ON ASYMPTOTICALLY $\Delta^m$ LACUNARY STATISTICAL EQUIVALENT SEQUENCES

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Abstract. This paper presents the following definition which is a natural combination of the definition for asymptotically equivalent, statistically limit and lacunary sequences. Let $\theta$ be a lacunary sequence; the two nonnegative sequences $x = (x_k)$ and $y = (y_k)$ are said to be asymptotically $\Delta^m$ lacunary statistical (defined in [2]) equivalent of multiple $L$ provided that for every $\epsilon > 0$,

$$\lim_{r \to \infty} \frac{1}{n_r} \left\{ \text{the number of } k \in I_r : \frac{\Delta^m x_k}{\Delta^m y_k} - L \geq \epsilon \right\} = 0$$

(denoted by $x S^L \sim \Delta^m y$), and simply $\Delta^m$-lacunary asymptotically statistical equivalent if $L = 1$. In section 3 are given some properties of $\Delta^m$-statistical asymptotically equivalent sequences and $\Delta^m$-Cesaro asymptotically equivalent sequences. In Theorems 3.1, 3.3, 3.4, 3.6, 3.7 are given inclusion cases of those classes and in Theorems 3.5, 3.8 and 3.9 are given equivalent conditions of those classes. In section 4 are given $\Delta^m$-Cesaro Orlicz asymptotically equivalent sequences and their relationship with other classes. In Theorems 4.3, 4.4, 4.5 and 4.6 are given inclusion cases of this class of sequences and classes defined in section 3. By Theorems 4.7, 4.8 and 4.9 are given equivalent conditions for those classes.

AMS Mathematics Subject Classification (2010): 40A05, 40C05, 46A45.
Key words and phrases: Difference sequence; Lacunary sequence; Statistical convergence, Asymptotically equivalent, Orlicz function.

1. Introduction

In 1993, Marouf presented definitions for asymptotically equivalent sequences and asymptotic regular matrices. In 2003, Patterson extend these concepts by presenting an asymptotically statistical equivalent analog of these definitions and natural regularity conditions for nonnegative summability matrices. In [12], Patterson and Savas extend these concepts to asymptotically lacunary statistical equivalent sequences. In this paper we will introduce the asymptotically $\Delta^m$ lacunary statistical equivalent sequences, where concept of $\Delta^m$ lacunary statistical sequences is introduced in [2]. This concept is extension of the previous concepts of asymptotically sequences. In addition to these definition, natural inclusion theorems shall also be presented. Also we will present Cesaro Orlicz asymptotically statistical equivalent and lacunary Cesaro Orlicz asymptotically statistical equivalent sequences.
2. DEFINITIONS AND NOTATIONS

Definition 2.1. (Maroof, [10]) Two nonnegative sequences \((x_k)\) and \((y_k)\) are said to be asymptotically equivalent if

\[
\lim_{k} \frac{x_k}{y_k} = 1,
\]

and is denoted by \(x \sim y\).

Definition 2.2. (Fridy, [5]) The sequence \((x_k)\) has statistic limit \(L\), denoted by \(st \lim x_k = L\) provided that for every \(\epsilon > 0\),

\[
\lim_{n} \frac{1}{n} \left\{ \text{the number of } k \leq n : |x_k - L| \geq \epsilon \right\} = 0.
\]

In context of the above definitions are given several definition by many authors: Patterson in [11], Patterson and Savas [12]. Let \(w\) be the set of all sequences of real or complex numbers and \(l_\infty\), \(c\) and \(c_0\) be, respectively, the Banach spaces of bounded, convergent and null sequences \(x = (x_k)\) with the usual norm \(\|x\| = \sup_{k} |x_k|\).

The idea of difference sequences was introduced by Kizmaz [8]. This idea is extended by Et and Nuray (see [2]) to \(\Delta^m\) differences and based in that we can define the following sequence spaces \(l_\infty(\Delta^m) = \{ x = (x_k)_i \in l_\infty, \epsilon(\Delta^m) = \{ x = (x_k) : \Delta^m x \in c \}, c_0(\Delta^m) = \{ x = (x_k) : \Delta^m x \in c_0 \}\) where \(\Delta^m x_k = \Delta^{m-1}(x_k) - \Delta^{m-1}(x_{k+1})\) and showed that these are Banach spaces with norm: \(\|x\|_\Delta = \sum_{i=1}^{\infty} |x_i| + \|\Delta^m x\|_\infty\).

We call these sequence spaces \(\Delta^m\)-bounded, \(\Delta^m\)-convergent and \(\Delta^m\)-null sequences. Subsequently difference sequence spaces have been discussed in Colak [2], Et and Basarir [3]. This paper presents new definitions which are a natural combination of the definition for asymptotically equivalence and \(\Delta^m\)-lacunary statistically convergence. In addition to these definitions, some connections between \(\Delta^m\)-lacunary statistical asymptotically equivalence and \(\Delta^m\)-lacunary asymptotically equivalence have also been presented.

Based in definition of \(\Delta^m\)-lacunary sequences several definitions of sequences spaces are given like as [2], [7], [13]. Following notations given above we will introduce the following concepts:

Definition 2.3. Two nonnegative sequence \((x_n)\) and \((y_n)\) are said to be \(\Delta^m\)-asymptotically statistical equivalent of multiple \(L\) provided that for every \(\epsilon > 0\),

\[
\lim_{n} \frac{1}{n} \left\{ \text{the number of } k \leq n : \left| \frac{\Delta^m x_k}{\Delta^m y_k} - L \right| \geq \epsilon \right\} = 0
\]

(denoted by \(x \sim_{L}^{\Delta^m} y\), and simply \(\Delta^m\)-asymptotically statistical equivalent if \(L = 1\)).

