Viscosity effects on flows of generalized Newtonian fluids through curved pipes

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Abstract

This paper is concerned with the application of finite element methods to obtain solutions for steady fully developed generalized Newtonian flows in a curved pipe of circular cross-section and arbitrary curvature ratio, under a given axial pressure gradient.

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1. Introduction

It is known since the pioneer experimental works of Williams et al. [1], Grindley and Gibson [2], and Eustice [3, 4] that flows in curved pipes are very challenging and considerably more complex than flows in straight pipes. Due to fluid inertia, a secondary motion appears in addition to the primary axial flow. It is induced by an imbalance between the cross-stream pressure gradient and the centrifugal force and consists of a pair of counter-rotating vortices, which appear even for the most mildly curved pipe.

Steady fully developed viscous flows in curved pipes of circular, elliptical and annular cross-section of both Newtonian and non-Newtonian fluids, have been studied theoretically by several authors (see e.g. [5–12]) following the fundamental work of Dean [13,14] for circular cross-section pipes. He obtained analytical solutions in the case of Newtonian fluids, using regular perturbation methods where the perturbation parameter is the curvature ratio defined as the cross-sectional radius of the pipe divided by the radius of curvature of the pipe centerline. These results based on perturbation solutions have been extended for a larger range of curvature ratio and Reynolds number, showing the existence of additional pairs of vortices and multiple solutions (see e.g. [15,16]).

Interest in the study of curved pipe flows is due to its wide range of applications in engineering (e.g. hydraulic pipe systems related to corrosion failure) and in biofluid dynamics, such as blood flow in the vascular system, where the distribution of velocities, pressure and wall shear stresses may help to provide some understanding of the genesis

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of atherosclerosis and other arterial lesions. The secondary flows induced by the centrifugal forces developed at the curvature sites will result in asymmetrical wall stresses with higher-shear and low pressure regions. This is particularly reviewed in Berger et al. [17] (see also [18] for additional information).

The aim of this paper is to present a numerical study for steady fully developed flows of generalized Newtonian fluids in curved pipes with circular cross-section and arbitrary curvature ratio. After introducing the governing equations in non-dimensional polar coordinates, to describe the curved pipe geometry, we outline their discretization using a finite element method to obtain approximate solutions to the original problem, and the methodology used for solving the resulting system of non-linear algebraic equations. Finally, Section 6 contains numerical results. We compare the quantitative and qualitative behavior of the axial velocity, the secondary streamlines and the wall shear stress for both Newtonian and generalized Newtonian flows, performing computations for different values of the Reynolds number, the curvature ratio and of the non-dimensional viscosity parameters involved in the governing equations.

In particular, we observe interesting viscosity effects that, as far as we know, have never been discussed in the literature: for small curvature ratio and within a certain range of viscosity parameters, the secondary streamlines contours undergo a counter-clockwise rotation and lose symmetry.

Conclusions are summarized at the end of the paper.

2. Governing equations

We consider flows of incompressible generalized Newtonian fluids with shear dependent viscosity (of Carreau–Yasuda type) in a curved pipe \( \Omega \subset \mathbb{R}^3 \) with boundary \( \partial \Omega \). For these fluids, the extra-stress tensor is related to the kinematic variables through

\[
\tau = 2(\eta^* + \eta^{**} (1 + |Du|^2)^q) Du,
\]

where \( u \) is the velocity field, \( Du = \frac{1}{2}( \nabla u + \nabla u^t ) \) denotes the symmetric part of the velocity gradient, \( q \) is a real number, \( \eta^* \) and \( \eta^{**} \) are non-negative real numbers satisfying \( \eta^* + \eta^{**} > 0 \). The Cauchy stress tensor is given by \( \mathbf{T} = -pI + \tau \), where \( p \) represents the pressure. The equations of conservation of momentum and mass hold in the domain \( \Omega \),

\[
\rho \left( \frac{\partial u}{\partial t} + u \cdot \nabla u \right) + \nabla p = \nabla \cdot \tau, \quad \nabla \cdot u = 0,
\]

where \( \rho > 0 \) is the (constant) density of the fluid. We consider the dimensionless form of this system by introducing the following quantities:

\[
x = \frac{\tilde{x}}{L}, \quad t = \frac{\tilde{t}}{\bar{U}}, \quad u = \frac{\tilde{u}}{\bar{U}}, \quad p = \frac{\bar{p}L}{(\eta^* + \eta^{**})\bar{U}},
\]

where the symbol \( \sim \) is attached to dimensional parameters (\( L \) represents a reference length and \( U \) a characteristic velocity of the flow). We also set \( \eta = \frac{\eta^{**}}{\eta^* + \eta^{**}} \), and introduce the Reynolds number \( Re = \frac{\rho U L}{\eta^* + \eta^{**}} \). The dimensionless system takes the form

\[
\begin{cases}
-\nabla \cdot 2 \left( 1 - \eta + \eta \left( 1 + |Du|^2 \right)^q \right) Du + Re \left( \frac{\partial u}{\partial t} + u \cdot \nabla u \right) + \nabla p = 0, \\
\nabla \cdot u = 0.
\end{cases}
\]

This system is supplemented with a Dirichlet homogeneous boundary condition

\( u = 0 \) on \( \partial \Omega \).

In this paper, we look for steady solutions of the previous problem defined in \( \Omega \). The corresponding system reads as

\[
\begin{cases}
-\nabla \cdot 2 \left( 1 - \eta + \eta \left( 1 + |Du|^2 \right)^q \right) Du + Re u \cdot \nabla u + \nabla p = 0 \quad \text{in } \Omega, \\
\nabla \cdot u = 0 \quad \text{in } \Omega, \\
u_{|\partial \Omega} = 0.
\end{cases}
\]

(2.3)
Fig. 1. A segment of curved pipe with centreline radius $R$ and cross-sectional radius $r_0$. The polar toroidal coordinates $(r, \phi, s)$ are also represented.

