A comment on “$L_\infty$ optimal control of SISO continuous-time systems”

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Abstract

The paper “$L_\infty$ optimal control of SISO continuous-time systems” by Wang and Sznaier (Wang & Sznaier (1997). Automatica, 33 (1), 85–90) studies the problem of designing a controller that optimally minimizes the peak absolute value of system output, due to a fixed input signal. With a newly defined function space $A$, it was claimed that the set of all $L_\infty$-bounded outputs could be parameterized and that the problem could be transformed to a minimal distance problem on $L_\infty$ space. We believe, however, their formulation has essential flaws.

Keywords: $L_\infty$ control; Controller parameterization; Optimization; Duality

1. A summary of the $L_\infty$ problem formulation

We provide a short summary of the $L_\infty$-problem formulation developed in Wang and Sznaier (1997). The same framework was also used in Wang and Sznaier (1994, 1996)

$L_\infty$ denotes the space of essentially bounded measurable function $f(\cdot)$ on $R_+$, the set of nonnegative real numbers. The space $A_\infty$ is defined as the set of Laplace transforms of all elements in $L_\infty$.

**Definition 1.** A system $H(s)$ is said to be $L_\infty$ stable if $H(s) \in A_\infty$.

Under the system configuration in Fig. 1, the design objective is to find an $L_\infty$ internally stabilizing controller such that $||\phi||_\infty$ is minimized, i.e.,

$$\mu = \inf_{L_\infty\text{-stabilizing } K} ||\phi||_\infty.$$  

The authors Wang and Sznaier (1997) claimed that, by a slight modification of the YJBK controllers parameterization (Youla, Jabr & Bongiorno, 1976), the set of all closed-loop transfer functions achievable with $L_\infty$ stabilizing controllers could be parameterized as

$$\Phi(s) = H(s) - U(s)Q(s) \quad \text{over } Q \in A_\infty$$  

with stable rational functions $H$ and $U$. Here $U$ was assumed to have distinct finite $n$ zeros $\{z_i, i = 1, \ldots, n\}$ in the open right half-plane (RHP) and no zeros on the imaginary axis.

It was also claimed that the next lemma could be proved using arguments similar to those in Dahleh and Pearson (1987).

**Lemma 1.** Let $M(s) = U(s)Q(s)$. Then $Q \in A_\infty$ if and only if $M \in \hat{T}$ where

$$\hat{T} = \{M \in A_\infty; M(z_i) = 0, i = 1, 2, \ldots, n\}.$$  

Based on these two asserted results; parameterization of $L_\infty$-bounded outputs and Lemma 1, the original problem (1) was transformed to the next minimal distance problem on $L_\infty$ space,

$$\mu = \min_{m \in \hat{S}^\times} ||h - m||_\infty.$$  

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where \( S \) is a finite-dimensional subspace of \( L_1 \) determined by \( U \) and \( S^\perp \), the annihilator subspace of \( S \), is a subspace of \( L_\infty = L_1^* \), the dual space of \( L_1 \).

2. Comments on \( L_\infty \) formulation

2.1. On Lemma 1

Let \( H_\infty \) denote the usual Hardy space and \( RH_\infty \subset H_\infty \) denote the set of stable rational functions. \( G \in RH_\infty \) is called \textit{biproper} if the relative degree of \( G \) is zero. And let \( \mathcal{L} \) denote the Laplace transform and \( \mathcal{L}^{-1} \) denote the inverse transform of \( \mathcal{L} \).

We first note that the constant function \( G(s) \equiv 1 \in RH_\infty \) does not belong to \( A_\infty \) since \( \mathcal{L}^{-1}(1) = \delta(\cdot) \notin L_\infty \). Similarly every biproper \( RH_\infty \) function does not belong to \( A_\infty \) since its inverse Laplace transform has \( \delta \) function.

