital channel in terms of the Mahler measure of the plant. Time-varying and nonlinear encoding and decoding algorithms are used to establish the data rate bound. The recent work [20] further discussed the minimal data rate problem for stabilizing an unstable plant over a lossy channel. The results in [8] and [9] showed a relationship between the optimal quantization density for quadratic stabilization and the Mahler measure of a single-input linear time-invariant (LTI) plant. The work [14] presented limitations on the ability to stabilize an SISO unstable plant over a signal-to-noise ratio constrained channel using static state feedback, and the solution is given in terms of the Mahler measure of the plant. Moreover, additional network resource would be required for stabilizing a nonminimum phase plant when dynamic output feedback is concerned [14], and the availability of channel feedback plays a key role in reducing the minimal signal-to-noise ratio for stabilizability [21]. Elia [11] proved that under stochastic channel fading, the minimal mean square capacity for stabilizing a single-input unstable plant via state feedback is again given in terms of the Mahler measure of the plant.

Note that the aforementioned results on the network requirement for stabilizability in [9], [11], [14], [21] are derived for single-input plants only, and the extension to multi-input plants is interesting and challenging. Motivated by logarithmic quantization [9], Gu and Qiu [16] considered the state feedback stabilization problem for an LTI discrete-time multi-input plant with sector-bounded time-varying uncertainties in the parallel input channels. They showed that a tighter lower bound on the product of sector sizes which measure quantization densities of the input channels can be quantified in terms of the Mahler measure of the plant if the product can be allocated among...
the channels. Following [14], the authors of [22] studied the best achievable tracking performance of an MIMO LTI plant over parallel output signal-to-noise ratio constrained channels, where the plant is assumed to be minimum phase.

The present work follows the line of research presented in [11] where only the minimal capacity for state feedback stabilization of a single-input plant over a single fading channel is given. In this paper, we focus on the network requirement for both state feedback and output feedback stabilization of MIMO plants over multiple fading channels. The mean square stabilization problem for a discrete-time NCS over an input network is formulated in Section II, followed by a preliminary lemma on stabilization over predefined fading channels. Section III presents the network requirement for stabilizability via state feedback where necessary and sufficient conditions on the network are given as one of the main results of this paper. Then the network requirement under the serial transmission strategy (STS) and the parallel transmission strategy (PTS) is further discussed. Under PTS, the overall mean square capacity of the network is assumed to be fixed and can be allocated to the parallel channels. An explicit tight lower bound on the overall mean square capacity for mean square stabilizability under PTS is given in terms of the Mahler measure of the plant, while an optimization may be required to compute the lower bound under STS. Section IV deals with the network requirement for stabilizability via output feedback. It is shown that additional channel capacity is required for stabilizability which can be exactly quantified in terms of the anti-stable poles and nonminimum phase zeros when the plant under consideration is SISO and has a relative degree greater than or equal to 1. Based on the triangular decoupling results [23], [24], the output feedback stabilization is further studied for MIMO plants under PTS where necessary and sufficient conditions on the overall mean square capacity of the network are derived, respectively. Section V further analyzes the possible benefits of introducing the pre- and post-channel filters and channel feedback to the NCS. It is shown that the feedback of the channel plays an important role in reducing the network requirement for stabilizability when output feedback is considered. The extension to stabilization over output fading channels is given in Section VI, followed by an application in multi-vehicle platooning. Section VII concludes the paper. A preliminary version of the paper on the state feedback case has been communicated in [25].

The notation used in this paper is mostly standard. The symbol ≜ means "defined as". The set of real numbers is denoted by \( \mathbb{R} \). Let \( |\lambda| \) and \( \lambda \) be the magnitude and the conjugate of the complex number \( \lambda \). Denote by \( [A]_{ij} \), \( A' \), \( A^H \), \( A^{-1} \), \( \text{tr}(A) \), \( \det(A) \) and \( \rho(A) \), respectively the entry that lies in the \( i \)-th row and the \( j \)-th column, the transpose, the conjugate transpose, the inverse, the trace, the determinant and the spectral radius of the matrix \( A \). When \( X \) and \( Y \) are real symmetric matrices, the notation \( X \succeq Y \) (respectively, \( X \succ Y \)) indicates that \( X - Y \) is positive semidefinite (positive definite). All matrices and vectors are assumed to be compatible for algebraic operations whenever their dimensions are not explicitly stated. The Kronecker product is denoted by \( \otimes \), and the Hadamard product by \( \circ \). Let vec\( (X) \) denote the vector formed by stacking the columns of \( X \) into one higher dimensional vector, and diag\( [a_1, a_2, \ldots, a_m] \) stand for the diagonal matrix having \( a_1, a_2, \ldots, a_m \) as its diagonal elements. Furthermore, the mathematical expectation operator is denoted by \( \mathbb{E}(\cdot) \). We say that \( G(z) \in \mathcal{H}_\infty \), if the discrete-time transfer function \( G(z) \) is proper and stable. The mean square norm of \( G(z) \) with dimension \( p \times m \), if exists, is defined as

\[
\|G(z)\|_{\text{MS}} \triangleq \sqrt{\max_{i=1,2,\ldots,p} \int_{-\pi}^{\pi} |G(e^{j\omega})|^2 d\omega}. \tag{1}
\]

The Mahler measure of an LTI plant \( G(z) \) with any detectable and stabilizable realization \( \begin{bmatrix} A & B \\ C & D \end{bmatrix} \) can be defined in terms of the Mahler measure of the square matrix \( A \in \mathbb{R}^{n \times n} \) as

\[
\mathcal{M}(G) \triangleq \mathcal{M}(A) = \prod_{i=1}^{n} \max\{|\lambda_i(A)|, 1\}, \tag{2}
\]

where \( \lambda_1(A), \lambda_2(A), \ldots, \lambda_n(A) \) are the eigenvalues of \( A \) counting algebraic multiplicity.

II. PROBLEM FORMULATION

Consider a discrete-time NCS as depicted in Fig. 1, where an unreliable network is placed in the path from the controller to the plant.

![Feedback control over an input network.](image)

Assume that the plant \( G(z) \) is strictly proper and has a state-space representation:

\[
x_G(t + 1) = Ax_G(t) + Bu(t),
\]

\[
y(t) = Cx_G(t), \tag{3}
\]

where \( x_G(t) \in \mathbb{R}^n \) is the plant state, \( u(t) \in \mathbb{R}^m \) is the plant input, and \( y(t) \in \mathbb{R}^p \) is the measured output. Without loss of generality, assume that \( A \) is unstable, \( B = [B_1 B_2 \cdots B_m] \) has full-column rank, \( C = [C_1' C_2' \cdots C_p']' \) has full-row rank, and the triple \((A, B, C)\) is stabilizable and detectable.

The controller is assumed to be time invariant, and has a proper transfer function \( K(z) \) and a state-space realization:

\[
x_K(t + 1) = A_K x_K(t) + B_K y(t),
\]

\[
v(t) = C_K x_K(t) + D_K y(t), \tag{4}
\]

where \( v(t) \in \mathbb{R}^m \) is the controller output. The dimension of the controller state \( z(t) \) is not specified \textit{a priori}.

The model of the fading channel(s) is given in the following memoryless multiplicative form:

\[
u(t) = \xi(t) v(t), \tag{5}
\]
where \( u(t) \) is directly applied to the plant\(^1\), and \( \xi(t) \in \mathbb{R}^{m \times m} \) represents the channel fading\(^2\) and has the diagonal structure:

\[
\xi(t) = \text{diag}\{\xi_1(t), \xi_2(t), \ldots, \xi_m(t)\}.
\]

Assume that \( \xi_i(t) \), \( i = 1, 2, \ldots, m \), are scalar-valued white noise processes with

\[
\mu_i \triangleq \mathcal{E}\{\xi_i(t)\}, \quad \sigma_{ij} \triangleq \mathcal{E}\{\xi_i(t) - \mu_i)(\xi_j(t) - \mu_j)\},
\]

satisfying \( \mu_i > 0, \sigma_{ii} > 0, \) and \( \sigma_{ij} = \sigma_{ji} \) for all \( i, j = 1, 2, \ldots, m \). We further denote \( \Sigma = \{\sigma_{ij}\}_{i,j=1,2,\ldots,m} \) and \( \Sigma = \sqrt{\sigma_{ii}} \), and

\[
\Pi = \text{diag}\{\mu_1, \mu_2, \ldots, \mu_m\}, \quad \Lambda = \text{diag}\{\sigma_1, \sigma_2, \ldots, \sigma_m\}.
\]

It is easy to see that \( \Sigma \) is positive semidefinite.

**Remark 2.1:** Different from the general MIMO communication system with multiple receiving and transmitting antennas [26], the input and the output of the network (5)-(6) are assumed to have the same dimension since our main focus is to investigate the effect of fading on stabilizability rather than information recovery. In model (5)-(6), if the \( i \)-th and \( j \)-th components of \( u(t) \) are sent over the same fading channel, whose channel fading is constant over each time step of the underlying discrete-time system\(^3\), then we can set \( \xi_i(t) = \xi_j(t) \). Moreover, the fading experienced by different channels may or may not be correlated depending on whether a non-orthogonal or an orthogonal access scheme [3] is adopted.