3. Formulation in polar toroidal coordinates

Since we are concerned with steady flows in curved pipe with circular cross section, it is more convenient to use the polar toroidal coordinate system (see Fig. 1), in the variables $(\tilde{r}, \theta, \tilde{s})$, defined with respect to the rectangular cartesian coordinates $(\tilde{x}, \tilde{y}, \tilde{z})$ through the relations

$$\tilde{r} = \sqrt{\tilde{z}^2 + (\sqrt{\tilde{x}^2 + \tilde{y}^2} - R)^2}, \quad \tilde{\theta} = \arctan \frac{\tilde{z}}{\sqrt{\tilde{x}^2 + \tilde{y}^2} - R},$$

$$\tilde{s} = R \arctan \frac{\tilde{y}}{\tilde{x}},$$

and the inverse relations

$$\tilde{x} = (R + \tilde{r} \cos \theta) \cos \frac{\tilde{s}}{R}, \quad \tilde{y} = (R + \tilde{r} \cos \theta) \sin \frac{\tilde{s}}{R}, \quad \tilde{z} = \tilde{r} \sin \theta,$$

with $0 < r_0 < R$, $0 \leq \theta < 2\pi$ and $0 \leq \tilde{s} < \pi R$. Introducing the axial variable and the pipe curvature ratio

$$s = \frac{\tilde{s}}{r_0}, \quad \delta = \frac{r_0}{R},$$

we see that the corresponding non-dimensional coordinate system is given by

$$r = \sqrt{z^2 + \left(\sqrt{x^2 + y^2} - \frac{1}{\delta}\right)^2}, \quad \theta = \arctan \frac{z}{\sqrt{x^2 + y^2} - \frac{1}{\delta}},$$

$$s = \frac{1}{\delta} \arctan \frac{y}{x},$$

and the inverse relations

$$x = \left(\frac{1}{\delta} + r \cos \theta\right) \cos(s\delta), \quad y = \left(\frac{1}{\delta} + r \cos \theta\right) \sin(s\delta), \quad z = r \sin \theta,$$

with $\delta < 1$, $0 \leq \theta < 2\pi$ and $0 \leq s < \frac{\pi}{\delta}$. Let us now formulate problem (2.3) in this new coordinate system. To simplify the notation we set

$$\beta_1 \equiv \beta_1(r, \theta) = r\delta \sin \theta, \quad \beta_2 \equiv \beta_2(r, \theta) = r\delta \cos \theta,$$

$$\beta \equiv \beta(r, \theta) = 1 + r\delta \cos \theta.$$

By using standard arguments, we first rewrite problem (2.3) in the toroidal coordinates $(r, \theta, s)$, and obtain
Find \((u, v, w, p)\) solution of

\[
\begin{align*}
\bigg[ -\bigg( \nabla \cdot \left( 2 \left( 1 - \eta + \eta \left( 1 + |Du|^2 \right)^q \right) Du - \nabla \cdot \mathbf{R} \mathbf{e} \mathbf{u} \otimes \mathbf{u} \bigg) \bigg)_r + \frac{\partial p}{\partial r} &= 0, \\
\bigg[ -\bigg( \nabla \cdot \left( 2 \left( 1 - \eta + \eta \left( 1 + |Du|^2 \right)^q \right) Du - \mathbf{R} \mathbf{e} \mathbf{u} \otimes \mathbf{u} \bigg) \bigg)_\theta + \frac{1}{r} \frac{\partial p}{\partial \theta} &= 0, \\
\bigg[ -\bigg( \nabla \cdot \left( 2 \left( 1 - \eta + \eta \left( 1 + |Du|^2 \right)^q \right) Du - \mathbf{R} \mathbf{e} \mathbf{u} \otimes \mathbf{u} \bigg) \bigg)_s + \frac{1}{\partial \theta} \frac{\partial p}{\partial \theta} &= 0, \\
\frac{\partial}{\partial r} (r \beta u) + \frac{\partial}{\partial \theta} (\beta v) + \frac{\partial}{\partial s} (rw) &= 0,
\end{align*}
\]

where

\[
\nabla \mathbf{u} = \begin{pmatrix}
\frac{\partial u}{\partial r} & \frac{\partial v}{\partial r} & \frac{\partial w}{\partial r} \\
\frac{1}{r} \frac{\partial u}{\partial \theta} - \frac{\partial v}{\partial r} & \frac{1}{r} \frac{\partial v}{\partial \theta} + \frac{1}{r} u & \frac{1}{r} \frac{\partial w}{\partial \theta} \\
\frac{1}{r} \frac{\partial u}{\partial s} - \frac{\beta_2}{r^2} w & \frac{\beta_1}{r^2} w + \frac{1}{r^2} w & \frac{\beta_1}{r^2} u + \frac{1}{r^2} u - \frac{1}{r^2} v
\end{pmatrix},
\]

and where \(\nabla \cdot \mathbf{\sigma}\) is given by

\[
\begin{align*}
(\nabla \cdot \mathbf{\sigma})_r &= \frac{1}{r \beta} \left( \frac{\partial}{\partial r} (r \beta \sigma_{rr}) + \frac{\partial}{\partial \theta} (\beta \sigma_{r\theta}) + \frac{\partial}{\partial s} (r \sigma_{rs}) - \beta_2 \sigma_{ss} - \beta \sigma_{\theta\theta} \right), \\
(\nabla \cdot \mathbf{\sigma})_\theta &= \frac{1}{r \beta} \left( \frac{\partial}{\partial r} (r \beta \sigma_{r\theta}) + \frac{\partial}{\partial \theta} (\beta \sigma_{\theta\theta}) + \frac{\partial}{\partial s} (r \sigma_{s\theta}) + \beta_1 \sigma_{ss} + \beta \sigma_{r\theta} \right), \\
(\nabla \cdot \mathbf{\sigma})_s &= \frac{1}{r \beta} \left( \frac{\partial}{\partial r} (r \beta \sigma_{rs}) + \frac{\partial}{\partial \theta} (\beta \sigma_{s\theta}) + \frac{\partial}{\partial s} (r \sigma_{ss}) - \beta_1 \sigma_{s\theta} + \beta_2 \sigma_{rs} \right).
\end{align*}
\]

In order to avoid singularities appearing in the calculations when \(r\) is close to zero, we rewrite the problem (3.4) in a more suitable form. Setting

\[
\gamma = 2 \max(q, 0) + 1, \quad \psi_p = (r \beta)^p \quad p > 0,
\]

we observe that for every \(\mathbf{\sigma} \in \mathbb{M}^{3 \times 3}\), we have

\[
\psi_{\gamma} \left( (1 - \eta) \mathbf{\sigma} + \frac{\eta}{(r \beta)^{2q}} \left( (r \beta)^2 + |r \beta \mathbf{\sigma}|^2 \right)^q \mathbf{\sigma} \right) = \left( (1 - \eta) \psi_{\gamma - 1} + \eta \psi_{\gamma - 2q - 1} \left( (r \beta)^2 + |r \beta \mathbf{\sigma}|^2 \right)^q \right) r \beta \mathbf{\sigma}.
\]