In [3], Fast introduced the idea of statistical convergence. This ideas was later studied by Connor [1], Freedman and Sember [4] and many others. A sequence of positive integers \(\theta = (k_r)\) is called lacunary if \(k_0 = 0, 0 < k_r < k_{r+1}\) and \(k_r = k_r - k_{r-1} \to \infty\) as \(r \to \infty\). We will denote by \(I_r = (k_{r-1}, k_r)\) and \(q_r = \frac{k_r}{k_{r-1}}\).

Definition 2.4. Let \(\theta\) be a lacunary sequence. Two nonnegative sequence \((x_n)\) and \((y_n)\) are said to be \(\Delta^m\)-lacunary asymptotically statistical equivalent of multiple \(L\)
provided that for every $\varepsilon > 0$, 
\[ \lim_{r \to \infty} \frac{1}{h_r} \left\{ \text{the number of } k \in I_r : \left| \frac{\Delta^m x_k}{\Delta^m y_k} - L \right| \geq \varepsilon \right\} = 0 \]
(denoted by $x^{S^m}_L(\Delta^m) \sim y$), and simply $\Delta^m$-lacunary asymptotically statistical equivalent if $L = 1$.

**Definition 2.5.** Two nonnegative sequences $(x_n)$ and $(y_n)$ are $\Delta^m$-Cesaro asymptotically equivalent of multiple $L$ provided that 
\[ \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \left( \frac{\Delta^m x_k}{\Delta^m y_k} - L \right) = 0, \]
(denoted by $x^{\sigma^m}_L(\Delta^m) \sim y$), and simply $\Delta^m$-Cesaro asymptotically equivalent if $L = 1$.

**Definition 2.6.** Two nonnegative sequences $(x_n)$ and $(y_n)$ are $\Delta^m$-strongly Cesaro asymptotically equivalent of multiple $L$ provided that 
\[ \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \left| \frac{\Delta^m x_k}{\Delta^m y_k} - L \right| = 0, \]
(denoted by $x^{|\sigma^m|}_L(\Delta^m) \sim y$), and simply $\Delta^m$-strongly Cesaro asymptotically equivalent if $L = 1$.

**Definition 2.7.** Let $\theta$ be a lacunary sequence. Two nonnegative sequences $(x_n)$ and $(y_n)$ are $\Delta^m$-lacunary Cesaro asymptotically equivalent of multiple $L$ provided that 
\[ \lim_{r \to \infty} \frac{1}{h_r} \sum_{k \in I_r} \left( \frac{\Delta^m x_k}{\Delta^m y_k} - L \right) = 0, \]
(denoted by $x^{\sigma^m,N^m}_L(\Delta^m) \sim y$), and simply $\Delta^m$-lacunary Cesaro asymptotically equivalent if $L = 1$.

**Definition 2.8.** Let $\theta$ be a lacunary sequence. Two nonnegative sequences $(x_n)$ and $(y_n)$ are $\Delta^m$-lacunary strongly Cesaro asymptotically equivalent of multiple $L$ provided that 
\[ \lim_{r \to \infty} \frac{1}{h_r} \sum_{k \in I_r} \left| \frac{\Delta^m x_k}{\Delta^m y_k} - L \right| = 0, \]
(denoted by $x^{\sigma^m,N^m}_L(\Delta^m) \sim y$), and simply $\Delta^m$-lacunary strongly Cesaro asymptotically equivalent if $L = 1$.

**Definition 2.9.** Two nonnegative sequences $(x_n)$ and $(y_n)$ are $\Delta^m$-strongly almost asymptotically equivalent of multiple $L$ provided that 
\[ \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \left| \frac{\Delta^m x_{k+m}}{\Delta^m y_{k+m}} - L \right| = 0, \]
uniformly in $m$ (denoted by $x^{AC^m}_L(\Delta^m) \sim y$), and simply $\Delta^m$-strongly almost asymptotically equivalent if $L = 1$. 

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Definition 2.10. Let $\theta$ be a lacunary sequence. Two nonnegative sequences $(x_n)$ and $(y_n)$ are \(\Delta^m\)-lacunary strongly almost asymptotically equivalent of multiple $L$ provided that
\[
\lim \frac{1}{hr} \sum_{k \in I_r} \frac{\Delta^m x_{k+m}}{\Delta^m y_{k+m}} - L = 0,
\]
uniformly in $m$ (denoted by \(x|\Delta^m| \sim L(y|\Delta^m)\), and simply \(\Delta^m\)-lacunary strongly almost asymptotically equivalent if $L = 1$.

3. Results

Theorem 3.1. If $x$ and $y$ are \(\Delta^m\) bounded sequences and \(\Delta^m\)-statistically asymptotically equivalent of multiple $L$ then they are \(\Delta^m\)-Cesaro asymptotically equivalent of multiple $L$. Conversely is not true.