**Remark 2.2:** Besides the fading phenomenon, the model (5) can describe the uncertainties of a digital network such as quantization errors [9] and packet dropouts [7]. In particular, for the logarithmic quantizer considered in [9], \( \xi(t) \) is a scalar-valued or diagonal matrix-valued sector-bounded time-varying gain, which is used to model uncertainties arising from quantization. In the case of packet dropout studied in [7], \( \xi(t) \) is a 0-1 binary-valued scalar and represents the packet-loss process.

Denote the overall system state by \( x(t) = [x_G(t)\, x_K(t)]' \), then the closed loop of the NCS in Fig. 1 can be written into

\[
x(t+1) = [\bar{A} + \bar{B}\xi(t)\bar{C}]x(t),
\]

where

\[
\bar{A} = \begin{bmatrix} A & 0 \\ B_KC & A_K \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} B \\ 0 \end{bmatrix}, \quad \bar{C} = [D_KC \, C_K].
\]

Throughout this paper, we shall concentrate on the mean square stabilization defined next.

**Definition 2.1 (91):** Consider the NCS in Fig. 1, and denote \( \Xi(t) = \mathcal{E}\{x(t)x'(t)\} \). The plant (3) is stabilized in the mean square sense by the controller (4), if, for any initial state \( x(0) \), \( \Xi(t) \) is well defined for all \( t \geq 0 \) and \( \lim_{t \to \infty} \Xi(t) = 0 \).

The model of the fading channel(s) (5)-(6) is said to be predefined if \( \mu_i, \sigma_{ij} \) in (7) are fixed and known for all \( i, j = 1, 2, \ldots, m \). Let \( \bar{B} = [\bar{B}_1 \, \bar{B}_2 \cdots \bar{B}_m] \) and \( \bar{C} = [\bar{C}_1' \, \bar{C}_2' \cdots \bar{C}_m']' \). The next lemma will be used in the later developments, which extends the stability criterion for systems with independent multiplicative noises presented in [11] and Chapter 9 of [27] to that with possibly correlated multiplicative noises.

**Lemma 1:** Consider the NCS in Fig. 1 with predefined fading channel(s) modeled by (5)-(6). The following statements are equivalent.

(i) The plant (3) or the triple \((A, B, C)\) can be stabilized in the mean square sense via controller of the form (4).

(ii) There exists a set of \( A_K, B_K, C_K, D_K \) such that the sequence \{\( \Xi(t) \)\}_{t \geq 0} computed by

\[
\Xi(t+1) = (\bar{A} + \bar{B}\Pi \bar{C})\Xi(t)(\bar{A} + \bar{B}\Pi \bar{C})' + \sum_{i=1}^{m} \sum_{j=1}^{m} \sigma_{ij} \bar{B}_i \bar{C}_j \Xi(t)\bar{C}_i' \bar{B}_j' \]

with any \( \Xi(0) \geq 0 \) is convergent to 0 as \( t \) approaches \( \infty \).

(iii) There exists a set of \( A_K, B_K, C_K, D_K \) such that \( \rho(\Psi) < 1 \), where

\[
\Psi = (\bar{A} + \bar{B}\Pi \bar{C}) \otimes (\bar{A} + \bar{B}\Pi \bar{C})' + \sum_{i=1}^{m} \sum_{j=1}^{m} \sigma_{ij} (\bar{B}_i \bar{C}_j) \otimes (\bar{B}_j \bar{C}_i). \]

(iv) There exist a set of \( A_K, B_K, C_K, D_K \) and \( P > 0 \) such that

\[
P > (\bar{A} + \bar{B}\Pi \bar{C})' P(\bar{A} + \bar{B}\Pi \bar{C}) + \bar{C}' J \bar{C},
\]

where \( J = \Sigma \otimes (\bar{B}' P \bar{B}) > 0 \).

(v) There exist a set of \( A_K, B_K, C_K, D_K \) and \( W > 0 \) such that

\[
W > (\bar{A} + \bar{B}\Pi \bar{C}) W(\bar{A} + \bar{B}\Pi \bar{C})' + \bar{B} H \bar{B}',
\]

where \( H = \Sigma \otimes (\bar{C} W C') \).

**Proof:** See Appendix A.

**Remark 2.3:** It is easy to show that the mean square stabilizability is invariant under similarity transformations on the triple \((A, B, C)\).

**III. NETWORK REQUIREMENT FOR STABILIZABILITY VIA STATE FEEDBACK**

Consider the state feedback case with \( C = I \). In view of Remark 2.3, we take \((A, B)\) to be of the form:

\[
A = \begin{bmatrix} A_s & 0 \\ 0 & A_u \end{bmatrix}, \quad B = \begin{bmatrix} B_s \\ 0 \end{bmatrix},
\]

where \( A_s \) is stable, all the poles of \( A_u \) are either on or outside the unit circle, and \((A_s, B_s)\) is controllable. We can derive the next lemma.

**Lemma 2:** Consider the NCS in Fig. 1 with state feedback and predefined fading channel(s) modeled by (5)-(6). The following statements are equivalent.

(i) The plant (3) or the pair \((A, B)\) can be stabilized in the mean square sense via state feedback.

(ii) The pair \((A, B)\) can be stabilized in the mean square sense via dynamic state feedback.

\(^1\)The NCS with pre- and post-channel filters will be further discussed in Section V.

\(^2\)We only consider the real-valued channel fading with the phase component being estimated and compensated for at the receiver; see, e.g., [12].

\(^3\)This is a valid assumption when the fading is coherent over the sampling interval of the system [2].
There exists $P > 0$ such that
\[
P > A'PA - A'PBB^{-1}PBPA, \tag{16}
\]
where $J$ is as defined in (13) with $B = B$.

ii) For any $R > 0$, there exists $P > 0$ such that
\[
P = A'PA - A'PBB^{-1}PBPA + R. \tag{17}
\]

v) The pair $(A_a, B_a)$ can be stabilized in the mean square
sense via static or dynamic state feedback.

Furthermore, if any of the conditions i)-v) is true, then a state
feedback gain ensuring the mean square stability of the closed
loop is given by
\[
K = -(J + PB'B^{-1}P)BPA, \tag{18}
\]
where $P > 0$ is any solution to (16) or (17).

Proof: See Appendix B.

Remark 3.1: By adopting similar ideas as in the proof of
Theorem 6 in [4], one solution $P > 0$ to (17), if exists, can be
obtained by solving the following optimization:
\[
P = \arg \max \limits_{P > 0} \text{tr}(P) \tag{19}
\]
subject to the linear matrix inequality constraint
\[
\begin{bmatrix}
P - A'P & -BPA \\
PBP & - \Sigma \odot (B'PB) - PBPA
\end{bmatrix} \leq 0.
\]
However, the feasibility of the above linear matrix inequality
cannot ensure the solvability of (17). For general cases, (16) and
(17) are not numerically easily testable. In what follows, we
aim to obtain some explicit conditions on the network over
which an unstable plant can be stabilized.

In view of Lemma 2, we limit our attention to the static state
feedback case in the rest of this section and assume that all the
eigenvalues of $A$ are either on or outside the unit circle without
loss of generality. Based on Remark 2.3, the pair $(A, B)$ is
further assumed to have the following Wonham decomposition
[28]:
\[
A = \begin{bmatrix}
A_1 & \cdots & \ast \\
0 & A_2 & \cdots & \ast \\
\vdots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & A_m
\end{bmatrix}, \quad B = \begin{bmatrix}
b_1 & \cdots & \ast \\
0 & b_2 & \cdots & \ast \\
\vdots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & b_m
\end{bmatrix}, \tag{20}
\]
where $\ast$ represents the part which will not be used in the
derivation, $A_i \in \mathbb{R}^{n_i \times n_i}, b_i \in \mathbb{R}^{n_i \times 1}, \sum_{i=1}^{m} n_i = n,$ and
each pair $(A_i, b_i)$ is controllable. The next theorem provides
necessary and sufficient conditions on the network for stabiliz-
ability via state feedback.

Theorem 3.1: The NCS in Fig. 1 is mean square stabilizable
via state feedback if there exists a diagonal matrix $\Sigma_1$ satisfying
$\Sigma_1 \geq \Sigma$, such that
\[
1 + \frac{\mu_i^2}{[\Sigma_1]_{ii}} > \mathcal{M}(A_i)^2, \quad \forall i = 1, 2, \ldots, m, \tag{21}
\]
and only if for any diagonal matrix $\Sigma_2$ satisfying $0 \leq \Sigma_2 \leq \Sigma$,
\[
\prod_{i=1}^{m} \left(1 + \frac{\mu_i^2}{[\Sigma_2]_{ii}}\right) > \mathcal{M}(A)^2. \tag{22}
\]

Proof: We will first prove the necessity of the condition
(22). Suppose that the closed loop is mean square stable under
a static state feedback controller. Based on the property of
Hadamard product in Problem 3 of [29, p. 475], we have
$J = J$ with $J = \Sigma_2 \odot (B'PB)$ for any diagonal matrix $\Sigma_2$
satisfying $0 \leq \Sigma_2 \leq \Sigma$. The condition (22) obviously holds
when any $[\Sigma_2]_{ii} = 0$, otherwise it follows from (16) that
\[
\det(P) > \det(A') \det(I - PBB^{-1}P) \times \det(P) \det(A) = \det(A)^2 \det(P) \det(I - (J + PB'B^{-1}P)) = \det(A)^2 \det(P) \det(I + \Sigma \odot (B'PB)) = \det(A)^2 \det(P) \left(\det(I + P^{1/2}B\hat{J}^{-1/2}B'PB^{1/2})\right)^{-1} \geq \det(A)^2 \det(P) \left(\det(I + \hat{J}^{-1/2}B'PB\hat{J}^{-1/2})\right)^{-1} = \det(A)^2 \det(P) \left(\det(I + J^{-1/2}B'PB\hat{J}^{-1/2})\right)^{-1} \geq \mathcal{M}(A)^2 \det(P) \prod_{i=1}^{m} \left(\frac{[\Sigma_2]_{ii}}{[\Sigma_2]_{ii} + \mu_i}\right). \tag{23}
\]
The inequality (23) follows from Hadamard’s inequality, i.e.,
the determinant of any positive-semidefinite matrix is not
greater than the product of all its diagonal elements; see
Theorem 7.8.1 of [29]. The proof of necessity is completed
since (23) implies (22).