On the other hand, taking into account the definition of \(\nabla \cdot \mathbf{\sigma}\) given above and using standard calculations, we can show that

\[
\psi_{\gamma + 1} (\nabla \cdot \mathbf{\sigma})_\alpha = A_{\gamma, \alpha} \left( \psi_\gamma \mathbf{\sigma} \right), \quad \alpha = r, \theta, s,
\]

where

\[
A_{\gamma, \alpha} (\mathbf{\sigma}) = r \beta (\nabla \cdot \mathbf{\sigma})_\alpha - \gamma (\beta + \beta_2) \sigma_{\alpha r} - \beta_1 \sigma_{\alpha \theta}.
\]
Therefore, multiplying Eq. \((3.4)\) by \(\psi_{r+1}\), we easily see that \((u, p)\) is solution of the following system:

\[
\begin{aligned}
-2A_{r, r} \left( \psi_r Du \right) + \psi_{r+1} \frac{\partial p}{\partial r} &= A_{r, r} \left( 2\eta \Phi^{\gamma} (r\beta Du) - Re\psi_r u \otimes u \right), \\
-2A_{\theta, r} \left( \psi_r Du \right) + \beta \psi_r \frac{\partial p}{\partial \theta} &= A_{\theta, r} \left( 2\eta \Phi^{\gamma} (r\beta Du) - Re\psi_r u \otimes u \right), \\
-2A_{s, r} \left( \psi_r Du \right) + r \psi_r \frac{\partial p}{\partial s} &= A_{s, r} \left( 2\eta \Phi^{\gamma} (r\beta Du) - Re\psi_r u \otimes u \right), \\
\frac{\partial}{\partial r} (r\beta u) + \frac{\partial}{\partial \theta} (\beta v) + \frac{\partial}{\partial s} (r w) &= 0, \\
u|_{\partial \Sigma} &= 0,
\end{aligned}
\]  

\begin{equation}
(3.8)
\end{equation}

where \(\Phi^{\gamma} : M^{3 \times 3} \rightarrow M^{3 \times 3}\) is a mapping defined by

\[
\Phi^{\gamma}(\sigma) = \left( \psi_{r-1} \psi_{r-2} \right) (r\beta)^2 + (|\sigma|^2)^\gamma \right) \sigma.
\]  

\begin{equation}
(3.9)
\end{equation}

4. Fully developed flows

We consider flows which are fully developed. Hence, the components of the velocity are then independent of the variable \(s\), i.e.

\[
\frac{\partial u}{\partial s} = \frac{\partial v}{\partial s} = \frac{\partial w}{\partial s} = 0.
\]  

\begin{equation}
(4.1)
\end{equation}

Consequently the axial component of the pressure gradient is a constant:

\[
\frac{\partial p}{\partial s} = -p^*.
\]  

\begin{equation}
(4.2)
\end{equation}

Taking into account \((4.1)\) and \((4.2)\) and replacing in system \((3.8)\), we easily see for fully developed flows, that problem is defined in the set

\[
\Sigma = \{(r, \theta) \in \mathbb{R}^2 \mid 0 < r < 1, 0 < \theta \leq 2\pi\},
\]  

\begin{equation}
(4.3)
\end{equation}

and reads as follows:

\[
\begin{aligned}
-2A_{r, r} \left( \psi_r Du \right) + \psi_{r+1} \frac{\partial p}{\partial r} &= A_{r, r} \left( 2\eta \Phi^{\gamma} (r\beta Du) - Re\psi_r u \otimes u \right), \\
-2A_{\theta, r} \left( \psi_r Du \right) + \beta \psi_r \frac{\partial p}{\partial \theta} &= A_{\theta, r} \left( 2\eta \Phi^{\gamma} (r\beta Du) - Re\psi_r u \otimes u \right), \\
-2A_{s, r} \left( \psi_r Du \right) &= A_{s, r} \left( 2\eta \Phi^{\gamma} (r\beta Du) - Re\psi_r u \otimes u \right), \\
\frac{\partial}{\partial r} (r\beta u) + \frac{\partial}{\partial \theta} (\beta v) &= 0, \\
u|_{\partial \Sigma} &= 0.
\end{aligned}
\]  

\begin{equation}
(4.4)
\end{equation}

where \(A_{r, r}\) and \(\Phi^{\gamma}\) are respectively given by \((3.7)\) and \((3.9)\).

**Remark 1.** The choice of the previous non-dimensionalization form of our problem enables us to consider the limiting cases of vanishing Reynolds number and vanishing viscosity exponent.

5. Numerical approximation

In this section we use finite element methods to obtain approximate solutions to system \((4.4)\) and study generalized Newtonian flows through curved pipes with circular cross-section. Let \(\{T_h\}_{h>0}\) be a family of regular triangulations defined over the rectangle \(\Sigma\), and consider the following finite element spaces:

\[
X_h = \{v_h \in C(\Sigma) \cap H^1_0(\Sigma) \mid v_h|_K \in P_2(K), \forall K \in T_h\},
\]

\[
Q_h = \{q_h \in C(\Sigma) \cap L^2_0(\Sigma) \mid q_h|_K \in P_1(K), \forall K \in T_h\}.
\]
This pair of spaces \((X_h, Q_h)\) corresponds to the so-called Hood–Taylor finite element method, and verifies a compatibility condition known as the discrete LBB (or inf–sup) condition. \cite{19}. For \(\delta \in [0, 1]\), let us set

\[
V_{\delta,h} \equiv (V_{\delta,h})^3 = \{ v_h \in X_h \mid \nabla' \cdot (\beta v_h) = 0 \}.
\]

System (4.4) is approximated by the following problem:

Find \((u_h, p_h) \equiv (u, p) \in X_h \times Q_h\) solution of

\[
\begin{align*}
-2A_{r,\gamma}(\psi_r D\mathbf{u}) + \psi_{r+1} \frac{\partial p}{\partial r}, \phi_1 & = (A_{r,\gamma}(2 \eta \Phi''(r \beta D\mathbf{u}) - \Re e \psi_r \mathbf{u} \otimes \mathbf{u}), \phi_1), \\
-2A_{\beta,\gamma}(\psi_r D\mathbf{u}) + \beta \psi_r \frac{\partial p}{\partial \theta}, \phi_1 & = (A_{\beta,\gamma}(2 \eta \Phi''(r \beta D\mathbf{u}) - \Re e \psi_r \mathbf{u} \otimes \mathbf{u}), \phi_1), \\
-2A_{s,\gamma}(\psi_r D\mathbf{u}), \phi_1 & = (p^* r \psi_r + A_{s,\gamma}(2 \eta \Phi''(r \beta D\mathbf{u}) - \Re e \psi_r \mathbf{u} \otimes \mathbf{u}), \phi_1), \\
\frac{\partial}{\partial r}(r \beta u) + \frac{\partial}{\partial \theta}(\beta v), \varphi & = 0,
\end{align*}
\] (5.5)

for all \((\phi_1, \phi_2, \phi_3, \varphi) \in V_{\delta,h} \times Q_h\).