From Lemma 1 and Example 1 of Wang and Sznaier (1997), it is obvious that the space \( A_\infty \) was endowed with the usual multiplication operation of transfer functions. However, since \( 1 \notin A_\infty \), it is clear that \( A_\infty \) does not have an \textit{identity} element of this operation. This deficiency provides a simple counter-example to Lemma 1.

Counter-example 1. By definition of \( \hat{T} \), it holds that \( U \in \hat{T} \) of itself whenever \( U \in A_\infty \). Thus we can choose \( M = U \) and obtain \( Q \equiv 1 \notin A_\infty \).

Note that if \( U \) is biproper then, since \( M = U \notin A_\infty \), the above reasoning fails. However, it is a fundamental fact that \( U \in A_\infty \) because the parameterization asserted in Wang and Sznaier (1997) was done on \( A_\infty \).

In fact, even with biproper \( U \), it is doubtful that Lemma 1, assuming that it is true, can be proved in a similar way to those of Dahleh and Pearson (1987). This is because the latter proof is a simple corollary of Theorem 2.2 of Callier and Desoer (1978), which comes from the properties of the convolution Banach algebra \( \mathcal{A} \) (Desoer & Vidyasagar, 1975, Appendix D) consisting of the element

\[
\delta(\cdot) \rightarrow P \rightarrow \phi(\cdot)
\]

Specifically, the fact \( \mathcal{L}(\mathcal{A}) \subset H_\infty \) was implicitly used in the proof of Theorem 2.2 of Callier and Desoer (1978) but we have \( A_\infty \notin H_\infty \) (see Fact 1 below).

Remark 1. In the \( L_1 \)-optimal control problem, the underlying set is not the \( L_1 \) space but the normed algebra \( \mathcal{A} \), and the cost function we should minimize is not \( \| f \|_{L_1} \) but the norm of \( \mathcal{A} \) defined as \( \| f \|_{\mathcal{A}} := \| f \|_{L_1} + \| \{ f_i \} \|_{L_1} \).

The failure of Lemma 1 implies that the minimal distance problem (4) is not equivalent to the \( L_\infty \) minimization problem over \( Q \in A_\infty \) in (2). To be concrete, it may be possible that \( Q_{\text{opt}} : = (H - \Phi_{\text{opt}})/U \notin A_\infty \) where \( \Phi_{\text{opt}} \) is the optimal solution of (4). Actually this happened in the numerical example of Wang and Sznaier (1997, Example 1). In that example, the parameterization was given by

\[
\Phi(s) = H - U Q = \frac{6(s-1)}{(s+1)(s+2)} - \frac{(s-1)(s-2)}{(s+1)(s+2)} Q(s)
\]

(6)

over \( Q \in A_\infty \) and the optimal solution \( \Phi_{\text{opt}} \) of the corresponding minimum distance problem turned out to be

\[
\Phi_{\text{opt}}(s) = \frac{2}{s} (1 - 22^{-s+1}).
\]

(7)

Thus \( Q_{\text{opt}} = (H - \Phi_{\text{opt}})/U \) becomes

\[
Q_{\text{opt}}(s) = \left( \frac{(s+1)(s+2)}{(s-1)(s-2)} \right)^2 \times \left[ \frac{6(s-1)}{(s+1)(s+2)} - \frac{2}{s} (1 - 2 \times 2^{-s+1}) \right].
\]

(8)

After simple algebraic computations we obtain

\[
Q_{\text{opt}}(s) = 4(1 + 2^{-s}) - \frac{2}{s} (1 - 22^{-s+1})
\]

(9)

and its time-domain counterpart

\[
q_{\text{opt}}(t) := \mathcal{L}^{-1}(Q_{\text{opt}}) = 4\delta(t) + 4\delta(t - \ln 2) - 4u(t) + 8u(t - \ln 2)
\]

(10)

where \( u(\cdot) \) denotes the unit step function. It is obvious that

\[
q_{\text{opt}}(t) \notin L_\infty \quad \text{and} \quad Q_{\text{opt}}(s) \notin A_\infty.
\]

In conclusion, \( \Phi_{\text{opt}} \) does not give the optimal solution of the minimization problem (6) over \( Q \in A_\infty \).