Proof. Let us suppose that $x$ and $y$ are in $l_{\infty}(\Delta^m)$ and $x \sim_S L(y|\Delta^m)$. Then we can assume that
\[
\frac{\Delta^m x_k - L}{\Delta^m y_k} \leq M \text{ for almost all } k.
\]
Given $\epsilon > 0$
\[
\left| \frac{1}{n} \sum_{k=1}^{n} \left( \frac{\Delta^m x_k}{\Delta^m y_k} - L \right) \right| \leq M \cdot \frac{1}{n} \left\{ \text{the number of } k \leq n : \left| \frac{\Delta^m x_k}{\Delta^m y_k} - L \right| \geq \epsilon \right\} \leq \frac{1}{n} \cdot n\epsilon.
\]
Hence $x \sim_{\Delta^m} y$.

Let us consider that $m = 1$ and $L = \frac{1}{2}$. Let $x = (x_k) = (0, -1, -1, -2, -2, -3, -3, -4, -4, \cdots)$ be a sequence of numbers and $\Delta y_k = 1$, for all $k$. It is easy to prove that $x \sim_{\sigma_1^L(\Delta)} y$.

Corollary 3.2. If $x$ and $y$ are \(\Delta^m\) bounded sequences and \(\Delta^m\)-statistical asymptotically equivalent of multiple $L$ then they are \(\Delta^m\)-strongly Cesaro asymptotically equivalent of multiple $L (x \sim_{\sigma_1^L(\Delta)} y)$.

Theorem 3.3. Let $\theta = (k_r)$ be a lacunary sequence with lim inf $q_r > 1$ then $x \sim_{\sigma_1^L(\Delta)} y$ implies $x \sim_{\Delta^m} y$.

Proof. Suppose first that lim inf $q_r > 1$, then there exists a $\delta > 0$ such that $q_r \geq 1 + \delta$ for sufficiently large $r$, which implies
\[
\frac{h_r}{k_r} \geq \frac{\delta}{\delta + 1}.
\]
If $x \sim_{\sigma_1(\Delta^m)} y$, then for every $\epsilon > 0$ and for sufficiently large $r$, we have
We define \( \Delta^k \). To prove this fact it is enough to find a sequence that
\[
 x / k_k \text{ for } k \\
\]

Let \( \theta = (k_r) \) be a lacunary sequence with \( \sup, q_r < \infty \), then
\[
x \sim S^L(\Delta^m) \text{ y implies } x \sim S^L(\Delta^m) \text{ y.}
\]

Proof. Proof of the Theorem is similar to Theorem 3.3 in [12].

Theorem 3.5. Let \( \theta = (k_r) \) be a lacunary sequence then

1: (a) If \( x \sim S^L(\Delta^m) \text{ y then } x \sim S^L(\Delta^m) \text{ y.}
\)
(b) \( S^L(\Delta^m) \) is a proper subset of \( |\sigma_1|N^L(\Delta^m) \).

2: If \( x \in l_\infty(\Delta^m) \) and \( x \sim S^L(\Delta^m) \text{ y then } x \sim S^L(\Delta^m) \text{ y.}
\)

3: \( S^L(\Delta^m) \cap l_\infty(\Delta^m) = |\sigma_1|N^L(\Delta^m) \cap l_\infty(\Delta^m) \).

1. (a). If \( \epsilon > 0 \) and \( x \sim |\sigma_1|N^L(\Delta^m) \text{ y then }
\]
\[
\sum_{k \in I_r} \left\{ \frac{\Delta^m x_k}{\Delta^m y_k} - L \right\} \geq \sum_{k \in I_r, k \geq \epsilon} \left\{ \frac{\Delta^m x_k}{\Delta^m y_k} - L \right\} \geq \epsilon \left\{ \text{the number of } k \in I_r : \left| \frac{\Delta^m x_k}{\Delta^m y_k} - L \right| \geq \epsilon \right\},
\]

hence \( x \sim S^L(\Delta^m) \text{ y.}
\]

[1](b). To prove this fact it is enough to find a sequence \( x = (x_k) \) such that
\[
x \in S^L(\Delta^m) \text{ and } x \notin |\sigma_1|N^L(\Delta^m) \). Let \( \theta = (k_r) \) be a lacunary sequence and \( L = 1 \).
\]

We define \( \Delta^m x_k \) to be
\[
1, 2, \ldots, \lfloor \sqrt{h_r} \rfloor
\]
for \( k = k_{r-1} + i, \text{ for } i \in \{1, 2, \ldots, \lfloor \sqrt{h_r} \rfloor \} \), and \( \Delta^m x_k = 1 \) otherwise. \( \Delta^m y_k = 1 \) for all \( k \).

It seems that \( x = (x_k) \) is not \( \Delta^m \)-bounded. In what follows we will prove that \( x \in S^L(\Delta^m) \) and \( x \notin |\sigma_1|N^L(\Delta^m) \).

Let \( \epsilon > 0 \) be given, then we have:
\[
\frac{1}{h_r} \left\{ \text{the number of } k \in I_r : \left| \frac{\Delta^m x_k}{\Delta^m y_k} - L \right| \geq \epsilon \right\} = \frac{[\sqrt{h_r}]}{h_r} \rightarrow 0,
\]
when \( r \rightarrow \infty \), hence \( x \in S^L(\Delta^m) \). On the other hand
\[
\frac{1}{h_r} \sum_{k \in I_r} \left| \frac{\Delta^m x_k}{\Delta^m y_k} - L \right| = 0 + 1 + 2 + \cdots + \left( \lfloor \sqrt{h_r} \rfloor - 1 \right) = \frac{[\sqrt{h_r}]}{2h_r} \left( [\sqrt{h_r}] - 1 \right),
\]
from which follows that \( x \notin |\sigma_1|N^L(\Delta^m) \).