Using straightforward calculations, we can prove that for all \(\varphi \in V_{\delta,h}\), the following equality holds:

\[
- \left( \frac{\partial}{\partial r}(r \beta \sigma_{ar}) + \frac{\partial}{\partial \theta}(\beta \sigma_{a\theta}) + \gamma (\beta + \beta_2) \sigma_{ar} - \beta_1 \sigma_{a\theta}, \varphi \right)
\]

\[
= \left( \sigma_{ar}, r \beta \frac{\partial \varphi}{\partial r} + \gamma (\beta + \beta_2) \varphi \right) + \left( \sigma_{a\theta}, \beta \frac{\partial \varphi}{\partial \theta} - \gamma \beta_1 \varphi \right).
\]

Taking into account the definition of the operator \(A_{\alpha,\gamma}\), we deduce that

\[
- (A_{r,\gamma}(\sigma), \varphi) = K_{\gamma}(\sigma_{rr}, \varphi) + L_{\gamma}(\sigma_{r\theta}, \varphi) + (\beta_2 \sigma_{ss} + \beta \sigma_{s\theta}, \varphi),
\] (5.6)

\[
- (A_{\beta,\gamma}(\sigma), \varphi) = K_{\gamma}(\sigma_{r\theta}, \varphi) + L_{\gamma}(\sigma_{\theta\theta}, \varphi) - (\beta_1 \sigma_{ss} + \beta \sigma_{s\theta}, \varphi),
\] (5.7)

\[
- (A_{s,\gamma}(\sigma), \varphi) = K_{\gamma}(\sigma_{ss}, \varphi) + L_{\gamma}(\sigma_{s\theta}, \varphi) + (\beta_1 \sigma_{ss} - \beta_2 \sigma_{s\theta}, \varphi),
\] (5.8)

where

\[
K_{\gamma}(\sigma, \varphi) = \left( \sigma, r \beta \frac{\partial \varphi}{\partial r} + \gamma (\beta + \beta_2) \varphi \right),
\] (5.9)

\[
L_{\gamma}(\sigma, \varphi) = \left( \sigma, \beta \frac{\partial \varphi}{\partial \theta} - \gamma \beta_1 \varphi \right).
\] (5.10)

Therefore, by setting \(\sigma = \psi_r D\mathbf{u}\) and \(\sigma = \psi_r \mathbf{u} \otimes \mathbf{u}\) in (5.6), we respectively obtain

\[
- (A_{r,\gamma}(\psi_r D\mathbf{u}), \varphi) = K_{\gamma}(\psi_r \frac{\partial u}{\partial r}, \varphi) + \frac{1}{2} L_{\gamma}(\beta \psi_{r-1} \frac{\partial u}{\partial \theta}, \varphi) + \left( \psi_{r-1} \left( \beta^2 + \beta_2^2 \right) u, \varphi \right),
\] (5.11)

\[
- (A_{r,\gamma}(\psi_r \mathbf{u} \otimes \mathbf{u}), \varphi) = K_{\gamma}(\psi_r u^2, \varphi) + L_{\gamma}(\psi_r u \mathbf{v}, \varphi) + \left( \psi_{r-1} \left( \beta^2 u^2 + \beta v^2 \right), \varphi \right).
\] (5.12)

Similarly, (5.7) and (5.8) imply that

\[
- (A_{\beta,\gamma}(\psi_r D\mathbf{u}), \varphi) = \frac{1}{2} K_{\gamma}(\psi_{r-1} \left( \beta^2 \frac{\partial v}{\partial r} - \beta v \right), \varphi) + L_{\gamma}(\beta \psi_{r-1} \frac{\partial v}{\partial \theta}, \varphi)
\]

\[
+ \left( \psi_{r-1} \left( \left( \beta^2 + \beta_1^2 \right) v - \frac{\beta_2^2 \partial u}{2 \partial r} \right), \varphi \right) + \frac{1}{2} K_{\gamma}(\beta \psi_{r-1} \frac{\partial u}{\partial \theta}, \varphi)
\]

\[
+ L_{\gamma}(\beta \psi_{r-1} u, \varphi) - \left( \psi_{r-1} \left( \beta_1 \beta_2 u + \frac{\beta^2 \partial u}{2 \partial \theta} \right), \varphi \right).
\] (5.13)
\[-(A_{\psi,\gamma}(\psi u \otimes u), \varphi) = K_{\gamma}(\psi uv, \varphi) + \mathcal{L}_{\gamma}(\psi v^2, \varphi) - \left(\psi_{\gamma} \left(\beta_1 w^2 + \beta uv\right), \varphi\right), \tag{5.14}\]

and

\[-(A_{\psi,\gamma}(\psi D u), \varphi) = \frac{1}{2} K_{\gamma}(\psi_{\gamma-1}(r^2 \beta \frac{\partial w}{\partial r} - \beta_2 w), \varphi) + \frac{1}{2} \mathcal{L}_{\gamma}(\psi_{\gamma-1}(\beta \frac{\partial w}{\partial \theta} + \beta_1 w), \varphi) + \frac{1}{2} \left(\psi_{\gamma-1}\left((r \delta)^2 w + \beta \beta_1 \frac{\partial w}{\partial \theta} - r \beta \beta_2 \frac{\partial w}{\partial r}\right), \varphi\right), \tag{5.15}\]

\[-(A_{\psi,\gamma}(\psi u \otimes u), \varphi) = K_{\gamma}(\psi uvw, \varphi) + \mathcal{L}_{\gamma}(\psi v w, \varphi) - (\psi_{\gamma} (\beta_2 u - \beta_1 v) w, \varphi). \tag{5.16}\]

Finally, a standard integration by part shows that

\[-\left(\psi_{\gamma+1} \frac{\partial p}{\partial r}, \varphi\right) = \left(\psi_{\gamma} p, (\gamma + 1)(\beta + \beta_2)\varphi + r \beta \frac{\partial \varphi}{\partial r}\right) = K_{\gamma+1} (\psi_{\gamma} p \varphi), \tag{5.17}\]

\[-(\beta \psi_{\gamma} \frac{\partial \varphi}{\partial \theta}, \varphi) = \left(\psi_{\gamma} p, -(\gamma + 1)\beta_1 \varphi + r \beta \frac{\partial \varphi}{\partial \theta}\right) = \mathcal{L}_{\gamma+1} (\psi_{\gamma} p, \varphi). \tag{5.18}\]

