Control of Inter-agent Distances in Cyclic Polygon Formations

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Abstract—This paper considers $n$-agent cyclic polygon formations of mobile autonomous agents in the plane. The aim of each agent is to maintain a desired distance from its neighbor. Since the formation graph is directed, only one agent is responsible for maintaining each distance. The control law for each agent is designed in distributive manner. Hence, it utilizes only local information. We use gradient-like control and Lyapunov-based stability analysis. Since there exist invariant subsets which include undesirable equilibrium state, the inter-agent distances are not globally stable under the given control law. However, once the initial locations of the agents are not inside the subsets, the agents do not approach them.

I. INTRODUCTION

Recently, there have been many publications on the formation control of multi-agent systems [1]-[15]. Based on double integrator model, an approach using artificial potential function to making a control law is introduced by Olfati-Saber and Murray [1]. The concept is extended to more remarkable general cases by Krick et al. [7]. By using center manifold theory, they show that local asymptotic stability of infinitesimally rigid formations formed by gradient control is achieved, and the result is applicable to directed formations constructed using the Henneberg insertion procedure. Dörfler and Francis use another method to handle the formation problem. They use differential geometric approach to show instability of invariant submanifolds, and apply the result to the triangular cyclic polygon formation [8]. Guo et al. investigate the leader-follower formation control problem where the group of robots move with a constant velocity. They use adaptive control schemes to control the robots. Oh and Ahn introduce localization-based formation control strategy [13]. They also propose sequential control law for cycle-free persistent formations† [14]. Another novel control strategy which is more effective than previous strategies is proposed in [15].

Especially in 3-agent case, very intuitive control law is proposed by Anderson et al. [3]. Cao et al. also consider the three-agent formation based on gradient control [5]. Later, Cao et al. propose more generalized controller encompassing previous results [10]. Although Smith et al. mention $n$-agent polygon formations, the analysis is restricted to 3-agent case [2].

In this paper, we deal with $n$-agent cyclic polygon formations. The objective is to make each agent restore a desired inter-agent distance. We take gradient control law used in [7] to control the inter-agent distances. Analysis is carried out via Lyapunov-based stability analysis. We categorize the situations where the derivative of Lyapunov function could be zero into four different cases, and analyze these cases. An interesting problem takes place when all agents are located on a single straight line. As a tool representing the situation, Cao et al. use a determinant of a matrix consisting of two relative position vectors for three agents in the plane, viz., three agents are collinear if and only if the determinant is zero. We take this idea and extend the determinant to a matrix form to handle $n$ agents. Note that the goal of this paper is to maintain inter-agent distances, not specific formation because cyclic polygon formation consisting more than four agents is not persistent. Maintaining particular formation needs concept of rigidity (in undirected formations) or persistence (in directed formations). We hope to extend our result to more general cases by combining those concepts together later.

The rest of the paper is organized as follows. In Section II, the problem treated is presented. Specifically cyclic $n$-agent polygon formations and corresponding realizability conditions are addressed. Stability analysis is presented in Section III and simulations supporting our results are presented in Section IV. Section V contains conclusions and future works.

II. PROBLEM DESCRIPTION

We consider formations formed by mobile autonomous $n$ agents in the plane. Each agent is required to keep distance from its neighbor. In our problem, we assume that each agent has only one neighbor. If we describe agents as vertices in the plane and distances as edges among them, it can be depicted as Fig. 1.

In the sequel, we use $p_i$ for the Cartesian coordinate vector of agent $i$ based on some global coordinate system in the plane. We assume that the velocity of each agent is equal to the control input of itself; so the dynamics is as follows

$$\dot{p}_i(t) = u_i(t)$$

(1)
where \( u_i \) is the control input of agent \( i \). Note that in (1), \( i \) is from 1 to \( n \), i.e., \( (i \in \{1, 2, \cdots, n\}) \). To avoid a confusion, with the subscript \( i + 1 \), when \( i = n \), \( i + 1 \) means 1 in the following equations. Without loss of generality, define agent \( i + 1 \) as a neighbor of agent \( i \). Let relative position vectors be represented as follows
\[
z_i(t) = p_{i+1}(t) - p_i(t).
\]
Using expression of \( z_i \), we can write the goal of this \( n \)-agent systems as
\[
\|z_i\| \to d_i, \quad d_i > 0 \quad \forall i \in \{1, 2, \cdots, n\}
\]
where \( d_i \) is a desired distance that agent \( i \) must maintain from its neighbor, and \( \|\cdot\| \) denotes Euclidean norm.

