An Algorithmic Approach to Multi Objective Fuzzy Linear Programming Problem

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Abstract
In this Paper, a ranking procedure based on fuzzy numbers distance is applied to
Multi -objective Linear Programming Problem with fuzzy coefficients. Here the
problem is transformed into a Crisp problem and then solved by Preemptive
optimization method. Numerical Examples are provided to illustrate its feasibility.

Keywords: Distance, Ranking, Triangular and Trapezoidal fuzzy numbers, Multi-objective
Linear Programming, Preemptive Optimization.

1. Introduction

Multi-objective optimization is the process of simultaneously optimizing two or more
conflicting objectives subject to certain constraints. In many real world problems, there are
situations where multiple objectives may be more appropriate rather than considering single
objective. However, in such cases emphasis is on efficient solutions, which are optimal in a
certain multi-objective sense. Bellman and Zadeh proposed the concepts of decision making
in fuzzy environments [1]. Qiu - Peng Gu, and Bing - Yuan Cao solved Fuzzy Linear
programming problems based on Fuzzy numbers distance [5]. Zimmermann [8] presented a
fuzzy approach to solve multi-objective linear programming problems. Tong Shaochien[7]
focused on the fuzzy linear programming problems with interval numbers. In this paper, an
attempt is made to solve Multi-objective Linear Programming Problem (MOFLPP) under
constraints with Fuzzy coefficients. Here, the MOFLPP is transformed into a Multi-objective
Linear Programming Problem (MOLPP) and it can be solved accordingly. The paper is
organized as follows: Certain definitions and Notations are provided in the next section. The
Ranking algorithm is given in section 3. Section 4 presents the method for solving a
MOFLPP. In section 5, numerical examples are discussed. The last section draws some concluding remarks.

2. Definitions and Notations

2.1 Interval Number

Let $R$ be the set of real numbers. Then closed interval $[a, b]$ is said to be an interval number, where $a, b \in R, a \leq b$.

2.2 Distance between Interval Numbers

Let $a = [a_1, a_2], b = [b_1, b_2]$ be two interval numbers. Then the distance between $a, b$, denoted by $d(a, b)$, is defined by

$$d(a, b) = \int_{-\frac{1}{2}}^{\frac{1}{2}} \left[ \frac{a_1 + a_2}{2} + x(a_2 - a_1) \right] - \left[ \frac{b_1 + b_2}{2} + x(b_2 - b_1) \right] dx$$

2.3 Fuzzy Number and Membership Function

The fuzzy set $\tilde{A}$ in the set of real numbers is called a fuzzy number. If its membership function is

$$\mu_{\tilde{A}}(x) = \begin{cases} 
L \left( \frac{a_2 - x}{a_2 - a_1} \right), & a_1 \leq x \leq a_2, \\
1, & a_2 \leq x \leq a_3, \\
R \left( \frac{x - a_3}{a_4 - a_3} \right), & a_3 \leq x \leq a_4, \\
0, & x < a_1, x > a_4,
\end{cases} \quad (2.1)$$

where $L$ and $R$ are strictly decreasing functions in $(0,1)$, and satisfy

$L(x) = R(x) = 1 \ (x \leq 0)$; $L(x) = R(x) = 0 \ (x \geq 1)$.

When $L(x) = R(x) = 1 - x$, fuzzy number defined in (2.1) is a Trapezoidal fuzzy number, denoted by $\tilde{A} = (a_1, a_2, a_3, a_4)$; when $L(x) = R(x) = 1 - x$ and $a_2 = a_3$, fuzzy number defined in (2.1) is a Triangular fuzzy number, denoted by $\tilde{A} = (a_1, a_2, a_3)$.

2.4 $\lambda$ - Level cut

For any $\lambda \in [0, 1]$, $\lambda$ - level cut of fuzzy number $\tilde{A}$ is an interval number $\tilde{A}_\lambda = [a_{L\lambda}, a_{R\lambda}]$. Let $L^{-1}_\lambda \lambda$ denote the inverse function of $\tilde{A}_\lambda = L \left( \frac{a_2 - x}{a_2 - a_1} \right)$ about the variant $\frac{a_2 - x}{a_2 - a_1}$; $R^{-1}_\lambda \lambda$ denote the inverse function of $\tilde{A}_\lambda = R \left( \frac{x - a_3}{a_4 - a_3} \right)$ about the variant $\frac{x - a_3}{a_4 - a_3}$. So, $a_{L\lambda} = a_2 - (a_2 - a_1) L^{-1}_\lambda \lambda$; $a_{R\lambda} = a_3 - (a_4 - a_3) R^{-1}_\lambda \lambda$. 