Combining (5.11)–(5.18), we deduce that problem (5.5) can be rewritten in the following form:

(P\(_h\)) Find \((u_h, p_h) \equiv (u, p) \in X_h \times Q_h\) solution of

\[
2K_{\gamma}\left(\psi_{\gamma} \frac{\partial u}{\partial r}, \phi_1\right) + \mathcal{L}_{\gamma}\left(\beta \psi_{\gamma-1} \frac{\partial u}{\partial \theta}, \phi_1\right) + 2\left(\psi_{\gamma-1}(\beta^2 + \beta_2^2) u, \phi_1\right)
\]

\[+ \mathcal{L}_{\gamma}\left(\beta \psi_{\gamma-1} \left(r \frac{\partial v}{\partial r} - v\right), \phi_2\right) + 2\left(\psi_{\gamma-1}(\beta^2 - \beta_1^2) v, \phi_2\right) - K_{\gamma}(\psi_{\gamma} p, \phi_2), \tag{5.19}\]

\[
- \mathcal{R}_e\left(K_{\gamma}\left(\psi_{\gamma} u^2, \phi_1\right) + \mathcal{L}_{\gamma}\left(\psi_{\gamma} u v, \phi_1\right) + \left(\psi_{\gamma} (\beta_2 w^2 + \beta v^2), \phi_1\right)\right)
\]

\[+ K_{\gamma}\left(\psi_{\gamma} \frac{\partial v}{\partial r}, \phi_2\right) - \mathcal{L}_{\gamma}(\psi_{\gamma-1}(\beta \frac{\partial u}{\partial \theta} + 2 \beta_1 \beta_2 u), \phi_2) - \mathcal{L}_{\gamma+1} (\psi_{\gamma} p, \phi_2) = 0. \tag{5.20}\]

for every \((\phi_1, \phi_2, \phi_3, \varphi) \in (V_{3,h})^3 \times Q_h\) with \(K_{\gamma}, \mathcal{L}_{\gamma}\) and \(\psi_{\gamma}\) given by (5.9), (5.10) and (3.9). The algorithm we consider to solve problem (P\(_h\)) is based on Newton’s method, the nonlinear part being explicitly given at each iteration.
step. Our aim here is to write the linear systems at the iteration $k$. To simplify the notation, we will consider the case of creeping non-Newtonian flow, which corresponds to $\text{Re} = 0$.

Given $u^k$ and expressing the corresponding approximate solutions $u^k$, $v^k$, $w^k$ and $p^k$ in the basis of $V_h$ and $M_h$, respectively

\[
 u^k = \sum_{i=1}^{n_h} u_i^k \phi_1^i, \quad v^k = \sum_{i=1}^{n_h} v_i^k \phi_2^i, \quad w^k = \sum_{i=1}^{n_h} w_i^k \phi_3^i, \quad p^k = \sum_{i=1}^{n_h} p_i^k \phi_4^i,
\]

we obtain the following linear system:

\[
 \begin{pmatrix}
 A_1 & A_2 & 0 & A_3 \\
 A_4 & A_5 & 0 & A_6 \\
 0 & 0 & A_7 & 0 \\
 A_8 & A_9 & 0 & 0
 \end{pmatrix}
 \begin{pmatrix}
 u^k \\
 v^k \\
 w^k \\
 p^k
 \end{pmatrix} =
 \begin{pmatrix}
 b_1^k \\
 b_2^k \\
 b_3^k \\
 0
 \end{pmatrix}
\]

where

\[
 (A_{1})_{ij} = 2K_y \left( \psi_y \frac{\partial \phi_j^i}{\partial r}, \phi_1^i \right) + L_y \left( \beta \psi_y - \frac{\partial \phi_j^i}{\partial \theta}, \phi_1^i \right), \quad (5.20)
\]

\[
 (A_{2})_{ij} = L_y \left( \psi_y \left( r \beta \frac{\partial \phi_j^2}{\partial r} - \beta \phi_2^j \right), \phi_1^i \right) + 2L_y \left( \psi_y \left( \beta^2 \frac{\partial \phi_j^2}{\partial \theta} - \beta_1 \beta_2 \phi_2^j \right), \phi_1^i \right), \quad (5.21)
\]

\[
 (A_{3})_{ij} = -K_y+1 \left( \psi_y \phi_1^j, \phi_1^i \right), \quad (5.22)
\]

\[
 (A_{4})_{ij} = K_y \left( \beta \psi_y - \frac{\partial \phi_j^2}{\partial \theta}, \phi_2^j \right) + 2L_y \left( \beta \psi_y - \frac{\partial \phi_j^2}{\partial \theta}, \phi_2^j \right) - \left( \psi_y - \frac{\partial \phi_j^2}{\partial \theta} + 2 \beta_1 \beta_2 \phi_2^j \right), \quad (5.23)
\]

\[
 (A_{5})_{ij} = K_y \left( \psi_y \left( r \beta \frac{\partial \phi_j^2}{\partial r} - \beta \phi_2^j \right), \phi_2^j \right) + 2L_y \left( \beta \psi_y - \frac{\partial \phi_j^2}{\partial \theta}, \phi_2^j \right) \\
 + \left( \psi_y \left( \beta^2 + \beta_2 \right) \phi_2^j - r \beta^2 \frac{\partial \phi_j^1}{\partial r} \right), \quad (5.24)
\]

\[
 (A_{6})_{ij} = -L_y+1 \left( \psi_y \phi_1^j, \phi_2^j \right), \quad (5.25)
\]

\[
 (A_{7})_{ij} = K_y \left( \psi_y \left( r \beta \frac{\partial \phi_j^3}{\partial r} - \beta_2 \phi_3^j \right), \phi_3^j \right) + K_y \left( \psi_y \left( \beta \frac{\partial \phi_j^3}{\partial \theta} + \beta_1 \phi_3^j \right), \phi_3^j \right) \\
 + \left( \psi_y \left( r \delta \phi_3^j + \beta \beta_1 \phi_3^j \varphi - r \beta \phi_2 \frac{\partial \phi_j^1}{\partial r} \right), \phi_3^j \right), \quad (5.26)
\]

\[
 (A_{8})_{ij} = \left( r \beta \frac{\partial \phi_j^1}{\partial r} + \beta + \beta_2 \phi_1^j, \phi_1^j \right), \quad (5.27)
\]

\[
 (A_{9})_{ij} = \beta \frac{\partial \phi_j^1}{\partial \theta} - \beta_1 \phi_2^j, \varphi \right), \quad (5.28)
\]

\[
 \left( b_1^k \right)_i = -K_y \left( \Phi_{rr} \left( r \beta D \mathbf{u}^{k-1}, \phi_1^j \right) - L_y \left( \Phi_{r \theta} \left( r \beta D \mathbf{u}^{k-1}, \phi_1^j \right) \right) - \left( \beta_2 \Phi_{ss} \left( r \beta D \mathbf{u}^{k-1} \right) + \beta \Phi_{\theta \theta} \left( r \beta D \mathbf{u}^{k-1} \right), \phi_1^j \right), \quad (5.29)
\]
6. Numerical results