Next, the sufficiency of the condition (21) will be shown.
According to Corollary 8.4 in [11] and Lemma 1, if (21) is
true for some $\Sigma_1$ satisfying $\Sigma_1 \geq \Sigma$, then there exist $P_i > 0$
and $\mu_i \in \mathbb{R}^{1 \times n_i}$ such that
\[
P_i > (A_i + b_i\mu_i k_i)P_i(A_i + b_i\mu_i k_i) + [\Sigma_1]_{ii} k_i b_i' P_i b_i k_i.
\]
It is direct to show that the closed loop is mean square stable
for $m = 1$. Next, we consider the case $m = 2$. Adopting
the similar technique in the proof of the equivalence between
i) and v) in Lemma 2, we can show that there exist $P = \text{diag}\{P_1, \gamma_1 P_2\}$ and $K = \text{diag}\{k_1, k_2\}$ with sufficiently large
$\gamma_1$ such that
\[
P > (A + B\hat{K})' (A + B\hat{K}) + K' (\Sigma_1 \odot (B'PB)) K, \tag{24}
\]
where the pair $(A, B)$ has the form given in (20). By
induction, for the case $m > 2$, there exist $P = \text{diag}\{P_1, \gamma_1 P_2, \ldots, \gamma_{m-1} P_m\}$ and $K = \text{diag}\{k_1, k_2, \ldots, k_m\}$
with sufficiently large $\gamma_1, \gamma_2, \ldots, \gamma_{m-1}$ such that (24) holds.
It follows from $\Sigma_1 \odot (B'PB) \geq J$ that (13) holds with
$A = A, B = B, C = K$. Therefore, the closed-loop NCS
is mean square stable via state feedback.

Remark 3.2: Note that for the single-input case, the bounds
presented in (21) and (22) are consistent by taking $\Sigma_1 = \Sigma_2 =
\Sigma$, and Theorem 3.1 is reduced to Corollary 8.4 of [11].
For multi-input case, the tightness of (21) and (22) is affected
by the structure of $\Sigma$, and (21) is also related to a particular
Wonham decomposition in (20) which is not unique in general.

Example 3.1: Let the system parameters of the plant (3) be
given by $A = \begin{bmatrix} 1.1 & 1 \\ 0 & -1.2 \end{bmatrix}$, $B = I$, $C = I$. Further suppose
that $\Sigma = \begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix}$ and $\mu_1 = \mu_2 = \mu$. Based on (21) and $\Sigma_1 \geq \Sigma$, we can conclude that a sufficient condition for mean square stabilizability via state feedback is $\mu > 0.718$. Note that $[\Sigma_1]_{11} = 0.5$ solves the optimization problem $\min_{\Sigma_2 \geq \Sigma} \prod_{i=1}^{2} \left( 1 + \frac{\mu_i^2}{\Sigma_{2i}^2} \right)$. In view of (22), a necessary condition for mean square stabilizability via state feedback is $\mu > 0.4$.

As we can see from the above example, there exists a gap between the conditions (21) and (22), which motivates us to further refine Theorem 3.1 under the following two typical transmission policies for the network (5)-(6).

- **Parallel transmission strategy (PTS):** Each element of $v(t)$ is sent across an individual fading channel, and $\xi_1(t), \xi_2(t), \ldots, \xi_m(t)$ are uncorrelated with each other.

- **Serial transmission strategy (STS):** All elements of $v(t)$ are transmitted over the same fading channel one after another, and $\xi_1(t) = \xi_2(t) = \cdots = \xi_m(t)$.

**Remark 3.3:** In some industrial and military applications where a group of aerial, ground, and/or underwater vehicles/robots are geographically separated from each other and from the remote controller, the control signal to each vehicle/robot is sent across an individual channel, which motivates the consideration of PTS. As mentioned in Remark 2.1, $\xi_1(t), \xi_2(t), \ldots, \xi_m(t)$ can be made to be uncorrelated with each other in PTS by adopting an orthogonal access scheme. For STS, the condition $\xi_1(t) = \xi_2(t) = \cdots = \xi_m(t)$ is valid in applications when the fading is coherent over the sampling interval. Note that the network model with parallel channels or a single channel has been widely used in the literature on networked estimation and networked control; see, e.g., [4]–[6], [16], [22].

Note that for a single channel ($m = 1$), the mean square capacity was introduced in [11] as $C_{MS} \triangleq \frac{1}{2} \ln \left( 1 + \frac{\mu^2}{\sigma^2} \right)$. Under PTS or STS, define the overall mean square capacity of the network as

$$C_{MS} \triangleq \sum_{i=1}^{m} \frac{1}{2} \ln \left( 1 + \frac{\mu_i^2}{\sigma_i^2} \right).$$

**Remark 3.4:** Different from the mean square capacity which is given in terms of fading statistics, the Shannon capacity of a fading channel when only the distribution of $\xi(t)$ is known at the transmitter and the receiver cannot be explicitly given in general [3].

Further denote $C_{MSI} \triangleq \frac{1}{2} \ln \left( 1 + \frac{\mu^2}{\sigma^2} \right)$, $g_i \triangleq 1 + \frac{\mu_i^2}{\sigma_i^2}$ and $g \triangleq \prod_{i=1}^{m} g_i$, then it follows that $C_{MSI} = \frac{1}{2} \ln g_i$, $C_{MS} = \frac{1}{2} \ln g$, and the larger the $g_i$, the larger the $C_{MSI}$. Note that under STS, $C_{MS} = m C_{MSI}$. Next, we will present a tight lower bound on the overall mean square capacity of the network for mean square stabilizability under PTS and STS, respectively.

### A. PTS

Under PTS, the overall mean square capacity is considered as the network resource and assumed to satisfy the following assumption.

**Assumption 1:** The overall mean square capacity of the network is fixed and can be allocated among the parallel channels.

**Remark 3.5:** It is worth mentioning that the resource (e.g., power, code, time and frequency) allocation technique has been employed extensively for studying capacity maximization in communications [30], [31], and recently it has attracted the interests from the control community as well [16], [17].

A relationship between the overall mean square capacity and the Mahler measure of the plant for ensuring the mean square stabilizability via state feedback under PTS is characterized by the next proposition.

**Proposition 3.1:** Under PTS and Assumption 1, the NCS in Fig. 1 is mean square stabilizable via state feedback if and only if

$$C_{MS} > \ln M(A).$$

**Proof:** Note that under PTS, $\sigma_{ij} = 0$ for all $i \neq j$, i.e., $\Sigma$ is diagonal. The necessity of (26) for mean square stabilizability via state feedback follows from (22) in Theorem 3.1 by taking $\Sigma_1 = \Sigma$.

On the other hand, under Assumption 1 on capacity allocation and $C_{MS} > \ln M(A)$, i.e., $g > M(A)^2 = \prod_{i=1}^{m} M(A_i)^2$, we can always choose $g_i > M(A_i)^2$ such that the condition (21) in Theorem 3.1 is true by selecting $\Sigma_1 = \Sigma$.

Therefore, there exist a stabilizing state feedback gain $K = \text{diag}\{k_1, k_2, \ldots, k_m\}$ as in Theorem 3.1 and an allocation $\{C_{MS1}, C_{MS2}, \ldots, C_{MSm}\}$ satisfying $C_{MS} = \sum_{i=1}^{m} C_{MSi}$ such that the closed-loop system is mean square stable.

**Remark 3.6:** Similar to the well-known data rate theorem [13] which quantifies the minimal data rate necessary for stabilizing an unstable system, Proposition 3.1 characterizes the minimal overall mean square capacity of the network in order to stabilize an unstable system in the mean square sense. The result also pinpoints a relationship between the minimal overall mean square capacity and the unstable eigenvalues of the system matrix $A$.

### B. STS

Based on Schur complement, the inequality (13) in Lemma 1 (iv) under STS with $A = A, B = B, C = K$ is equivalent to

$$\begin{bmatrix}
-S & (AS + BY)' (\sqrt{\frac{1}{\rho(A)}} BY)' \\
\sqrt{\frac{1}{\rho(A)}} BY & 0
\end{bmatrix} < 0$$

with $S = P^{-1}/\mu$, $Y = KP^{-1}$. Then, we can deduce the following result.