Proof is similar to proof of the Theorem 3.1.

3. Proof follows from part [1] and [2].

Theorem 3.6. For any lacunary sequence \( \theta = (k_r) \)
\[
x \sim |\sigma_1|N^L(\Delta^m) \text{ y implies } x \sim |\sigma_1|N^L(\Delta^m) \text{ y}
\]
and converse relation is not true.
Proof. If \( x \mid_{AC}^\sim y \) and \( \epsilon > 0 \) then there exists a \( n_0 > 0 \) such that
\[
\frac{1}{n} \sum_{i=m+1}^{m+n} \left| \frac{\Delta^m x_i}{\Delta^m y_i} - L \right| < \epsilon
\]
for every \( n > n_0 \) and \( m = 1, 2, 3, \ldots \). Since \( \theta \) is lacunary sequence we can choose \( M > 0 \) such that \( r \geq M \) implies \( h_r > n_0 \), respectively
\[
\frac{1}{h_r} \sum_{i \in I_r} \left| \frac{\Delta^m x_i}{\Delta^m y_i} - L \right| < \epsilon,
\]
hence \( x \mid_{\sigma, i}^\sim y \). In what follows we will show that in general case the converse implication is not true. Let us consider that
\[
\liminf_{r \to \infty} x_{(i)}\not= \left\{ \begin{array}{ll}
3 & \text{if } k_{r-1} < i \leq k_{r-1} + \left[ \sqrt{r} \right] \\
2 & \text{otherwise}
\end{array} \right.
\]
and \( \Delta^m y_i = 1 \) for all \( i \). Then it easy to prove that \( x \mid_{\sigma, i}^\sim y \) and \( x \mid_{AC}^\sim y \).

\textbf{Theorem 3.7.} Let \( \theta = (k_r) \) be a lacunary sequence, then the following relations are valid:

(i): If \( \liminf q_r > 1 \) then \( x \mid_{\sigma, i}^\sim y \) \( \Rightarrow x \mid_{\sigma, i}^\sim y \).

(ii): If \( \limsup q_r < \infty \) then \( x \mid_{\sigma, i}^\sim y \) \( \Rightarrow x \mid_{\sigma, i}^\sim y \).

(iii): If \( 1 < \liminf q_r \leq \limsup q_r < \infty \), then \( x \mid_{\sigma, i}^\sim y \) \( \iff x \mid_{\sigma, i}^\sim y \).

(i). Suppose that \( \liminf q_r > 1 \). There exists a \( \delta > 0 \) such that \( q_r \geq 1 + \delta \) for sufficiently large \( r \). We have:
\[
\frac{h_r}{k_r} > \frac{\delta}{1 + \delta}.
\]
From \( x \mid_{\sigma, i}^\sim y \) we get:
\[
\frac{1}{k_r} \sum_{k=1}^{k_r} \left| \frac{\Delta^m x_k}{\Delta^m y_k} - L \right| \geq \frac{1}{k_r} \sum_{k \in I_r} \left| \frac{\Delta^m x_k}{\Delta^m y_k} - L \right| = \frac{h_r}{k_r} \cdot \frac{1}{k_r} \sum_{k \in I_r} \left| \frac{\Delta^m x_k}{\Delta^m y_k} - L \right| \geq \frac{\delta}{1 + \delta} \frac{1}{k_r} \sum_{k \in I_r} \left| \frac{\Delta^m x_k}{\Delta^m y_k} - L \right|,
\]
hence \( x \mid_{\sigma, i}^\sim y \).

(ii) If \( \limsup q_r < \infty \) then there exists a \( N > 0 \) such that \( q_r < N \) for all \( r \). Let \( \epsilon > 0 \). From \( x \mid_{\sigma, i}^\sim y \) we can find constants \( M \) and \( K \) such that
\[
\sup_{n \geq M} \frac{1}{h_n} \sum_{i \in I_r} \left| \frac{\Delta^m x_k}{\Delta^m y_k} - L \right| < \epsilon \quad \text{and} \quad \frac{1}{h_n} \sum_{i \in I_r} \left| \frac{\Delta^m x_k}{\Delta^m y_k} - L \right| < K \quad \text{for all } i = 1, 2, \ldots.
\]
Then if \( t \) is any integer such that \( k_{r-1} < t \leq k_r \), where \( r > M \), we have:
which completes proof.

\[ (iii) \] Follows directly from \[(i)\] and \[(ii)\].

**Theorem 3.8.** Let \( \theta = (k_r) \) be a lacunary sequence, then the following relations are valid:

(i): If \( \lim \inf q_r > 1 \) then \( x \overset{\sigma_1}{\sim} (\Delta^m) y \Rightarrow x \overset{\sigma_1 N^\sigma}{\sim} (\Delta^m) y. \)

(ii): If \( \lim \sup q_r < \infty \) then \( x \overset{\sigma_1 N^\sigma}{\sim} (\Delta^m) y \Rightarrow x \overset{\sigma_1}{\sim} (\Delta^m) y. \)

(iii): If \( 1 < \lim \inf q_r \leq \lim \sup q_r < \infty \) then \( x \overset{\sigma_1 N^\sigma}{\sim} (\Delta^m) y \iff x \overset{\sigma_1}{\sim} (\Delta^m) y. \)