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2.5 Distance between Fuzzy Numbers

Let $\tilde{A}, \tilde{B}$ be two fuzzy numbers. The distance between the fuzzy numbers $\tilde{A}$ and $\tilde{B}$ is defined by

$$D(\tilde{A}, \tilde{B}) = \int_0^1 d(\tilde{A}_\lambda, \tilde{B}_\lambda) d\lambda. \quad (2.2)$$

3. Ranking Algorithm

Qiu – peng Gu and Bing – Yuan Cao [1] developed a ranking procedure by defining a distance between fuzzy numbers (see [1]). Based on this ranking procedure, a ranking algorithm is developed for Triangular and Trapezoidal Fuzzy numbers. Moreover, it is applied to MOFLPP under constraints with fuzzy coefficients.

Algorithm:

Step 1: Consider the fuzzy numbers $\tilde{A}$ & $\tilde{B}$ which are either triangular or trapezoidal.

Step 2: Find supremum $M = \sup (s(A) \cup s(B))$, where $s(A)$ = support set of $\tilde{A}$ and $s(B)$ = support set of $\tilde{B}$.

Step 3: If $\tilde{A}$ and $\tilde{B}$ are triangular fuzzy numbers, then go to step 6.

Step 4: Otherwise, Take $\tilde{A} = (a_1, a_2, a_3, a_4)$ & $\tilde{B} = (b_1, b_2, b_3, b_4)$ as trapezoidal fuzzy numbers.

Step 5: Calculate $D(\tilde{A}, M) = M - \frac{a_1 + a_2 + a_3 + a_4}{4}$ & $D(\tilde{B}, M) = M - \frac{b_1 + b_2 + b_3 + b_4}{4}$ and go to step 8.

Step 6: Take $\tilde{A} = (a_1, a_2, a_3)$ and $\tilde{B} = (b_1, b_2, b_3)$.

Step 7: Calculate $D(\tilde{A}, M) = M - \frac{a_1 + 2a_2 + a_3}{4}$ & $D(\tilde{B}, M) = M - \frac{b_1 + 2b_2 + b_3}{4}$

Step 8: If $D(\tilde{A}, M) < D(\tilde{B}, M)$ then $\tilde{A}$ is greater than $\tilde{B}$

If $D(\tilde{A}, M) = D(\tilde{B}, M)$ then $\tilde{A}$ is equal to $\tilde{B}$

If $D(\tilde{A}, M) > D(\tilde{B}, M)$ then $\tilde{A}$ is less than $\tilde{B}$

Step 9: Stop.

4. Method of Solving Multi-objective Fuzzy Linear Programming Problem

In this paper, we discuss a multi-objective model in the constraint conditions with fuzzy coefficients. Although a multi-objective model may have several objective criteria, they are rarely of equal importance. Hence, they can be considered in priority order with respect to the problem undertaken. Moreover, the objectives may be of all minimization type or all maximization type or mixed with both minimization and maximization types. In this paper, we discuss a model whose standard form is

Maximize $Z_1 = c_1 x$

Minimize $Z_2 = c_2 x$, .........

subject to $\tilde{A}X \leq \tilde{b}$, $x \geq 0$,

where $c = (c_1, c_2, ......., c_m)$ is an n-dimensional crisp row vector, $\tilde{A} = \tilde{a}_i$ is an m x n fuzzy number matrix, $\tilde{b} = \tilde{b}_1, \tilde{b}_2, ......., \tilde{b}_m$ is an m-dimensional fuzzy line vector and $X = (x_1, x_2, ......., x_n)^T$ is an n-dimensional decision variable vector.
We now consider a bi-objective Fuzzy Linear Programming Problem with constraints having fuzzy coefficients is given by

Maximize $Z_1 = c_{1i}x_1 + c_{12}x_2 + \ldots + c_{in}x_n$
Minimize $Z_2 = c_{21}x_1 + c_{22}x_2 + \ldots + c_{2n}x_n$
subject to $\bar{a}_{i1}x_1 + \bar{a}_{i2}x_2 + \ldots + \bar{a}_{in}x_n \leq \bar{b}_i$,
$x_1, x_2, \ldots, x_n \geq 0, i = 1, 2, \ldots, m$.

where fuzzy numbers are triangular fuzzy ones, that is, $\bar{a}_{i1} = a_{i11}, a_{i12}, a_{i13}$,
$\bar{a}_{i2} = a_{i21}, a_{i22}, a_{i23}$, \ldots, $\bar{a}_{in} = a_{in1}, a_{in2}, a_{in3}$; $\bar{b}_i = b_{i1}, b_{i2}, b_{i3}$.