The domain is defined by $\Sigma = (0, 1) \times (0, 2\pi)$ and is discretized using triangles. A Navier–Stokes system is solved for $(u, v, p)$ and a Poisson equation for $w$. The velocity is set to zero on the lateral surface of the pipe.

In order to study the behaviour of the velocity and the existence of secondary motions, a non-dimensional stream function $\psi$ is often considered in the case of incompressible fully developed flows. This function can be written with respect to the components $u$ and $v$, as

$$u = -\frac{1}{r\beta} \frac{\partial \psi}{\partial \theta}, \quad v = \frac{1}{\beta} \frac{\partial \psi}{\partial r}.$$

Another interesting criterion when analysing the effect of the (non-constant) viscosity is the behaviour of the wall shear stress. For the velocity fields considered here, the stress vector acting on the inner walls of the tube can be written as

$$t = \tau_w \lambda + p_e n,$$

where $n$ is the outward unit normal to the fluid surface and $\lambda$ is the tangent to the fluid surface. The quantities $\tau_w$ and $p_e$ are usually referred to as the wall shear stress (WSS) and the mechanical wall pressure, and can be defined relatively to the Cauchy stress tensor $T$ by the following expressions:

$$\tau_w = -(T \cdot n) \cdot \lambda |_{r=1}, \quad p_e = -(T \cdot n) \cdot n |_{r=1}.$$

Straightforward calculations show that in our case, the wall shear stress is given by

$$\tau_w = -2 \left( (1 - \eta) + \eta \left( 1 + |Du|_{r=1}^2 \right)^q \right) (D_{rr}u - D_{\theta\theta}u) |_{r=1} \sin \theta \cos \theta$$

$$\times -2 \left( (1 - \eta) + \eta \left( 1 + |Du|_{r=1}^2 \right)^q \right) D_{\theta r}u |_{r=1} (\sin^2 \theta - \cos^2 \theta).$$

In order to study the effect of the generalized viscosity, we compare the qualitative behaviour of the axial velocity, the stream function and the wall shear stress of both Newtonian ($q = 0$ or $\eta = 0$) and generalized Newtonian fluids. We analyse the behaviour of the flow for different values of the parameters involved in the governing equations (the Reynolds number $Re$, the curvature ration $\delta$, the non-dimensional viscosity parameter $\eta$ and the exponent $q$ appearing in the power-law type viscosity). A continuation method is carried out to implement these different tests.

When the Reynolds number is set to zero (creeping flows), there is no secondary motion and no wall shear stress and as expectable, the solution is of Poiseuille type in both cases. The contours of the axial velocity $u$ are circles, centred about the central axis in the case of a small curvature and are shifted away from the center when the curvature ratio increases. The only remarkable difference between the Newtonian and generalized Newtonian flows is related to the maximal value achieved by the axial velocity. Indeed, the maximum increases with the absolute value of the exponent $q$, before stabilizing for some value of this parameter.

6.1. Newtonian flows

For inertial Newtonian flows ($Re \neq 0$), it is well known that a “slight curvature” of the pipe axis induces centrifugal forces on the fluid which forms a secondary flow, sending fluid outward along the symmetry axis and returning along the upper and lower curved surfaces. A pair of symmetrical vortices is then superposed to the axial Poiseuille flow, and strengthen when the Reynolds number increases. The effect of the curvature ratio is similar to the case of creeping
flows. The flow is “stable” in the sense that its qualitative behaviour is the same. In Figs. 2 and 3, we plot the contours of the stream function and the wall shear stress for $Re = 1, 15, 30, 70$ and $\delta$ equal to 0.001 and 0.1. As can be observed, the contours of the stream function show a shift from the center when $\delta = 0.1$. The behavior of the wall shear stress is identical. The only difference is related to the amplitude which clearly depends on the Reynolds number and on the curvature ratio.

6.2. Generalized Newtonian flows

Inertial generalized Newtonian flows are more complex. The contours of the stream function, as well as the wall shear stress vary with respect to the non-dimensional parameter $\eta$ and the exponent $q$, appearing in the definition of the non-linear viscosity function, and with respect to the Reynolds number $Re$. Because of the highly non-linear nature of the model, the interaction between the corresponding terms, namely the convective and the viscosity terms, is difficult to handle.

Considering first the case of shear-thickening flows, we fix $\eta$ and $Re$ and apply a continuation method with respect to the exponent $q$. The maximum value for which the convergence was ensured is $q_{\text{max}} = 0.18$, independently of the values of the other parameters. The behaviour of the shear-thickening flows, at least for the achieved tests, seems to be homogeneous. The only remarkable fact is related to a small rotation to the right of the contours when the viscosity parameter $q$ increases. Similarly, the contours of the wall shear stress show a slight shift to the right. The magnitude slightly decreases when $q$ increases, and in contrast to the Newtonian case, the values of $\tau_w$ at $\theta = 0$ and $\theta = 2\pi$ are different from zero.

The case of shear-thinning flows is much more complex and is the aim of the rest of the paper.
Table 1
Maximum values for the exponent $|q|$.