**Proposition 3.2:** Under STS, the NCS in Fig. 1 is mean square stabilizable via state feedback if and only if

$$C_{MS} > \frac{m}{2} \ln g_{1e},$$

where

$$g_{1e} = \begin{cases} M(A)^2, & \text{if } m = 1; \\ \rho(A)^2, & \text{if } m = n; \\ \inf_{s>0} g_1, & \text{subject to (27), otherwise.} \end{cases}$$


Furthermore, it holds that
\[
\frac{m}{2} \ln g_{1c} \geq \ln M(A). \tag{30}
\]

Proof: First, we will show the expression for \(g_{1c}\) in (29). When \(m = 1\), the STS is the same as the PTS, thus the critical value \(g_{1c}\) for mean square stabilizability follows directly from Proposition 3.1. Note that (16) in Lemma 2 iii) under STS becomes
\[
P > A'PA - \frac{\mu_1^2}{\sigma_1^2 + \mu_1^2} A'PB'PB^{-1}B'PA. \tag{31}
\]

When \(m = n\), the inequality (31) is further reduced to
\[
P > \left(g_1^{-\frac{1}{2}}A\right) P \left(g_1^{-\frac{1}{2}}A\right). \tag{32}
\]

In this case, the mean square stabilizability is equivalent to \(\rho(g_1^{-\frac{1}{2}}A) < 1\), and thus the critical value \(g_{1c}\) is given by \(\rho(A)^2\). For general cases, we can use (27) as the necessary and sufficient condition for mean square stabilizability. Observe that if (27) is true for some \(g_1 = g_{1a} > 1\), then it holds for all \(g_1 \geq g_{1a}\). Therefore, the critical value \(g_{1c}\) can be obtained via the minimization of \(g_1\) over (27).

Next, we will prove (30) for an arbitrary \(m\). Since all the eigenvalues of \(A\) are assumed to be on or outside the unit circle, it follows from (31) that
\[
\det(P) > \det(A) \det \left( I - \frac{\mu_1^2}{\sigma_1^2 + \mu_1^2} PB'B^{-1}B' \right) \times \\
\det(P) \det(A') \\
= \det(A)^2 \det(P) \left( I_m - \frac{\mu_1^2}{\sigma_1^2 + \mu_1^2} I_m \right) \\
= M(A)^2 \det(P) g_{1c}^{-m},
\]

which gives (30). In particular, the equality in (30) holds for \(m = 1\), while for \(m = n\), we have \(\frac{m}{2} \ln g_{1c} = \ln \rho(A)^n \geq \ln M(A)\).

Remark 3.7: Note that Proposition 3.2 is consistent with Lemma 5.4 of [7] where the packet-loss issue is considered. Except for some special cases, the critical value \(g_{1c}\) is, in general, not connected with the system matrix \(A\) explicitly. From Propositions 3.1 and 3.2, we can conclude that if the overall mean square capacity is allocatable among fading channels, i.e., additional flexibility of designing the communication component is added, then the optimization in (29) is avoided, and the requirement on overall mean square capacity would be reduced when \(m > 1\).

The next example shows that Propositions 3.1 and 3.2 refine the result of Theorem 3.1 under PTS and STS, respectively.

Example 3.2: Reconsider the plant given in Example 3.1. For the case of PTS (e.g., when an orthogonal access scheme is adopted), we let \(\Sigma = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\) and \(\mu_1 = \mu_2 = \mu\). According to Theorem 3.1, a sufficient condition for mean square stabilizability via state feedback is \(\mu > 0.6633\) and a necessary condition is \(\mu > 0.5657\). Under Assumption I on capacity allocation and according to Proposition 3.1, the above necessary condition is also sufficient for mean square stabilizability via state feedback.

For the case of STS, we let \(\Sigma = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\) and \(\mu_1 = \mu_2 = \mu\). By applying Theorem 3.1, a sufficient condition for mean square stabilizability via state feedback is \(\mu > 0.8062\) and a necessary condition is \(\mu > 0.6633\). Based on Proposition 3.2, a necessary and sufficient condition for mean square stabilizability via state feedback is \(\mu > 0.6633\).

IV. NETWORK REQUIREMENT FOR STABILIZABILITY VIA OUTPUT FEEDBACK

In the case of output feedback, we will base our analysis on both the state-space and stable coprime factorization approaches. The frequency variable \(z\) will be omitted whenever no confusion is caused.

In view of the framework of fading channels in [11], the network (5)-(6) under PTS is equivalent to the structure shown in Fig. 2, where \(z(t) = v(t), w(t) = \Delta(t)z(t), \text{ and the uncertainty block } \Delta(t)\text{ is defined as } \Delta(t) = \Lambda^{-1}(\xi(t) - \Pi).\)

It is easy to verify that the mean square stability of the NCS in Fig. 1 with the network (5)-(6) always implies the internal stability of the corresponding closed-loop system with \(w(t) = 0\). Denote the set of all proper controllers achieving the above internal stability by \(\mathcal{K}\). The transfer function of the closed loop from \(w(t)\) to \(z(t)\), without considering the uncertainty block \(\Delta(t)\), is given by
\[
T(z) = (I - K(z)G(z))^{-1}K(z)G(z). \tag{32}
\]

As proved in [11], under PTS with given \(\Pi\) and \(\Lambda\) in (8), the mean square stabilization of the NCS in Fig. 1 is equivalent to that
\[
\inf_{K(z) \in \mathcal{K}, \theta > 0, \text{diag}} \|\Theta^{-1}T(z)\Theta\|^2_{\text{MS}} < 1, \tag{33}
\]
where \(T(z)\) is given in (32) and \(\Theta \in \mathbb{R}^{m \times m}\) is a positive-definite diagonal matrix. Similarly to other robust control problems, the search for the optimal controller \(K(z)\) on the left-hand side of (33) is generally nonconvex in \(\Theta\), and the minimal overall capacity for stabilizability has no explicit solution in general.

Let a pair of right and left coprime factorizations of \(G(z)\) be given in the familiar way, namely
\[
G = NM^{-1} = \tilde{M}^{-1}\tilde{N}, \tag{44}
\]

where \(N, M, \tilde{N}, \tilde{M} \in \mathcal{H}_\infty\), and satisfy the double Bezout identity
\[
\begin{bmatrix} \tilde{Y} & -\tilde{X} \\ -\tilde{N} & M \end{bmatrix} \begin{bmatrix} X & Y \\ N & Y \end{bmatrix} = I
\]
for some \(X, Y, \tilde{X}, \tilde{Y} \in \mathcal{H}_\infty\). Then, the set of all proper controllers achieving the internal stability of the closed loop

![Network diagram](image-url)
in Fig. 1 with the network in Fig. 2 can be parameterized as
\[ K = \{ K(z) : K(z) = \Pi^{-1}(X + MQ)(Y + NQ)^{-1} \}
\]
\[ = \Pi^{-1}(\tilde{X} + QN)^{-1}(X + MQ), Q \in \mathcal{H}_\infty. \] (35)

In this situation, the transfer function \( T(z) \) in (32) becomes
\[ T(z) = \Pi^{-1}(X \tilde{N} + MQ \bar{N}) \Lambda, \] (36)
where \( \Lambda \) and \( A \) are given in (8).

A. SISO Plants
For SISO plants, we have \( m = p = 1, \) and \( \xi(t) = \xi_1(t) \) is a scalar-valued white noise signal. In this case, the STS and the PTS are the same.

Consider an SISO plant \( G(z) \) with \( n_\phi \) anti-stable poles \( \phi_1, \phi_2, \ldots, \phi_{n_\phi}, \) \( n_\zeta \) distinct\(^4\) nonminimum phase zeros \( \zeta_1, \zeta_2, \ldots, \zeta_{n_\zeta} \) and relative degree \( r \geq 1. \) It is easy to derive that
\[ \mathcal{M}(G) = \left\{ \prod_{i=1}^{n_\phi} |\phi_i|, \right. \quad \text{if} \quad n_\phi \geq 1; \]
\[ = 1, \quad \text{if} \quad n_\phi = 0, \] (37)
which is equal to \( \mathcal{M}(A) \) with \[ \begin{bmatrix} A & B \\ C & D_1 \end{bmatrix} \] being any detectable and stabilizable realization of \( G(z). \) The Blaschke product with respect to the anti-stable poles of \( G(z) \) is given by [14]
\[ B_\phi(z) \triangleq \prod_{i=1}^{n_\phi} \frac{z - \phi_i}{1 - z \phi_i}. \] (38)
Since \( B_\phi(z) \) is inner [34, p. 66], we can derive that \( B_\phi(z^{-1}) = \sum_{k=0}^\infty \beta_k z^{-k} \) with \( \beta_k \triangleq \frac{1}{\pi} \int \frac{B_\phi(z)}{z} \frac{dz}{z}. \) Further denote [35]
\[ \eta(G) \triangleq \left\{ \begin{array}{ll} \sum_{i=1}^{n_\psi} \gamma_k \zeta_k^{-i} - 1, & \text{if } \psi \neq 0, \\
0, & \text{if } \psi = 0, \end{array} \right. \] (39)
\[ \varphi(G) \triangleq \left\{ \begin{array}{ll} \sum_{i=1}^{n_\psi} (|\beta_i|^2 + |\psi_i|^2), & \text{if } \psi \neq 0, \\
0, & \text{if } \psi = 0, \end{array} \right. \] (40)
where \( \psi_i \triangleq \sum_{k=1}^{n_\psi} \gamma_k \zeta_k^{-i-1} \) and
\[ \gamma_k \triangleq \left(1 - |\zeta_k|^2 \right) \left( B_\phi(\zeta_k^{-1}) - \sum_{i=0}^{r-1} \beta_i \zeta_i^{-1} \right) \prod_{i=1, i \neq k}^{n_\psi} \frac{1 - \zeta_i \zeta_k}{\zeta_i - \zeta_k}. \]

**Proposition 4.1:** Assume that \( G(z) \) is SISO and has no or distinct (if any) nonminimum phase zeros. The NCS in Fig. 1 is mean square stabilizable via dynamic output feedback if and only if
\[ \text{CMS} > \frac{1}{2} \log \left\{ \mathcal{M}(G)^2 + \eta(G) + \varphi(G) \right\}, \] (41)
where \( \mathcal{M}(G), \eta(G), \varphi(G) \) are defined in (37), (39) and (40), respectively.