By Zadeh’s Extension principle, the sum of any triangular fuzzy numbers is still a triangular one.

The above formulation can be written as

Maximize $Z_1 = c_{11}x_1 + c_{12}x_2 + \ldots + c_{1n}x_n$
Minimize $Z_2 = c_{21}x_1 + c_{22}x_2 + \ldots + c_{2n}x_n$
subject to $(a_{i11}x_1 + a_{i12}x_2 + \ldots + a_{i1n}x_n) + 2(a_{i12}x_1 + a_{i22}x_2 + \ldots + a_{i2n}x_n) + a_{i13}x_1 + a_{i23}x_2 + \ldots + a_{in3}x_n \leq b_{i1} + 2b_{i2} + b_{i3}$,
$x_1, x_2, \ldots, x_n \geq 0, i = 1, 2, \ldots, m$.

By the ranking algorithm, the above MOFLPP is transformed into a MOLPP as follows:

Maximize $Z_1 = c_{11}x_1 + c_{12}x_2 + \ldots + c_{1n}x_n$
Minimize $Z_2 = c_{21}x_1 + c_{22}x_2 + \ldots + c_{2n}x_n$
subject to $(a_{i11}x_1 + a_{i12}x_2 + \ldots + a_{i1n}x_n) + 2(a_{i12}x_1 + a_{i22}x_2 + \ldots + a_{i2n}x_n) + 2(a_{i13}x_1 + a_{i23}x_2 + \ldots + a_{in3}x_n) \leq b_{i1} + 2b_{i2} + b_{i3}$,
$x_1, x_2, \ldots, x_n \geq 0, i = 1, 2, \ldots, m$.

By the same ranking algorithm, a MOFLPP using trapezoidal fuzzy numbers can similarly be transformed into a MOLPP as follows:

Maximize $Z_1 = c_{11}x_1 + c_{12}x_2 + \ldots + c_{1n}x_n$
Minimize $Z_2 = c_{21}x_1 + c_{22}x_2 + \ldots + c_{2n}x_n$
subject to $(a_{i11}x_1 + a_{i12}x_2 + \ldots + a_{i1n}x_n) + 2(a_{i12}x_1 + a_{i22}x_2 + \ldots + a_{i2n}x_n) + 2(a_{i13}x_1 + a_{i23}x_2 + \ldots + a_{in3}x_n) + (a_{i14}x_1 + a_{i24}x_2 + \ldots + a_{in4}x_n) \leq b_{i1} + 2b_{i2} + 2b_{i3} + b_{i4}$,
$x_1, x_2, \ldots, x_n \geq 0, i = 1, 2, \ldots, m$.

In this method, the most important objective subject to the fuzzy constraints is considered first. Using I or II, this can be converted into a single objective problem subject to constraints with transformed crisp number coefficients and hence solved accordingly. Then the next objective is optimized subject to a requirement that the first achieve its optimal value which is nothing but the preemptive optimization.

Similarly, multi-objective problems with more than two objectives can also be solved using the above procedure subject to a requirement that the previous objectives achieve its optimal value. Here, in the very first stage itself the problem is transformed into a crisp problem and afterwards there will be no more fuzziness in the constraints as well as in the problem.
5. Numerical Examples

Consider the following MOFLPP

Maximize \( z_1 = x_1 + x_2 \)

Minimize \( z_2 = 2x_1 + 3x_2 \)

subject to \( \bar{a}_{11}x_1 + \bar{a}_{12}x_2 \leq \bar{b}_1 \)
\( \bar{a}_{21}x_1 + \bar{a}_{22}x_2 \leq \bar{b}_2 \), \( x_1, x_2 \geq 0 \),

where \( \bar{a}_{11} = (2, 3, 4), \bar{a}_{12} = (1.5, 2, 3), \bar{b}_1 = (4, 5, 7), \bar{a}_{21} = (-1, 0, 1), \bar{a}_{22} = (0.5, 1, 1.5), \bar{b}_2 = (1.5, 2, 3) \).

We assume that the maximization objective has a higher priority. Therefore, we consider the problem first with the maximization objective.

Maximize \( z_1 = x_1 + x_2 \)

subject to \( (2, 3, 4)x_1 + (1.5, 2, 3)x_2 \leq (4, 5, 7) \)
\( (-1, 0, 1)x_1 + (0.5, 1, 1.5)x_2 \leq (1.5, 2, 3), \)

That is, \( 12x_1 + 8.5x_2 \leq 21, 4x_2 \leq 8.5, x_1, x_2 \geq 0 \).