<table>
<thead>
<tr>
<th>$\eta$</th>
<th>0.1</th>
<th>0.2</th>
<th>0.4</th>
<th>0.5</th>
<th>0.6</th>
<th>0.7</th>
<th>0.9</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Re$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>72.5</td>
<td>72.5</td>
<td>6.7</td>
<td>5.0</td>
<td>3.4</td>
<td>1.8</td>
<td>0.6</td>
</tr>
<tr>
<td>15</td>
<td>72.5</td>
<td>72.5</td>
<td>25.2</td>
<td>5.9</td>
<td>2.9</td>
<td>1.7</td>
<td>0.6</td>
</tr>
<tr>
<td>30</td>
<td>72.5</td>
<td>24.3</td>
<td>5.9</td>
<td>2.5</td>
<td>1.4</td>
<td>0.6</td>
<td></td>
</tr>
<tr>
<td>50</td>
<td>72.5</td>
<td>28.3</td>
<td>5.9</td>
<td>2.4</td>
<td>1.2</td>
<td>0.6</td>
<td></td>
</tr>
<tr>
<td>70</td>
<td>72.5</td>
<td>31.6</td>
<td>5.8</td>
<td>2.1</td>
<td>1.2</td>
<td>0.5</td>
<td></td>
</tr>
</tbody>
</table>

6.2.1. Small curvature ratio

In order to understand the combined effect of the parameters involved in the model, and especially those related to the generalized viscosity, we have implemented several tests. We first consider the case of a “slightly curved” pipe ($\delta = 0.001$). Fixing the Reynolds number $Re$ and the viscosity parameter $\eta$, we initiate a continuation process with respect to the exponent $q$.

For $\eta = 0$, we have the Newtonian regime. The effect of the convection can be observed, and the secondary flows appear. The idea is then to increase the parameter $\eta$ (together with $|q|$) in order to study the effect of the viscosity. As can be seen in Tables 2 and 3, tests for different values of $\eta$ and different values of $Re$ were done. One of the general observations is that the flow seems to behave in three different ways, depending on the value of the viscosity parameter, namely for $\eta$ taking values in the sub-intervals $[0, 0.4]$, $[0.4, 0.6]$ and $[0.6, 1]$.

Table 1 shows the maximum value $q_{\text{max}}$ obtained for each one of the cases considered. For a fixed $\eta$, we have globally the same range of values, independently of the Reynolds number. For $\eta \in [0, 0.4]$, $q_{\text{max}}$ is a constant. For
Table 2
Qualitative behaviour of the contours

<table>
<thead>
<tr>
<th>η</th>
<th>0.1</th>
<th>0.2</th>
<th>0.4</th>
<th>0.5</th>
<th>0.6</th>
<th>0.7</th>
<th>0.9</th>
</tr>
</thead>
<tbody>
<tr>
<td>Re</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>S</td>
<td>L–S</td>
<td>L</td>
<td>L</td>
<td>L</td>
<td>S</td>
<td>S</td>
</tr>
<tr>
<td>15</td>
<td>S</td>
<td>L–S</td>
<td>L–S</td>
<td>L</td>
<td>L</td>
<td>S</td>
<td>S</td>
</tr>
<tr>
<td>30</td>
<td>S</td>
<td>L–S</td>
<td>L–S</td>
<td>L</td>
<td>S</td>
<td>S</td>
<td>S</td>
</tr>
<tr>
<td>50</td>
<td>S</td>
<td>L–S</td>
<td>L–S</td>
<td>S</td>
<td>S</td>
<td>S</td>
<td>S</td>
</tr>
<tr>
<td>70</td>
<td>S</td>
<td>L–S</td>
<td>L–S</td>
<td>S</td>
<td>S</td>
<td>S</td>
<td>S</td>
</tr>
</tbody>
</table>

Table 3
Values for |q_var| initiating the rotation

<table>
<thead>
<tr>
<th>η</th>
<th>0.1</th>
<th>0.2</th>
<th>0.4</th>
<th>0.5</th>
<th>0.6</th>
<th>0.7</th>
<th>0.9</th>
</tr>
</thead>
<tbody>
<tr>
<td>Re</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>–</td>
<td>19</td>
<td>6.0</td>
<td>4.0</td>
<td>2.9</td>
<td>–</td>
<td>–</td>
</tr>
<tr>
<td>15</td>
<td>–</td>
<td>19</td>
<td>6.0</td>
<td>4.0</td>
<td>2.8</td>
<td>–</td>
<td>–</td>
</tr>
<tr>
<td>30</td>
<td>–</td>
<td>19</td>
<td>6.0</td>
<td>4.2</td>
<td>–</td>
<td>–</td>
<td>–</td>
</tr>
<tr>
<td>50</td>
<td>–</td>
<td>19</td>
<td>6.0</td>
<td>–</td>
<td>–</td>
<td>–</td>
<td>–</td>
</tr>
<tr>
<td>70</td>
<td>–</td>
<td>19</td>
<td>6.0</td>
<td>–</td>
<td>–</td>
<td>–</td>
<td>–</td>
</tr>
</tbody>
</table>

Fig. 4. Streamlines and wall shear stress for shear-thinning generalized Newtonian flows with η = 0.1 and Re = 70 (δ = 0.001).

η = 0.4 and η = 0.5, the maximum value increases with Re, while it decreases for η taking the values 0.6, 0.7 and 0.9. As we shall see below, one of the surprising effects of the non-constant viscosity on the secondary flows, verifiable at least for small curvature ratios, is the rotation of the contours. Table 2 deals with the global behaviour of these contours for different values of the parameters. Notation S and L refer respectively to symmetrical and left-rotating contours, while L–S refers to contours that rotate to the left and then to the right, before stabilizing in a symmetrical way. Table 3 presents the values of |q_var| corresponding to the viscosity exponent that initiates the rotation (when it occurs). It is clear that q_var decreases as η increases. Moreover, for η fixed, this exponent is constant and is independent of the values of the Reynolds number.

For η = 0.1, the contours are still symmetrical and the qualitative behaviour is of Newtonian type. The only difference lies in the values of the stream function, which increase in absolute value with the exponent q and with Re (see Fig. 19(a)), while the axial velocity is constant.

It is also interesting to note that for a fixed Reynolds number, the wall shear stress does not depend on q (see Fig. 4).
For $\eta = 0.2$ (respectively $\eta = 0.3$ and 0.4), interesting phenomena can be observed. Initially, the behaviour is identical to the previous case (Fig. 4(a)), but as $|q|$ increases, we observe a variation in the shape of the vortices, their displacement to the core, the concentration of the contours and the reduction of the global surface in this region. For $q = q_{\text{var}} = -19$ (respectively $q_{\text{var}} = -9$ and $q_{\text{var}} = -6$), the contours initiate a counter-clockwise rotation, augmenting with the viscosity exponent. At some level, a stabilization can be noticed, followed by a clockwise rotation where the inverse phenomenon occurs: weakening of the contours in the core region, distanceing of the vortices and decreasing the angle of rotation till the recovery of the symmetry (see Figs. 6–11 and Fig. 5(c) and (d)).
Fig. 8. Streamlines for $Re = 70$ and $\eta = 0.4$.