**Proof:** For \( m = 1, \) we have \[ \| T\eta^{-1}(T(z)\|_2^2 = \| T(z) \|_2^2 \]
with \( T(z) = (X \tilde{N} + MQ \bar{N})(g - 1)^{-\frac{1}{2}}. \) It follows that (33) is equivalent to
\[ g > \inf_{Q \in \mathcal{H}_\infty} \| X \tilde{N} + MQ \bar{N} \|_2^2 + 1. \] (42)

\(^4\)The assumption on distinct nonminimum phase zeros simplifies the subsequent analysis, and may be relaxed at the expense of more complex expressions [33].

The evaluation of the right-hand side of (42) follows from [14], [35].

**Remark 4.1:** Note that \( \eta(\cdot) \) and \( \varphi(\cdot) \) are always nonnegative. As we can see from Propositions 3.1 and 4.1, when the plant is SISO, the required capacity for mean square stabilizability via output feedback is no less than that via state feedback, and they are identical if the plant is minimum phase with relative degree 1.

B. Triangularly Decoupled Plants
Motivated by the observation that Wonham decomposition plays an important role in establishing the sufficiency part of Theorem 3.1, we adopt the next definition of triangular decoupling.

**Definition 4.1 ([23]):** \( G(z) \) is said to be triangularly decoupled by \( K(z) \in \mathcal{K}, \) if \( T(z) \) in (36) is either lower or upper triangular.

For \( G = NM^{-1} \) with \( M, N \in \mathcal{H}_\infty \) and \( m \leq p, \) there always exist suitable unimodular matrices \( U, M, N \) such that
\[ L_M = MU_M, \quad [L_N \quad 0] = UNNU_M, \quad UNG = \left[ L_N L_M^{-1} \quad 0 \right]. \] (43)

where \( L_M, L_N \) are respectively square and lower (or upper) triangular with compatible dimensions. We refer to \( L_G \triangleq L_N L_M^{-1} \) as the left triangular structure of \( G(z). \)

**Remark 4.2:** As shown in [23], (i) \( L_M \) and \( L_N \) are unique up to postmultiplication and/or premultiplication by any lower (or upper) triangular unimodular matrix, and are independent of the particular factorization of \( G(z) \) in (34); (ii) a necessary and sufficient condition for the solvability of triangular decoupling is the coprimeness between each diagonal entry of \( L_M \) and the corresponding diagonal entry of \( L_N. \)

We sum up the assumptions as follows.

**Assumption 2:**

i. The plant \( G(z) \) can be triangularly decoupled and satisfies \( m \leq p. \)

ii. The \( i \)-th diagonal entry of the left triangular structure, \( [L_G]_{ii}, \) has no or distinct (if any) nonminimum phase zeros for every \( i = 1, 2, \ldots, m. \)

Note that \( [L_G]_{ii} \) has relative degree \( r_i \geq 1, \) since \( U_N(z) \) in (43) is unimodular and \( G(z) \) is strictly proper. Suppose that \( U_N(z) \) has a minimal realization \[ \begin{bmatrix} A_U & B_u \\ C_U & D_U \end{bmatrix}, \] then \( \rho(A_U) < 1 \) since \( U_N(z) \) is unimodular. Under Assumption 2 i),
\[ \begin{bmatrix} A \\ B_u C A_U \end{bmatrix}, \]
is a stabilizable and detectable realization for \( U_N(z)G(z). \) It follows that
\[ \prod_{i=1}^{m} \mathcal{M}([L_G]_{ii}) = \mathcal{M}(U_N G) = \mathcal{M} \left[ \begin{bmatrix} A \\ B_u C A_U \end{bmatrix} \right] \]
\[ = \mathcal{M}(A) = \mathcal{M}(G). \] (44)
Theorem 4.1: Under PTS and Assumptions 1 and 2, the NCS in Fig. 1 is mean square stabilizable via dynamic output feedback if
\[
C_{MS} > \frac{1}{2} \sum_{i=1}^{m} \ln \left\{ \mathcal{M}([L_G]_{ii})^2 + \eta([L_G]_{ii}) + \varphi([L_G]_{ii}) \right\},
\]
and only if
\[
C_{MS} > \ln \mathcal{M}(A).
\]

Proof: We will first show the necessity. Assume that there exists \( K(z) \) such that the closed loop is mean square stable. Since any dynamic output feedback controller can always be constructed from a dynamic state feedback controller, the condition (46) follows directly from the equivalence between i) and ii) in Lemma 2 and Proposition 3.1.

Next, the sufficiency of (45) will be shown. Suppose that \( L_G \) is in a lower triangular form. Let a left stable coprime factorization of \( L_G \) be \( L_{\bar{G}} \). According to Assumption 2 i) and Remark 4.2, \([L_M]_{ii}\) and \([L_N]_{ii}\) are coprime, and thus there always exist lower triangular \( L_X, L_Y, L_{\bar{X}}, L_{\bar{Y}} \in \mathcal{H}_\infty \) such that
\[
\begin{bmatrix}
L_{\bar{G}} & -L_{\bar{X}} \\
-L_{\bar{N}} & L_N
\end{bmatrix}
\begin{bmatrix}
L_M & L_X \\
L_N & L_Y
\end{bmatrix} = I.
\]

It is easy to see that \([L_G]_{ii} = [L_N]_{ii}[L_M]_{ii}^{-1} = [L_{\bar{N}}]_{ii}^{-1}[L_{\bar{X}}]_{ii}\), and
\[
\begin{bmatrix}
[L_{\bar{G}}]_{ii} & -[L_{\bar{X}}]_{ii} \\
-[L_{\bar{N}}]_{ii} & [L_N]_{ii}[L_Y]_{ii}
\end{bmatrix} = I.
\]

If we construct right and left coprime factorizations of \( U_N G \) as
\[
U_N G = \begin{bmatrix}
L_N \\
0
\end{bmatrix}^{-1} L_M = \begin{bmatrix}
L_{\bar{N}} & 0 \\
0 & I
\end{bmatrix}^{-1} \begin{bmatrix}
L_{\bar{N}} \\
0
\end{bmatrix},
\]
then the set of all proper controllers achieving triangular decoupling can be parameterized as
\[
K_L \triangleq \left\{ [L_K] \mid U_N \in \mathcal{K} : L_K = \Pi^{-1}(L_X + L_M L_Q) \right\}.
\]

By choosing \( \Theta_1 = \Pi \Theta = \text{diag}\{\epsilon^{m-1}, \epsilon^{m-2}, \ldots, 1\} \) with a small real number \( \epsilon > 0 \), it follows that
\[
\Theta^{-1} T(z) \Theta = \Theta^{-1} \Pi^{-1}(L_X L_{\bar{N}} + L_M L_Q L_{\bar{N}}) \Pi \Theta
= \Theta^{-1}(L_X L_{\bar{N}} + L_M L_Q L_{\bar{N}}) \Lambda \Pi \Theta
= \text{diag}\left\{ \frac{1}{g_1-1}, \frac{1}{g_2-1}, \ldots, T_m(z) \sqrt{\frac{1}{g_m-1}} \right\} + o_\epsilon(\epsilon),
\]
where \( T_i(z) = [L_N]_{ii}[L_{\bar{N}}]_{ii} + [L_M]_{ii}[L_Q]_{ii}[L_{\bar{N}}]_{ii} \) with \([L_Q]_{ii} \in \mathcal{H}_\infty\). According to the proof of Proposition 4.1, it follows that \( \epsilon > 0 \).

Remark 4.3: Note that the proof of the necessity does not rely on Assumptions 1 and 2, thus (46) actually provides a uniform lower bound on the overall capacity for stabilizability via output feedback under PTS over all proper stabilizing controllers, which is also consistent with the minimum overall capacity for stabilizability via state feedback under PTS as shown in Proposition 3.1. In view of (44), the gap between (45) and (46) shrinks to zero if \([L_G]_{ii} \) is minimum phase with relative degree 1 for all \( i = 1, 2, \ldots, m \).

V. THE EFFECT OF PRE- AND POST-CHANNEL PROCESSING AND CHANNEL FEEDBACK

As pointed out in Remark 3.7, additional degree of freedom can benefit the NCS in reducing the capacity requirement for stabilizability. Here, we further analyze the possible advantages of introducing the pre- and post-channel filters and channel feedback into the NCS.