**Theorem 3.9.** Let \( \theta = (k_r) \) be a lacunary sequence, then the following relations are valid:

(i): If \( \lim \inf q_r > 1 \) then \( x \overset{|AC|^L}{\sim} (\Delta^m) y \Rightarrow x \overset{|AC|^L}{\sim} (\Delta^m) y. \)

(ii): If \( \lim \sup q_r < \infty \) then \( x \overset{|AC|^L}{\sim} (\Delta^m) y \Rightarrow x \overset{|AC|^L}{\sim} (\Delta^m) y. \)

(iii): If \( 1 < \lim \inf q_r \leq \lim \sup q_r < \infty \) then \( x \overset{|AC|^L}{\sim} (\Delta^m) y \iff x \overset{|AC|^L}{\sim} (\Delta^m) y. \)

4. Cesaro Orlicz Asymptotically Statistical Equivalent

In this section we will introduce Cesaro Orlicz asymptotically statistical equivalence between two sequences. Let \( P_s \) denotes the set of all subsets of \( \mathbb{N} \), that do not contain more than \( s \) elements. With \( (\phi_s) \) we will denote a nondecreasing sequence of positive real numbers such that \( \phi_s \to \infty, s \to \infty \) and \( \phi_s \leq s \) for every \( s \in \mathbb{N} \). The class of all the sequences \( (\phi_s) \) satisfying this property is denoted by \( \Phi \). An Orlicz function is a function \( M : (0, \infty) \to (0, \infty) \) which is continuous, nondecreasing and convex with \( M(0) = 0, M(x) > 0 \) for \( x > 0 \) and \( M(x) \to \infty \) as \( x \to \infty \). It is well known that if \( M \) is a convex function and \( M(0) = 0 \), then \( M(\lambda x) \leq \lambda M(x) \) for all \( \lambda \) with \( 0 < \lambda \leq 1 \). An Orlicz function \( M \) is said to satisfy the \( \Delta_2 \)-condition for
all values of \( u \), if there exists a constant \( L > 0 \) such that \( M(2u) \leq LM(u), u \geq 0 \) (see, Krasnosel’skii and Rutitsky [9]).

We will define the following asymptotic statistical equivalences:

**Definition 4.1.** Two nonnegative sequences \((x_n)\) and \((y_n)\) are \( M - \Delta^m \)-strongly Cesaro Orlicz asymptotically equivalent if

\[
\lim \frac{1}{n} \sum_{k=1}^{n} M \left( \frac{\Delta^m x_k}{\Delta^m y_k} - L \right) = 0,
\]

(denoted by \( x \overset{[\sigma]}{\sim}^M (M - \Delta^m) y \)), and simply \( M - \Delta^m \)-strongly Cesaro Orlicz asymptotically equivalent if \( L = 1 \).

**Definition 4.2.** Two nonnegative sequences \((x_n)\) and \((y_n)\) are \( M - \Delta^m \)-lacunary strongly Cesaro Orlicz asymptotically equivalent of multiple \( L \) provided that

\[
\lim \frac{1}{s} \sum_{k \in \sigma, \sigma \in P_n} M \left( \frac{\Delta^m x_k}{\Delta^m y_k} - L \right) = 0,
\]

(denoted by \( x \overset{[\sigma]}{\sim}^M (N^k - \Delta^m) y \)), and simply \( M - \Delta^m \)-lacunary strongly Cesaro Orlicz asymptotically equivalent if \( L = 1 \).

**Theorem 4.3.** Two nonnegative sequence \((x_n)\) and \((y_n)\) are said to be \( x \overset{S_L}{\sim}^M y \) of multiple \( L \) provided that for every \( \epsilon > 0 \),

\[
\lim_{s} \frac{1}{s} \frac{1}{\phi_s} \left\{ \text{the number of } i \in \sigma, \sigma \in P_s : \left| \frac{\Delta^m x_i}{\Delta^m y_i} - L \right| \geq \epsilon \right\} = 0.
\]

Then \( x \overset{S_L}{\sim}^M y \) implies \( x \overset{S_L}{\sim}^M y \).

**Proof.** From the definition of the sequences \( \phi_s \) it follows that \( \inf_{s} \frac{s}{s-\phi_s} \geq 1 \). Then there exist a \( \delta > 0 \), such that

\[
\frac{s}{\phi_s} \leq \frac{1 + \delta}{\delta}.
\]

If \( x \overset{S_L}{\sim}^M y \), then for every \( \epsilon > 0 \) and for sufficiently large \( s \), we have

\[
\frac{1}{\phi_s} \left\{ \text{the number of } i \in \sigma, \sigma \in P_s : \left| \frac{\Delta^m x_i}{\Delta^m y_i} - L \right| \geq \epsilon \right\} = \frac{1}{s} \cdot \frac{s}{\phi_s} \left\{ \text{the number of } i \leq s : \left| \frac{\Delta^m x_i}{\Delta^m y_i} - L \right| \geq \epsilon \right\} \leq \frac{1 + \delta}{\delta} \cdot \frac{1}{s} \left\{ \text{the number of } i \leq s : \left| \frac{\Delta^m x_{i_0}}{\Delta^m y_{i_0}} - L \right| \geq \epsilon \right\},
\]

where \( i_0 \in \{1, 2, \ldots, s\} \setminus \sigma, \sigma \in P_s \). This completes the proof. \( \square \)

**Proposition 4.4.** Let \((x_n)\) and \((y_n)\) be two nonnegative sequences. Then \( x \overset{S_L}{\sim}^M y \) implies \( x \overset{S_L}{\sim}^M y \), if \( \sup_{s} \frac{\phi_s}{\phi_{s-1}} < \infty \).
Theorem 4.5. Let $M$ be an Orlicz function which satisfies the $\Delta_2$ conditions.