(a) $q = -6.5$. (b) $q = -6.8$. (c) $q = -7$.

Fig. 9. Streamlines for $Re = 15$ and $\eta = 0.4$.

(a) $q = -8$. (b) $q = -11$. (c) $q = -12$.

Fig. 10. Streamlines for $Re = 30$ and $\eta = 0.4$.

(a) $q = -8$. (b) $q = -11$. (c) $q = -12$.

Fig. 11. Streamlines for $Re = 70$ and $\eta = 0.4$.

(a) $q = -8$. (b) $q = -11$. (c) $q = -12$. 
Let us also observe that even if the rotation initiates at the same exponent independently of the Reynolds number, the inertial forces seem to oppose resistance and clearly affect the maximum angle of rotation.

Besides this phenomenon, one can observe parallel variations in the behavior of the wall shear stress. In Fig. 12, we plot the corresponding curves for $\eta = 0.4$ and for different values of the Reynolds number. Initially, the behavior
is of Newtonian type, the curves showing symmetries relatively to the horizontal axis and to the vertical axis $\theta = \pi$, and with the wall shear stress vanishing at $\theta = 0, \pi$ and $2\pi$. As $|q|$ increases, some modifications can be observed when the viscosity exponent reaches $q_{\text{var}} = -6$, and are more visible for relatively small values of $Re$ ($Re = 15, 30$). At this level, there is lost of symmetry with respect to both axis and the wall shear stress takes positive values at $\theta = 0, 2\pi$ and negative values at $\theta = \pi$. In the neighbourhood of $q = -8$, there is a stabilization with recovering of the symmetry but for a different type of curve. Finally, the inverse behaviour initiates until we recover the original state.

Let us also observe that for a fixed $\eta$, the behavior of the maximum values for the axial velocity is independent of the Reynolds number and that it increases in absolute value with respect to $|q|$, while the maximum values of the stream function increases with $Re$ and, for a fixed viscosity parameter, have the same behaviour as the contours: they first increase, they stabilize and finally decrease (cf. Fig. 19(b)–(d)).

The case $\eta = 0.5$ is particular in the sense that it represents a limit for the behaviour of the flow. For $Re = 15$ and $Re = 30$, the case is similar to the previous ones: Newtonian type, followed by a counter-clockwise rotation and then a stabilization. However, at least for the achieved tests, no clockwise rotations have been observed. This fact together
with the shape of the vortices and the strong deformation of the contours in their neighbourhood suggest that we are in the presence of some “forces” in opposition to the rotation. This can be confirmed when considering the case of \( Re = 70 \), where the vortices shift away from the centre and where the streamlines initiate a slight clockwise slope (see Figs. 13–15 for the stream function, and Fig. 16 corresponding to the wall shear stress for \( Re = 15, 30 \) and 70).

Finally, the cases corresponding to \( \eta \geq 0.6 \) are more stable. Even if the maximum value of \( |q| \) achieved is relatively small, which limits our conclusions, a careful analysis of the different plots show that the opposition to the rotation is stronger in these cases. This could be seen in particular in the case of \( Re = 70 \) where the shift of the vortices occurs for small values of \( |q| \) (see Figs. 17 and 18).
6.2.2. Higher curvature ratio

Our aim in this section is to consider the behaviour of the stream function and of the wall shear stress for an intermediate curvature ratio. The idea is basically to compare with the case studied in the last section, and to analyse the effect of the curvature on the observed phenomenon. As previously, we made several tests involving different Reynolds numbers, different viscosity parameters and implementing a continuation method for the viscosity exponent $q$.

In contrast to the previous case, one of the direct conclusions when analysing the results corresponding to the intermediate curvature ratio is the fact that the behaviour of the flow is much more stable (see Tables 4–6 and Figs. 20–22).

Indeed, we did not observe any rotation and the contours of the stream function remain symmetrical. The only remarkable fact is related to variations in the shape of the vortices and their shift from the centre. This behaviour works for small values of the viscosity exponent $q$ and is more pronounced when the Reynolds number increases.

The maximum values for the stream function increase with $|q|$, with the viscosity parameter $\eta$ and with the Reynolds number $Re$ (cf. Fig. 23). The same observation can be done for the axial velocity which, contrarily to the case of small curvature ratio, is sensitive to the variations of $Re$. This stability can also be observed in the behaviour of the wall shear stress. The corresponding curves are of Newtonian type (with symmetries relatively to the axis) and amplitude (for a fixed $\eta$ and a fixed $Re$) increases with $|q|$. However, in contrast to the Newtonian type, for a fixed $q$ and a fixed $\eta$, the amplitude of the wall shear stress decreases when the Reynolds number increases (see, for example, Figs. 20–22).

Finally, let us observe that there exists an exception to this global behaviour. For $Re = 1$ with the viscosity parameter $\eta = 0.3, 0.4,$ and $0.5$ some differences can be observed. For $\eta = 0.3$ and $0.4$, the behaviour of the contours...
Fig. 19. Contours of the maximum values of the stream function ($\delta = 0.001$).

is analogous to the general case, but the maximum values of the stream function increase with $|q|$ and then decrease. For $\eta = 0.5$, the flow behaves as in the case of small curvature ratio with counter-clockwise rotation and stabilization followed by clockwise rotation. Similarly, the maximum values of the stream function increase with $|q|$ and then decrease.

7. Conclusions

The finite element numerical simulations presented in this work provide relevant information on the shear-thinning viscosity effects of steady flows of generalized Newtonian fluids through curved pipes of uniform cross-section. As stated above, the main feature of inertial curved pipe flows is the existence of secondary motions which are clearly related to small changes on the viscosity parameters and influence the distribution of the axial velocity and wall shear stress (WSS) for small and intermediate curvature ratio.
The complexity of the flow characteristics shown in the numerical tests presented here suggest that a further theoretical analysis is needed to study the existence of more than one solution and investigate the corresponding stability, for a range of appropriate non-dimensional parameters.

More detailed discussion and numerical results can be found in [8] where the generalized Newtonian flows are obtained as a particular case of generalized Oldroyd-B flows, in the limit of vanishing Weissenberg number (neglected viscoelastic effects).
Fig. 22. Streamlines and wall shear stress for $Re = 70$ ($\eta = 0.9, \delta = 0.1$).

Fig. 23. Contours of the maximum values of the stream function ($\delta = 0.1$).
Acknowledgements

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