Consider the NCS in Fig. 3 and limit our attention to the LTI case. The system interconnections are described as follows:

\[
\begin{array}{ccc}
\text{Controller} & \text{Channel Feedback} & \text{Network} \\
K(z) & F(z) & G(z)
\end{array}
\]

![Fig. 3. Feedback control over an input network with filtering and channel feedback.](image)

\[
y = G(z) u, \quad v = K(z) y, \quad s = F_1 v + F_1 h, \quad u = F_2 h, \quad h(t) = \xi(t) s(t),
\]

where \( F_1(z) = [F_1 v(z) F_1 h(z)], s(t) \in \mathbb{R}^m \) is the channel input, \( h(t) \in \mathbb{R}^m \) is the channel output, and \( \xi(t) \) has the form as in (6). We assume that the channel feedbacks back to the pre-filter with one-step delay, therefore \( F_1 h(z) \) is strictly proper.
Remark 5.1: The channel feedback in Fig. 3 is equivalent to sending the exact value of $\xi(t)$ which is also known as channel side information back to the pre-filter [3]. If $\xi(t)$ models the packet-loss process, then the channel feedback amounts to the packet acknowledgments in the TCP-like protocol [7]. For a fading channel with channel reciprocity\(^5\), the channel feedback can be realized by sending powerful pilot signals to the pre-filter, which allows the pre-filter to compute the channel fading [12].

Theorem 5.1: Consider the NCS in Fig. 3, where $K(z), F_1(z), F_2(z)$ are to be designed. Under PTS and Assumption 1, the networked system is mean square stabilizable via state feedback or dynamic output feedback if and only if $C_{MS} > \ln \mathcal{M}(A)$.

Proof: First, we will show the necessity for the dynamic state feedback case, which also implies the necessity for the static state feedback and dynamic output feedback cases. Note that Lemma 1 is still true with the overall system state $x(t) = [x_G(t)\ x_K(t)\ x_F_1(t)\ x_F_2(t)]'$, where $x_F_1(t)$ and $x_F_2(t)$ are the states of the pre-filter and the post-filter, respectively. Suppose that $F_2(z)$ has a state-space realization
\[
\begin{bmatrix}
A_F & B_F \\
C_F & D_F
\end{bmatrix}
\]
then the interconnection between the plant and the post-filter yields
\[
G(z)F_2(z) = \begin{bmatrix}
A & BC_F \\
0 & A_F \\
C & 0
\end{bmatrix}
\begin{bmatrix}
BD_F \\
0
\end{bmatrix}
\]

Considering $G(z)F_2(z)$ as an augmented plant and following similar lines of the proof of Theorem 4.1, we have
\[
\prod_{i=1}^{m} \left(1 + \frac{\mu_i^2}{\sigma_i^2}\right) \geq \mathcal{M}(A)^2 \mathcal{M}(A_F)^2 \geq \mathcal{M}(A)^2,
\]
i.e., $C_{MS} > \ln \mathcal{M}(A)$.

To show the sufficiency, a constructive proof will be given. Let $F_{1u}(z) = I, F_2(z) = I$, then we have $s = K(z)y + F_{1h}(z)u$. Construct an observer-based realization for $[K(z) F_{1h}(z)]$ as
\[
\dot{x}_G(t+1) = A\dot{x}_G(t) + L(y(t) - C\dot{x}_G(t)) + Bu(t),
\]
\[
s(t) = K\dot{x}_G(t),
\]
where $L$ and $K$ are respectively the observer gain and the state feedback gain to be designed. Then $F_{1h}(z)$ is strictly proper, and the closed loop with $x(t) = [x_G(t)\ \ddot{x}_G(t)\ ]'$ is given by
\[
x(t+1) = \begin{bmatrix}
A & 0 \\
\frac{1}{L}C & A - LC
\end{bmatrix}
\begin{bmatrix}
B \\
0
\end{bmatrix}
\begin{bmatrix}
\xi(t)[0\ K]
\end{bmatrix}
x(t)
\]
\[
= \begin{bmatrix}
I & 0 \\
I & I
\end{bmatrix}
\begin{bmatrix}
A & 0 \\
0 & A - LC
\end{bmatrix}
\begin{bmatrix}
B \\
0
\end{bmatrix}
\begin{bmatrix}
\xi(t)[0\ K]
\end{bmatrix}
\]
\[
= \begin{bmatrix}
I & 0 \\
I & I
\end{bmatrix}
^{-1}
x(t).
\]

VI. EXTENSION AND APPLICATION
A. Extension on Stabilization over Output Fading Channels

Now, consider an NCS with an output network as depicted in Fig. 4, where
\[
y = G(z)u, \quad u = K(z)v, \quad \xi(t) = \xi(t)y(t).
\]

Motivated by applications where a sensor network is used to monitor a phenomenon/object of interest (e.g., a moving vehicle to be tracked) and the sensors are geographically distributed and are not colocated with the fusion center and/or the remote controller, we let the fading gain $\xi(t)$ in (51) and the overall mean square capacity of the network be given by $\xi(t) = \text{diag} \{\xi_1(t), \xi_2(t), \ldots, \xi_p(t)\}$ and $C_{MS} = \sum_{m=1}^{p} C_{MS_m}$, respectively. In this subsection, we assume that the controller only knows the first and second moments but not the exact value of $\xi(t)$.

Note that for $G = \tilde{M}^{-1}\tilde{N}$ with $\tilde{M}, \tilde{N} \in \mathbb{H}_\infty$ and $p \leq m$, there exist suitable unimodular matrices $U_M, U_N$ such that $L_M = U_M M, [L_N_0] = U_N \tilde{N} U_M^*, G_U = [L_M^{-1} L_N^{-1}]$, where $L_M, L_N$ are respectively square and lower (or upper) triangular with compatible dimensions. We refer to $L_M^{-1} L_N^{-1}$ as the right triangular structure of $G(z)$.

The proposition that follows extends the previous results on the NCS in Fig. 1.

Proposition 6.1: Consider the NCS in Fig. 4.

Networks.
a) Assume that \( B = I \). Under PTS and Assumption 1, the networked system is mean square stabilizable via static output feedback if and only if \( C_{\text{MS}} > \ln \mathcal{M}(A) \).

b) Assume that \( G(z) \) is SISO and has no or distinct (if any) nonminimum phase zeros. The networked system is mean square stabilizable via dynamic output feedback if and only if \( C_{\text{MS}} > \frac{1}{2} \sum_{i=1}^{p} \ln \left\{ \mathcal{M}(L_{G}) + \eta(L_{G}) + \varphi(L_{G}) \right\} \), and only if \( C_{\text{MS}} > \ln \mathcal{M}(A) \).

c) Assume that \( G(z) \) is decoupled, \( p \leq m \), and \([L_{G}]_{ii}\) has no or distinct (if any) nonminimum phase zeros. Under PTS and Assumption 1, the networked system is mean square stabilizable via dynamic output feedback if \( C_{\text{MS}} > \frac{1}{2} \sum_{i=1}^{p} \ln \left\{ \mathcal{M}(L_{G}) + \eta(L_{G}) + \varphi(L_{G}) \right\} \), and only if \( C_{\text{MS}} > \ln \mathcal{M}(A) \).

Proof: The proof follows analogously to those of Propositions 3.1, 4.1 and Theorem 4.1.

The extensions of Proposition 3.2 and Theorem 5.1 to the case of output fading channels are omitted for brevity.

B. Application to Stabilization of a Finite Platoon

In automated highway systems, one basic issue is to move a platoon of closely spaced vehicles from one place to another [36], [37]. Suppose that there are \( p+1 \) vehicles in a platoon, and the leader, i.e., vehicle 0, generates a position trajectory \( \{x(t)\}_{t \geq 0} \) with \( x(0) = 0 \) according to its local reference signal. Each follower, i.e., vehicle \( i \), for all \( i = 1, 2, \ldots, p \), uses its ranging sensor and local controller to keep a fixed distance behind the preceding vehicle.

Given any time-domain signal \( x(t) \), denote its \( z \)-transform by \( X(z) \). Let \( x_i(t), u_i(t), G_i(z) \) be the position, the input and the dynamics of the \( i \)-th vehicle, and model the \( i \)-th vehicle starting from rest by

\[
X_i(z) = G_i(z)U_i(z) + \frac{z x_i(0)}{z - 1}, \quad i = 1, \ldots, p. \tag{52}
\]

We assume that \( G_i(z) \) is SISO and strictly proper. The vehicle separations \( e_i(t), i = 1, 2, \ldots, p \), are defined as \( e_i(t) = x_i(t) - x_{i-1}(t) + \tau \), where \( \tau > 0 \) is a constant target separation for all followers. Suppose that the initial position of the \( i \)-th vehicle, \( x_i(0) \), is \( -i\tau \) indicating that the platoon starts with zero spacing errors. It follows that

\[
E(z) = G(z)U(z) - [1 0 \cdots 0]' X_0(z)
\]

with \( E(z) = \begin{bmatrix} E_1(z) & E_2(z) & \cdots & E_p(z) \end{bmatrix}' \),

\[
U(z) = \begin{bmatrix} U_1(z) & U_2(z) & \cdots & U_p(z) \end{bmatrix}'.
\]