Two nonnegative sequence $(x_n)$ and $(y_n)$ are said to be $x \sim_{\Delta_m} y$ of multiple $L$ provided that for every $\epsilon > 0,
\lim_{\phi_s} \frac{1}{s} \left\{ \text{the number of } i \in \sigma, \sigma \in P_s : M \left( \frac{\Delta^m x_i}{\Delta^m y_i} - L \right) \geq \epsilon \right\} = 0.

Then $x \sim_{\Delta_m} y$ implies $x \sim_{\Delta_m} y$.

Proof. From the definition of the sequences $\phi_s$ it follows that $\inf_s \frac{s}{s-\phi_s} \geq 1$. Then there exist a $\delta > 0$, such that
\[
\frac{s}{\phi_s} \leq \frac{1 + \delta}{\delta}.
\]

If $x \sim_{\Delta_m} y$, then for every $\epsilon > 0$ and for sufficiently large $r$, we have
\[\frac{1}{\phi_s} \left\{ \text{the number of } i \in \sigma, \sigma \in P_s : M \left( \frac{\Delta^m x_i}{\Delta^m y_i} - L \right) \geq \epsilon \right\} = \frac{1}{s} \cdot \frac{s}{\phi_s} \left\{ \text{the number of } i \leq s : M \left( \frac{\Delta^m x_i}{\Delta^m y_i} - L \right) \geq \epsilon \right\} - \frac{1}{\phi_s} \left\{ \text{the number of } i \in \{1, 2, \cdots, s\} \setminus \sigma, \sigma \in P_s : M \left( \frac{\Delta^m x_i}{\Delta^m y_i} - L \right) \geq \epsilon \right\} \leq \frac{1 + \delta}{\delta} \cdot \frac{1}{s} \left\{ \text{the number of } i \leq s : M \left( \frac{\Delta^m x_i}{\Delta^m y_i} - L \right) \geq \epsilon \right\} - \frac{1}{\phi_s} \left\{ \text{the number of } i \in \{1, 2, \cdots, s\} \setminus \sigma, \sigma \in P_s : M \left( \frac{\Delta^m x_i}{\Delta^m y_i} - L \right) \geq \epsilon \right\} \leq \frac{1}{\phi_s} \left\{ M \left( \frac{\Delta^m x_i}{\Delta^m y_i} - L \right) \geq \epsilon \right\},\]

where $i_0 \in \{1, 2, \cdots, s\} \setminus \sigma, \sigma \in P_s$. In other side from fact that $M$ satisfies the $\Delta_2$ conditions it follows that
\[M \left( \frac{\Delta^m x_i}{\Delta^m y_i} - L \right) \leq K \cdot \frac{\Delta^m x_i}{\Delta^m y_i} - L \leq 1 \text{ and } \frac{\Delta^m x_i}{\Delta^m y_i} - L \geq 1.
\]

Really in first case it follows directly from definition of the Orlicz function. In second case we have
\[\frac{\Delta^m x_i}{\Delta^m y_i} - L = 2 \cdot L^{(1)} = 2^2 \cdot L^{(2)} = \cdots = 2^s \cdot L^{(s)},\]
such that $L^{(s)} \leq 1$. Now taking into consideration $\Delta_2$ conditions of Orlicz functions, we get the following estimation:
\[M \left( \frac{\Delta^m x_i}{\Delta^m y_i} - L \right) \leq T \cdot L^{(s)} \cdot M(1) = K \cdot \frac{\Delta^m x_i}{\Delta^m y_i} - L \leq 1,
\]

where $T$ and $K$ are constants. Now proof of Theorem follows from relations (1) and (2). \hfill \square

Proposition 4.6. Let $M$ be an Orlicz function and consider $k \in \mathbb{Z}$ such that $\phi_s \leq [\phi_s] + k$, $\sup_s [\phi_s + k \phi_{s-1}] < \infty$. Then for any two nonnegative sequences $(x_i)$ and $(y_i)$ we have: $x \sim_{\Delta_m} y$, implies $x \sim_{\Delta_m} y$. 

Proof. If \( \sup \frac{\phi_s + k}{\phi_{s-1}} < \infty \), then there exists \( K > 0 \) such that \( \frac{\phi_s + k}{\phi_{s-1}} < K \) for all \( s \geq 1 \). Let \( n \) be an integer number such that \( \phi_{s-1} < n \leq \phi_s \). Then we have

\[
\frac{1}{n} \left\{ \text{the number of } i \leq n : \left| \frac{\Delta^m x_i}{\Delta^m y_i} - L \right| \geq \epsilon \right\}
\leq \frac{1}{n} \left\{ \text{the number of } i \leq n : M \left( \left| \frac{\Delta^m x_i}{\Delta^m y_i} - L \right| \right) \geq M(\epsilon) \right\}
\]

\[
\leq \frac{1}{[\phi_s + k]} \cdot \frac{[\phi_s + k]}{[\phi_{s-1}]} \left\{ \text{the number of } i \leq \phi_s : M \left( \left| \frac{\Delta^m x_i}{\Delta^m y_i} - L \right| \right) \geq M(\epsilon) \right\} \leq \frac{K}{[\phi_s + k]} \left\{ \text{the number of } i \in \sigma, \sigma \in P_{[\phi_s + k]} : M \left( \left| \frac{\Delta^m x_i}{\Delta^m y_i} - L \right| \right) \geq M(\epsilon) \right\}.
\]

Now proof of the proposition follows from the last relation. \( \square \)

**Theorem 4.7.** Let \( M \) be an Orlicz function. The following relations are valid:

(i): \( x \mid \sigma_1^{k_0} (M-\Delta^m) y \Rightarrow x \mid \sigma_1^{N^k} (M-\Delta^m) y \).