2.2. On \( L_\infty \)-stability and parameterization

Note that the \textit{doing-nothing} system \( G(\cdot) = 1 \) is unstable in \( L_\infty \)-stability sense. Also, from the relation \( G = G \cdot 1 \), we can say that every stable system is a product of stable and unstable systems. Moreover, as shown below, a product of two stable systems can be unstable. Hence a serial connection of stable systems is not
provided that the terminology stability has usual physical interpretations. Hence we believe that the $L_{\infty}$ stability as used in the paper (Wang & Sznaier, 1997) is not an appropriate concept in system theoretical viewpoint.

In a mathematical sense, as a matter of fact, the situation is more skeptic. Since $1/s \in A_{\infty}$ but $1/s^2 \notin A_{\infty}$, it is clear that $A_{\infty}$ is not algebraically closed under the usual multiplication of transfer functions, i.e., we do not have a valid multiplication operation on $A_{\infty}$. Thus $A_{\infty}$ is merely a commutative group but not a ring or algebra under the addition and multiplication operations of transfer functions (Vidyasagar, 1987, Appendix A). From these results, it is hardly expected that the set of all $L_{\infty}$-bounded closed-loop outputs can be parameterized by a certain ring theoretic approach.

Also the space $A_{\infty}$ has no inclusion relations with $H_{\infty}$ or $RH_{\infty}$.

**Fact 1.** $A_{\infty} \not\subset RH_{\infty} \subset H_{\infty}$ and $RH_{\infty} \subset H_{\infty} \not\subset A_{\infty}$.

**Proof.** Obvious from $1/s \notin A_{\infty}$ and $1/s \notin RH_{\infty} \subset H_{\infty}$. □

Recall that the optimal solution $Q_{\text{opt}}$ of Example 1 of Wang and Sznaier (1997) does not belong to $A_{\infty}$. In fact, it is also true that $Q_{\text{opt}} \notin A_{\infty} \cup H_{\infty}$ since $|Q_{\text{opt}}(s)|$ in (9) is not bounded on $R_{+}$. This proves that the solution technique of Wang and Sznaier (1997) does not give the optimal solution even among the class of $H_{\infty}$ stabilizing controllers.

**Remark 2.** Each parameterization of $L_{\infty}$, $L_1$ and $RH_{\infty}$ optimal control problems was performed on three rings; $L_{\infty}$, $\mathcal{F}$ and $RH_{\infty}$, respectively (Dahleh & Pearson, 1988, 1987; Francis, 1987).

**Remark 3.** With the parameterization of $RH_{\infty}$ setting, Miller (1992) studied a similar $L_{\infty}$-problem with an additional constraint $\lim_{t \to \infty} \phi(t) = 0$ and obtained the same results as Theorem 2 of Wang and Sznaier (1997). In this work, however, suboptimal rational controllers were sought from a rational approximation theory of $L_{\infty}$ function.

### 3. Conclusion

We believe that the problem formulation of Wang and Sznaier (1997) has essential defects described above. In summary, the actual domain of optimization problem solved in Wang and Sznaier (1997) is not the set

$$\mathcal{W} := \{ \Phi; \Phi \text{ achievable with a } L_{\infty} \text{ stabilizing controller} \}$$

as claimed but the next set

$$\mathcal{W} := \{ \Phi; \Phi = H - M, M \in \mathbb{T} \}.$$  

(13)

The optimal solution sought among $\mathcal{W}$ does not provide a correct solution to the original $L_{\infty}$-optimal control problem because of defects in both the Lemma 1 and the parameterization of all $L_{\infty}$-bounded outputs.

### References