If we take the separation \( e_i(t) \) as the output of vehicle \( i \) and let its ranging sensor obtain \( e_i(t) \) through the \( i \)-th channel as \( v_i(t) = e_i(t) \), then without considering \( X_0(z) \), the platoon system fits in the model of Fig. 4 with \( m = p, y(t) = e(t) \). Since the local controller on vehicle \( i \) only uses \( v_i(t) \), the distributed controller in Fig. 4 has the form

\[
U(z) = \text{diag} \{ K_1(z), K_2(z), \ldots, K_p(z) \} V(z), \tag{54}
\]

where \( V(z) = [V_1(z) V_2(z) \cdots V_p(z)]' \) is the output vector of parallel channels. Assume that each channel experiences transmission failure and Nakagami fading [3] simultaneously:

\[
\xi_i(t) = \Omega_i(t) \Upsilon_i(t), \quad i = 1, 2, \ldots, p, \tag{55}
\]

and \( \xi_1(t), \xi_2(t), \ldots, \xi_p(t) \) are uncorrelated with each other. In (55), \( \Omega_i(t) \) is 0/1-valued (0 for “failure”, 1 for “success”) with probability distribution

\[
Pr\{\Omega_i(t) = 0\} = \alpha_i, \quad Pr\{\Omega_i(t) = 1\} = 1 - \alpha_i, \quad 0 \leq \alpha_i < 1,
\]

and \( \Upsilon_i(t) \) is Nakagami distributed with \( \alpha_i \) denoting the mean channel power gain and \( g_i \in [2, \infty) \) describing the severity of fading (i.e., the severity of fading decreases as \( g_i \) increases) [12]. Further assume that \( \Omega_i(t), \Upsilon_i(t) \) are uncorrelated, then it is easy to derive that

\[
g_i = \frac{g_i \Gamma(g_i)^2}{\Gamma(1) \Gamma(g_i) - \Gamma(1 + \frac{1}{g_i})^2}, \tag{56}
\]

where \( \Gamma(\cdot) \) is the gamma function. Based on the expression of \( g_i \) in (56) and \( C_{\text{MSi}} = \frac{1}{2} \ln g_i \), it is intuitive to see that the smaller the \( \alpha_i \), the larger the \( g_i \), the larger the \( \Gamma(\cdot) \) and \( C_{\text{MSi}} \). The effect of transmission failure and Nakagami fading disappears as \( \alpha_i \rightarrow 0 \) and \( g_i \rightarrow \infty \), respectively. In particular, we have \( \lim_{\alpha_i \rightarrow 0} g_i = \frac{1}{\alpha_i} \) since \( \lim_{g_i \rightarrow \infty} \frac{\Gamma(1) \Gamma(g_i)}{\Gamma(1 + \frac{1}{g_i})^2} = 1 \), which is reduced to the scenario considered in Corollary 5.1 of [25].

We have the next corollary on stabilization of a finite platoon.

Corollary 6.1: Assume that \( G_i(z) \), for all \( i = 1, 2, \ldots, p \), is SISO, strictly proper and has no or distinct (if any) nonminimum phase zeros. Then, the finite platoon system described above with \( p+1 \) vehicles is mean square stabilizable via dynamic output feedback over the network (55) if and only if \( C_{\text{MS}} > \frac{1}{2} \sum_{i=1}^{p} \ln \left\{ \mathcal{M}(G_i) + \eta(G_i) + \varphi(G_i) \right\} \), \( i = 1, 2, \ldots, p \), where \( C_{\text{MS}} = \frac{1}{2} \ln g_i \), and \( g_i \) is given in (56).

Proof: Based on the diagonal structure of \( K(z) \) in (54) and the lower triangular structure of \( G(z) \) in (53), without considering both \( X_0(z) \) and the uncertainty block, we have

\[
T(z) = G(z)(I - K(z)P(z))^{-1}K(z)A
\]

\[
= T_1 \cdots T_p - T_1 S_2 - T_2 S_3 - T_3 \cdots - T_{p-1} S_p - T_p \Pi^{\dagger}, \tag{57}
\]

where \( S_i \) = \( (1 - K_i K_i^{-1})^{-1} \) and \( T_i = G_i(1 - K_i K_i^{-1})^{-1} K_i K_i^{-1} \). Denote the set of all proper controllers...
achieving the internal stability by \( K \). It follows from the proofs of Propositions 6.1 b) and 4.1 that
\[
\inf_{K(z) \in \mathcal{K}} \| T_i(z) \|^2_F = \mathcal{M}(G_i)^2 + \eta(G_i) + \varphi(G_i) - 1. \tag{58}
\]

For any diagonal scaling matrix \( \Theta \), the condition (33) implies that \( g_i > \mathcal{M}(G_i)^2 + \eta(G_i) + \varphi(G_i) \), which shows the necessity.

For the sufficiency, a constructive proof will be given. By choosing \( K_i(z) \) based on (58), we can ensure that \( G_i(1 - K_i \mu_i G_i)^{-1}, (1 - K_i \mu_i G_i)^{-1}K_i \mu_i \) and \( S_i \) are all stable and proper. It is then easy to verify that \( K(z) = \text{diag}\{K_1(z), K_2(z), \ldots, K_p(z)\} \in \mathcal{K} \) by invoking the special structures of \( G(z) \) and \( K(z) \), i.e., the internal stability of the overall closed loop in Fig. 4 is guaranteed. Thus, by setting \( \Theta = \text{diag}\{\epsilon^p-1, \epsilon^p-2, \ldots, 1\} \) with sufficiently small \( \epsilon \), we have (33) is true. This completes the proof. \( \blacksquare \)

The numerical example below demonstrates the usefulness of Corollary 6.1.

**Example 6.1:** Consider a homogeneous platoon with 5 vehicles and
\[
G_i(z) = \frac{1}{(z - 1)^2}, \quad i = 1, 2, 3, 4.
\]
Suppose that the leader has a constant speed: 1 meter/second. It is easy to check that the infimum in (58) is 0 which is not achievable by any stabilizing controller due to the double marginally stable poles of \( G_i(z) \). If we approximate \( z = 1 \) by \( z = 1 - \varepsilon \) with a small real number \( \varepsilon > 0 \), then the design procedure in [35] yields the controller:
\[
K_i = \frac{2\varepsilon - (2\varepsilon + \varepsilon^2)z}{(z + 2\varepsilon)\mu_i}, \tag{59}
\]
which can approach the infimum in (58) with any desired accuracy by choosing a sufficiently small \( \varepsilon \).

Let \( \alpha_i = 0.2, \quad \omega_i = 2, \quad g_i = 2 \) for the \( i \)-th channel, then \( g_i \) in (56) is 3.4113. By setting \( \varepsilon = 0.1 \) in (59), we have
\[
\| T_i(z) \|^2_F = \| G_i(1 - K_i \mu_i G_i)^{-1}K_i \mu_i \|^2 = 0.1489.
\]
Since \( \| T_i(z) \|^2_F / (g_i - 1) = 0.0436 < 1 \) for all \( i = 1, 2, 3, 4 \), Corollary 6.1 implies that the platoon can be stabilized in the mean square sense by the controller (59) with \( \varepsilon = 0.1 \). The trajectories of the vehicles in the platoon for one sample of simulation are shown in Fig. 5.

**VII. CONCLUSIONS**

It has been shown in this paper that for a discrete-time LTI plant with LTI feedback over input fading channels under PTS and the assumption on capacity allocation, the minimal overall mean square capacity of the network for mean square stabilizability can be given in terms of the Mahler measure of the plant in the case of state feedback. The minimal capacity for stabilizability under STS in general can only be computed by optimization. The mean square stabilization of SISO plants or MIMO plants via dynamic output feedback has also been investigated. For SISO plants, the corresponding minimal mean square capacity for stabilizability is given in terms of the anti-stable poles, nonminimum phase zeros and relative degree of the plant. For triangurally decoupled MIMO plants, necessary and sufficient conditions have been provided. In addition, it has been shown that the channel feedback plays a key role in eliminating the limitation on stabilization induced by the nonminimum phase zeros and high relative degree of the plant. It is expected that the approach in this paper can be extended to the continuous-time case, sampled-data control and performance control. Other future work includes the stability robustness with respect to the fading parameters \( \Pi \) and \( \Lambda \), and the string instability analysis of an infinite platoon over fading channels.

**APPENDIX A**

**PROOF OF LEMMA 1**

(i)\(\Leftrightarrow\)(ii): It follows directly from Definition 2.1.

(ii)\(\Leftrightarrow\)(iii): Based on Lemma 4.3.1 in [38], the recursion (11) can be written into \( \text{vec}(\Xi(t + 1)) = \Psi \text{vec}(\Xi(t)) + \Psi^{t+1} \text{vec}(\Xi(0)) \). It is easy to see that \( \lim_{t \to \infty} \Xi(t) = 0 \) is equivalent to the existence of \( A_K, B_K, C_K, D_K \) such that \( \rho(\Psi) < 1 \).

(i)\(\Leftrightarrow\)(iv): First, suppose that (iv) is true. Introduce a Lyapunov function \( V(\Xi(t)) = \text{tr}(\Xi(t)P) \), then
\[
V(\Xi(t + 1)) = \text{tr}\left\{ (\hat{A} + \hat{B}P\hat{C})\Xi(t)(\hat{A} + \hat{B}P\hat{C})^T \right\} + \text{tr}\left( \sum_{i=1}^{m} \sum_{j=1}^{m} \sigma_{ij} \hat{B}_i\hat{C}_j \Xi(t)\hat{C}_j^T \hat{B}_i^T P \right)
\]
\[
< \text{tr}(\Xi(t)P) = V(\Xi(t)).
\]
It follows from Lyapunov theory that \( \lim_{t \to \infty} \Xi(t) = 0 \). On the other hand, assume that the closed loop is mean square stable. Then \( \rho(\Psi) < 1 \) based on (iii), which is equivalent to
\( \rho (\Psi ) < 1 \). Therefore, the sequence \( \{ \hat{z}(t) \}_{t \geq 0} \) computed by
\[
\hat{z}(t + 1) = (A + B \Pi C)^t \hat{z}(t) (A + B \Pi C)
+ \sum_{i=1}^{m} \sigma_{ij} \hat{C}_i^t \hat{B}_j \hat{z}(t) \hat{B}_j \hat{C}_j
\]
(60) is convergent to zero as \( t \) approaches infinity for any \( \hat{z}(0) \geq 0 \).