(ii): \( \sup_s \frac{\phi_s}{\phi_{s-1}} < \infty \), for every \( s \in \mathbb{N} \). Then

\( x \mid \sigma_1^{N^k} (M-\Delta^m) y \Rightarrow x \mid \sigma_1^{k_0} (M-\Delta^m) y \).

(iii): \( \sup_s \frac{\phi_s}{\phi_{s-1}} < \infty \), for every \( s \in \mathbb{N} \). Then \( x \mid \sigma_1^{k_0} (M-\Delta^m) y \iff x \mid \sigma_1^{N^k} (M-\Delta^m) y \).

**Proof.** (i) From definition of the sequences \( \phi_n \) it follows that inf \( \frac{s}{\phi_n} \geq 1 \). Then there exist a \( \delta > 0 \), such that

\[
\frac{s}{\phi_n} \leq 1 + \frac{\delta}{\sigma}.
\]

Then we get the following relation:

\[
\frac{1}{\phi_s} \sum_{k \in \sigma, \sigma \in P_s} M \left( \left| \frac{\Delta^m x_k}{\Delta^m y_k} - L \right| \right) = \frac{1}{\phi_s} \sum_{k \in \sigma, \sigma \in P_s} M \left( \left| \frac{\Delta^m x_k}{\Delta^m y_k} - L \right| \right) - \frac{1}{\phi_s} \sum_{k \in \{1, \ldots, s\} \setminus \sigma, \sigma \in P_s} M \left( \left| \frac{\Delta^m x_k}{\Delta^m y_k} - L \right| \right) \leq \frac{1 + \delta}{\phi_s} \sum_{k = 1}^{s} M \left( \left| \frac{\Delta^m x_k}{\Delta^m y_k} - L \right| \right) - \frac{1}{\phi_s} M \left( \left| \frac{\Delta^m x_{k_0}}{\Delta^m y_{k_0}} - L \right| \right),
\]

where \( k_0 \in \{1, \ldots, s\} \setminus \sigma \). Knowing that \( x \mid \sigma_1^{k_0} (M-\Delta^m) y \) and \( M \) is continuous, letting \( s \to \infty \) on last relation we obtain:

\[
\frac{1}{\phi_s} \sum_{k \in \sigma, \sigma \in P_s} M \left( \left| \frac{\Delta^m x_k}{\Delta^m y_k} - L \right| \right) \to 0.
\]
Hence $x^{\sigma_1 N^k (M - \Delta^m)} y$.

(ii) Suppose that $\sup_s \frac{\phi_s}{\phi_{s-1}} < \infty$, then there exists $B > 0$ such that $\frac{\phi_s}{\phi_{s-1}} < B$ for all $s \geq 1$. Let $x^{\sigma_1 N^k (M - \Delta^m)} y$ and $\epsilon > 0$, then there exist $R > 0$ such that for every $s \geq R$

$$
\frac{1}{\phi_s} \sum_{k \in \sigma, \sigma \in P_s} M \left( \frac{\Delta^m x_k}{\Delta^m y_k} - L \right) < \epsilon.
$$

We can also find a constant $K > 0$ such that

$$
\frac{1}{\phi_s} \sum_{k \in \sigma, \sigma \in P_s} M \left( \frac{\Delta^m x_k}{\Delta^m y_k} - L \right) < K
$$

for all $s \in \mathbb{N}$. Let $n$ be any integer with $\phi_{s-1} < n \leq [\phi_s]$, for every $s > R$. Then

$$
\frac{1}{n} \sum_{k=1}^n M \left( \frac{\Delta^m x_k}{\Delta^m y_k} - L \right) \leq \frac{1}{\phi_{s-1}} \sum_{k=1}^{[\phi_s]} M \left( \frac{\Delta^m x_k}{\Delta^m y_k} - L \right) =
$$

$$
\frac{1}{\phi_{s-1}} \sum_{k=1}^{[\phi_1]} M \left( \frac{\Delta^m x_k}{\Delta^m y_k} - L \right) + \frac{[\phi_2]}{\phi_{s-1}} \sum_{k=1}^{[\phi_1]} M \left( \frac{\Delta^m x_k}{\Delta^m y_k} - L \right) + \cdots + \frac{[\phi_s]}{\phi_{s-1}} \sum_{k=1}^{[\phi_1]} M \left( \frac{\Delta^m x_k}{\Delta^m y_k} - L \right) \leq
$$