The solution is \( x_1 = 0, x_2 = 2.4706 \) and \( \text{Max } z_1 = 2.4706 \).

Next, we proceed to solve the problem with minimization objective and an additional constraint \( x_1 + x_2 \geq 2.4706 \), obtained from previous step.

Minimize \( z_2 = 2x_1 + 3x_2 \)

subject to \( 12x_1 + 8.5x_2 \leq 21, 4x_2 \leq 8.5, x_1 + x_2 \geq 2.4706 \), \( x_1, x_2 \geq 0 \).

Now, the optimal solution is \( x_1 = 0, x_2 = 2.4706 \) and \( \text{Min } z_2 = 7.4118 \).

The feasibility of the solution will be explained as follows:

When all the fuzzy numbers are in the smallest possible values, the MOLPP becomes

Maximize \( z_1 = x_1 + x_2 \), Minimize \( z_2 = 2x_1 + 3x_2 \)

subject to \( 2x_1 + 1.5x_2 \leq 4, -x_1 + 0.5x_2 \leq 1.5, x_1, x_2 \geq 0 \).

The optimal solution to the above problem is \( x_1 = 0, x_2 = 2 \), \( \text{Max } z_1 = 8/3 \) and \( \text{Min } z_2 = 8 \).

When all the fuzzy numbers are in their peak values, the MOLPP takes the form

Maximize \( z_1 = x_1 + x_2 \), Minimize \( z_2 = 2x_1 + 3x_2 \)

subject to \( 3x_1 + 2x_2 \leq 5, x_1 \leq 2, x_1, x_2 \geq 0 \).

The optimal solution to the above problem is \( x_1 = 1/3, x_2 = 2 \), \( \text{Max } z_1 = 7/3 \) and \( \text{Min } z_2 = 20/3 \).

When all the fuzzy numbers are in the highest possible values, the MOLPP becomes

Maximize \( z_1 = x_1 + x_2 \), Minimize \( z_2 = 2x_1 + 3x_2 \)

subject to \( 4x_1 + 3x_2 \leq 7, x_1 + 1.5x_2 \leq 3, x_1, x_2 \geq 0 \).

The optimal solution to the above problem is \( x_1 = 1/2, x_2 = 5/3 \), \( \text{Max } z_1 = 13/6 \) and \( \text{Min } z_2 = 6 \).
Moreover, $x_1 \in [0,1/2]$, $x_2 \in [5/3,8/3]$, Max $z_1 \in [7/3,8/3]$ and Min $z_2 \in [6,8]$, so the solution is feasible and the same problem can also be solved with trapezoidal fuzzy numbers by using II.

Consider the following Bi-objective FLPP model with both the objectives of minimization type.

Minimize $z_1 = 10x_1 + 20x_2$

Minimize $z_2 = 20x_1 + 30x_2$

subject to $\bar{a}_{11}x_1 + \bar{a}_{12}x_2 \geq \bar{b}_1$

$\bar{a}_{21}x_1 + \bar{a}_{22}x_2 \geq \bar{b}_2$, $x_1, x_2 \geq 0$.

where $\bar{a}_{11} = (1,1,2)$, $\bar{a}_{12} = (0,1,2)$, $\bar{b}_1 = (4,5,7)$, $\bar{a}_{21} = (1,3,5)$, $\bar{a}_{22} = (3,5,5)$, $\bar{b}_2 = (6,7,8)$.

The optimal solution obtained by using the above procedure is given by $x_1 = 4.2$, $x_2 = 0$, Minimum Value = 84. The feasibility of the solution is verified for the smallest, peak and highest values of the fuzzy numbers. After analyzing, we get $x_1 \in [7/2,5]$, $x_2 \in [0,2/3]$ and the minimum lies in the interval $[70,100]$. So, the solution is feasible and the same problem can also be solved with trapezoidal fuzzy numbers.

6. Conclusion

In this paper, Multi-objective Fuzzy Linear Programming Problem under constraints with fuzzy coefficients is considered. A specific ranking method based on distance between fuzzy numbers is used for developing the ranking algorithm. By the Ranking algorithm, MOFLPP using triangular fuzzy numbers is transformed into MOLPP and then solved by Preemptive optimization method. Again it remains to research MOFLPP with general fuzzy numbers and MOFLPP with fuzziness in objective functions.

References