Let \( \hat{z}(0) > 0 \) and \( P(t) = \sum_{s=0}^{t} \hat{z}(s) \), then we have
\[
P(t + 1) = \hat{z}(0) + (A + B \Pi C)^t P(t) (A + B \Pi C)
+ \sum_{i=1}^{m} \sigma_{ij} \hat{C}_i^t \hat{B}_j P(t) \hat{B}_j \hat{C}_j
\]
\[
> (A + B \Pi C)^t P(t) (A + B \Pi C)
+ \sum_{i=1}^{m} \sigma_{ij} \hat{C}_i^t \hat{B}_j P(t) \hat{B}_j \hat{C}_j.
\]
(61)

It follows from the convergence of \( \{ \hat{z}(t) \}_{t \geq 0} \) and (61) that \( P = \lim_{t \to \infty} P(t) \) exists and satisfies (13) in (iv). It is easy to see that \( P > 0 \) since \( \hat{z}(0) > 0 \). In addition, according to Theorem 5.2.1 in [38], we have \( J = \Sigma \otimes (B'PB) > 0 \) since \( \Sigma \geq 0, B'PB > 0, \) and \( \Sigma \) has no diagonal entry that is equal to zero.

(iii) ⇔ (v): Assume that (v) holds. By selecting a Lyapunov function \( V(\hat{z}(t)) = \text{tr}(\hat{z}(t)W) \), we can easily prove that (v) implies the convergence of \( \{ \hat{z}(t) \}_{t \geq 0} \) computed by (60), which further implies (iv). The converse part follows by setting \( W = \lim_{t \to \infty} \sum_{s=0}^{t} \hat{z}(s) \) and \( \hat{z}(0) > 0 \).

APPENDIX B
PROOF OF LEMMA 2

i) ⇔ ii): For static state feedback, \( \hat{A}, \hat{B}, \hat{C} \) in (10) is reduced to \( \hat{A} = A, \hat{B} = B, \hat{C} = K \). Obviously, the stabilizability via static state feedback always implies the stabilizability via dynamic state feedback. In what follows, we will show the converse is also true. Suppose the closed loop is mean square stable under a dynamic state feedback controller, i.e., there exist a set of \( A_K, B_K, C_K, D_K \) and \( W > 0 \) such that (14) in Lemma 1 is true for \( C = I \). Partition \( W \) in accordance with \( \hat{A} \in \text{in} (10) \) as \( W = \begin{bmatrix} \hat{W}_{11} & \hat{W}_{12} & \hat{W}_{13} \\ \hat{W}_{21} & \hat{W}_{22} & \hat{W}_{23} \end{bmatrix} \). In this case, the 1 × 1 block of the inequality (14) yields
\[
\begin{align*}
W_{11} > (A + B \Pi D_K) W_{11} (A + B \Pi D_K) & + (A + B \Pi D_K) W_{12} \Pi B' \\
+ B (\Sigma \otimes (D_K W_{11} D_K' + D_K W_{12} D_K' + C_K W_{12} C_K')) B' + B \Pi C_K W_{12} C_K' \Pi B' & + B \Pi C_K W_{12} (A + B \Pi D_K)' \\
(62)
\end{align*}
\]
By setting \( K = D_K + C_K W_{12} W_{11}^{-1} \), it follows from (62), the property of Hadamard product in Problem 3 of [29, p. 475], and \( W_{22} - W_{21} W_{11}^{-1} W_{12} > 0 \) that
\[
W_{11} > (A + B \Pi K') W_{11} (A + B \Pi K') + B (\Sigma \otimes (K W_{11} K')) B'.
\]
(63)
From Lemma 1 (v), the inequality (63) is equivalent to the existence of a state feedback gain \( K \) such that the closed-loop system is mean square stable.

\[ i) \iff iii): \quad \text{By substituting } \hat{A} = A, \hat{B} = B, \hat{C} = K \text{ into (13),}
\] \[ \text{we can obtain that}
\]
\[ P > (A + B \Pi K') P (A + B \Pi K) + K' J K. \]
\[ \] (64)

The inequality (64) implies (65) by setting \( K \) as in (18).

i) ⇒ iii): By taking derivative with respect to \( u \), we can conclude that \( u = K x + K \) defined in (18) minimizes the function \( f(u) = -x' P x + (A x + B \Pi u') P (A x + B \Pi u) + u' J u \).

iii) ⇔ iv): By choosing the same \( P \), iv) obviously implies iii). Next, suppose that iii) holds. Define the operators
\[
\begin{align*}
L_1(Z) &= A' Z A - A' Z B \Pi (\Sigma \otimes (B' Z B) + \Pi B' Z B) \Pi B' Z A, \\
L_2(Z, K) &= (A + B \Pi K') Z (A + B \Pi K) \]
\[ + K' (\Sigma \otimes (B' Z B)) K. \]
\[ \] (65)

It follows that \( L_2(Z, K) \) is affine in \( Z \), and for any \( Z \geq 0 \) and \( K \),
\[ 0 \leq L_1(Z) = L_2(Z, K^*(Z)) \leq L_2(Z, K) \]
with \( K^*(Z) = - (\Sigma \otimes (B' Z B) + \Pi B' Z B) \Pi B' Z A. \) For any \( Z_2 > Z_1 \geq 0 \), we have \( L_1(Z_1) = L_2(Z_1, K^*(Z_1)) \leq L_2(Z_2, K^*(Z_2)) \leq L_2(Z_2, K^*(Z)) \leq L_1(Z_2) \). Let the sequence \( \{ Z(t) \}_{t \geq 0} \) computed by \( Z(t + 1) = L_1(Z(t)) + R \) with any initial condition \( Z(0) \geq 0 \). The condition iii) can be written into \( P > L_1(P) \) with \( P > 0 \). In this case, we can always choose \( c_1 \in [0, 1) \) and \( c_2 \in (0, \infty) \) such that \( L_1(P) \leq c_1 P, Z(0) \leq c_2 P, R \leq c_2 P \). Since \( L_1(c_2 P) = c_2 L_1(P) \), we have
\[ Z(1) = L_1(Z(0)) + R \leq c_1 c_2 P + c_2 P, \]
\[ Z(2) = L_1(Z(1)) + R \leq c_2^2 c_2 P + c_1 c_2 P + c_2 P. \]
By mathematical induction, we have \( Z(t) \leq \sum_{s=0}^{t} c_1 c_2 P \leq c_2 c_2^t P \), which implies that the sequence \( \{ Z(t) \}_{t \geq 0} \) is bounded. Let \( \hat{Z}(0) = 0 \), then it follows from the monotonicity and boundedness of \( \{ Z(t) \}_{t \geq 0} \) that \( Z = \lim_{t \to \infty} Z(t) \) exists and satisfies \( Z = L_1(Z) + R \). Note that \( Z > 0 \) since \( R > 0 \) and \( L_1(Z) \geq 0 \), which completes the proof.

i) ⇔ v): In view of the equivalence between i) and ii), we limit our attention to the static state feedback case in the proof. Let \( (A, B) \) be of the form (15). Since \( A_n \) in (15) is stable, there exists \( P_1 > 0 \) such that \( P_1 - A' P_1 A_n > 0 \). The pair \( (A_n, B_n) \) is mean square stabilizable, thus based on Lemma 1 (iv), there exist \( P_2 > 0 \) and \( K_n \) such that
\[ P_2 > (A_n + B_n P K_n) P_2 (A_n + B_n P K_n) + K_n' J_n K_n, \]
where \( J_n = \Sigma \otimes (B_n' P_2 B_n) \). It follows that, for some \( \beta > 0 \), the inequality
\[ P_2 > \beta I + (A_n + B_n P K_n) P_2 (A_n + B_n P K_n) + K_n' J_n K_n \]
is true. Let \( J_n = \Sigma \otimes (B_n' P_2 B_n) \). By choosing \( P = \text{diag}(P_1, \gamma P_2), K = K_n \) with a sufficiently large \( \gamma > 0 \) such that
\[ \gamma \beta I > K_n' P_2 B_n P K_n + K_n' J_n K_n \]
\[ - K_n' P_2 B_n P_1 A_n (P_1 - A_n' P_1 A_n) - A_n' P_1 A_n, \]
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and in view of \( J = J_0 + J_a \), we have (64) holds, i.e., the pair \((A, B)\) is mean square stabilizable.

\[ i \Rightarrow v): According to the condition iii), the mean square stabilizability of \((A, B)\) in the form of (15) ensures that there exists \( P = \begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix} > 0 \) such that (16) is true. After applying the linear coordinate transformation matrix \( \begin{bmatrix} I & -P_{11} \\ 0 & I \end{bmatrix} \), the \( 2 \times 2 \) block of the inequality (16) implies that

\[ P_2 > A_i'P_2A_i - A_i'P_2B_i(J_s + J_a + \Pi(J_s + \Pi'P_2B_iJ_a)\Pi^{-1}P_i'P_2A_i, \]

where \( P_2 = P_{22} - P_{12}'P_{12}^{-1}P_{12} > 0 \) and \( J_s = C \circ (P_i'P_2B_i) \).

Therefore, the pair \((A_i, B_i)\) is mean square stabilizable.

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