$$
\frac{\phi_1}{\phi_{s-1}} \left( \frac{1}{[\phi_1]} \sum_{k \in \sigma, \sigma \in P^{(1)}} M \left( \frac{\Delta^m x_k}{\Delta^m y_k} - L \right) \right) + \frac{\phi_2}{\phi_{s-1}} \left( \frac{1}{[\phi_2]} \sum_{k \in \sigma, \sigma \in P^{(1)}} M \left( \frac{\Delta^m x_k}{\Delta^m y_k} - L \right) \right) + \cdots + \frac{\phi_s}{\phi_{s-1}} \left( \frac{1}{[\phi_s]} \sum_{k \in \sigma, \sigma \in P^{(1)}} M \left( \frac{\Delta^m x_k}{\Delta^m y_k} - L \right) \right),
$$

where $P^{(t)}$ are sets of integer numbers which have more than $[\phi_t]$ elements for $t \in \{1, 2, \cdots, s\}$. Passing by limit on last relation, where $n \to \infty$ we get that

$$
\frac{1}{n} \sum_{k=1}^n M \left( \frac{\Delta^m x_k}{\Delta^m y_k} - L \right) \to 0,
$$

from this follows that $x^{\sigma_1 (M - \Delta^m)} y$.

(iii) Proof of this part follows directly from [(i)] and [(ii)].

\[ \square \]

**Theorem 4.8.** Let $M$ be an Orlicz function. Then

1. (a) If $x^{\sigma_1 (M - \Delta^m)} y$ then $x^{S_L(\Delta^m)} y$.
   (b) $S_L(\Delta^m)$ is a proper subset of $[\sigma_1] L (M - \Delta^m)$.

2. If $M$ satisfies the $\Delta_2$ condition and $x \in l_\infty (M - \Delta^m)$ such that $x^{S_L(\Delta^m)} y$, then $x^{\sigma_1 (M - \Delta^m)} y$.

3. If $M$ satisfies the $\Delta_2$ condition, then $S_L(\Delta^m) \cap l_\infty (M - \Delta^m) = [\sigma_1]^{L} (M - \Delta^m) \cap l_\infty (M - \Delta^m)$, where $l_\infty (M - \Delta^m) = \{ x = (x_k) : M(\Delta^m x_k) \in l_\infty \}$
Proof. 1: (a) If $\epsilon > 0$ and $x \sim [\sigma_1]^t(M - \Delta^m)$, then

$$1 \leq \frac{1}{n} \left\{ \text{the number of } k \leq n : \left| \frac{\Delta^m x_k}{\Delta^m y_k} - L \right| \geq \epsilon \right\} \leq \frac{1}{n} \sum_{k=1}^{n} M \left( \left| \frac{\Delta^m x_k}{\Delta^m y_k} - L \right| \right) \leq \frac{1}{n} \sum_{k=1}^{n} M \left( \left| \frac{\Delta^m x_k}{\Delta^m y_k} - L \right| \right) \geq M(\epsilon)$$

(b) To prove this fact it is enough to find a sequence $x = (x_k)$ such that $x \in S_L(\Delta^m)$ and $x \notin [\sigma_1]^t(M - \Delta^m)$. Let $L = 1$ and $M = x$. We define $\Delta^m x_k$ to be

$$1, 2, \cdots, \lfloor \sqrt{n} \rfloor$$

for $k = i$, for $i \in \{1, 2, \cdots, \lfloor \sqrt{n} \rfloor \}$, and $\Delta^m x_k = 1$ otherwise. $\Delta^m y_k = 1$ for all $k$. It seems that $x = (x_k)$ is not $(M - \Delta^m)$-bounded. In what follows we will prove that $x \in S_L(\Delta^m)$ and $x \notin [\sigma_1]^t(M - \Delta^m)$.

Let $\epsilon > 0$ be given, then we have:

$$\frac{1}{n} \left\{ \text{the number of } k \leq n : \left| \frac{\Delta^m x_k}{\Delta^m y_k} - L \right| \geq \epsilon \right\} = \frac{\lfloor \sqrt{n} \rfloor}{n} \to 0,$$

when $n \to \infty$, hence $x \in S_L(\Delta^m)$. On the other hand

$$\frac{1}{n} \sum_{k=1}^{n} \left| \frac{\Delta^m x_k}{\Delta^m y_k} - L \right| = 0 + 1 + 2 + \cdots + (\lfloor \sqrt{n} \rfloor - 1) = \frac{\lfloor \sqrt{n} \rfloor (\lfloor \sqrt{n} \rfloor - 1)}{2},$$

from which follows that $x \notin [\sigma_1]^t(M - \Delta^m)$.

2: Proof of this part is similar to proof of the Theorem 3.1 and Theorem 4.5.

3: Proof follows from part 1 and 2. \qed

Theorem 4.9. Let $M$ be an Orlicz function. Then

1: (a) If $x \sim [\sigma_1]^t(M - \Delta^m)$, then $x \sim [\sigma_1]^t(M - \Delta^m)$.

(b) $S^L_\phi(\Delta^m)$ is a proper subset of $[\sigma_1]^t(M - \Delta^m)$.

2: If $M$ satisfies the $\Delta_2$ condition and $x \in l_\infty(M - \Delta^m)$, such that $x \sim [\sigma_1]^t(M - \Delta^m)$, then $x \sim [\sigma_1]^t(M - \Delta^m)$.

3: If $M$ satisfies the $\Delta_2$ condition, then $S^L_\phi(\Delta^m) \cap l_\infty(M - \Delta^m) = [\sigma_1]^t(M - \Delta^m) \cap l_\infty(M - \Delta^m)$.

References