We assume that the desired distances are consistent. In other words, they satisfy polygonal inequality [17]. Hence,
\[
2 \max\{d_1, d_2, \cdots, d_n\} < \sum_{i=1}^{n} d_i. \tag{4}
\]

We assume that each agent can measure the relative position of its neighbor so we want to control the distances using only local information. Consider a control strategy
\[
u_i = e_i z_i \tag{5}
\]
where
\[
e_i(t) = \|z_i(t)\|^2 - d_i^2. \tag{6}
\]
For convenience in later, let us define concatenated vectors as
\[
p = [p_1^T \quad p_2^T \quad \cdots \quad p_n^T]^T, \tag{7}
\]
\[
z = [z_1^T \quad z_2^T \quad \cdots \quad z_n^T]^T, \tag{8}
\]
\[
u = [u_1^T \quad u_2^T \quad \cdots \quad u_n^T]^T, \tag{9}
\]
\[
e = [e_1^T \quad e_2^T \quad \cdots \quad e_n^T]^T. \tag{10}
\]

Apparantly, the control in (5) consists of only the relative measurements which are not dependent on global coordinate system. From (1) and (5), motion of agent \( i \) can be represented as
\[
p_i = e_i z_i. \tag{11}
\]
Notice that equilibrium points of (11) are \( p \) where \( e_i z_i = 0 \) for all \( i \). Combining (2) and (11), dynamics of the relative position vectors can be written by \( \dot{z}_i = -e_i z_i + e_{i+1} z_{i+1} \), and the overall system is described as
\[
\dot{p}_i = (\|p_{i+1} - p_i\|^2 - d_i^2)(p_{i+1} - p_i), \quad i \in \{1, 2, \cdots, n\}. \tag{12}
\]

Now, our problem is summarized as follows.

**Problem:** For given systems, requirements, conditions and control strategy in (1), (3), (4) and (5), analyze if \( e \) which is defined in (6) and (10) converges to 0.

To characterize a set of \( p \) where every agent lies on a single straight line, we use the following simple statements

**Lemma 1:** Points \( p_1, p_2, \cdots, p_n \in \mathbb{R}^2 \) lie on a single straight line if and only if \( \text{rank} \ Z < 2 \) where \( Z = [z_1 \quad z_2 \quad \cdots \quad z_n] \).

**Proof:** The simple proof is omitted. \( \blacksquare \)

Then, we define a set \( C = \{p : \text{rank} [z_1 \quad z_2 \quad \cdots \quad z_n] < 2\} \). The condition for \( C \) means that every relative position vector is parallel to each other.

### III. Analysis

To analyze convergence characteristic of \( e \), define a positive definite function as
\[
V = \frac{1}{2} \sum_{i=1}^{n} e_i^2. \tag{13}
\]
Then, time derivative of \( V \) is
\[
\dot{V} = \sum_{i=1}^{n} e_i e_{i+1} + \sum_{i=1}^{n} 2e_i z^T_i (-e_i z_i + e_{i+1} z_{i+1})
\]
\[
= -\sum_{i=1}^{n} \|e_i z_i - e_{i+1} z_{i+1}\|^2 \leq 0. \tag{14}
\]

Since \( \dot{V} \leq 0 \), \( V \) does not increase on \([0, \infty]\) along the trajectory of the solution of (12). Note that \( V \) is radially unbounded function of \( e \). Therefore, every \( e_i \) is bounded, which implies that every \( z_i \) is also bounded.

From now on, we are going to consider the cases such that \( \|e_1 z_1 + e_2 z_2\|^2 = 0 \). It takes place when

1) \( e = 0 \), or
2) \( z = 0 \), or
3) Some \( \|z_i\| \) are 0 and the others are equal to the desired distances, or
4) \( p \in C \) and \( e_i z_i - e_{i+1} z_{i+1} = 0, \forall i \in \{1, 2, \cdots, n\} \).

Case 1), 2) and 3) cover all equilibrium points of (11), and case 4) forms an invariant subset. Case 1) is desirable one, but the others are undesirable. If we can exclude those undesirable cases, then the desired case, i.e., case 1), can be achieved because of (13) and (14). We are going to analyze these cases in detail.

**A. Case 2)**

Since \( p \in C \) when \( z = 0 \), case 2) is a special one of case 4). Thus, discussion of this case is referred to Section III-C.

**B. Case 3)**

Define a set of points in this case as
\[
\mathcal{O} = \{p : \exists z_i = 0, \quad e_j = 0, \quad i \in \mathcal{I}_z, \quad j \in \mathcal{I}_e \}
\]
where \( \mathcal{I}_z \) and \( \mathcal{I}_e \) are subsets of integers such that \( \mathcal{I}_z \cap \mathcal{I}_e = \emptyset, \mathcal{I}_z \cup \mathcal{I}_e = \{1, 2, \cdots, n\}, \mathcal{I}_z \neq \emptyset, \quad \text{and } \mathcal{I}_e \neq \emptyset \).

An example of this case is depicted in Fig. 2. Agent 1 and 2 are co-located and the others are already achieving desired distances. If the agents are initially located on the plane forming this formation, they would not move anymore. The situation changes when the agent 1 is perturbed slightly from agent 2, however.
Assume the situation mentioned just before. Hence, $z_1 = \epsilon_1, d_1 \gg \|\epsilon_1\| > 0, \|z_2\| = d_2, \|z_3\| = d_3, \|z_4\| = d_4$, and $\|z_5\| = d_5$. From (11), $\dot{p}_2 = \dot{p}_3 = \dot{p}_4 = \dot{p}_5 = 0$, and $\dot{p}_1 = (\|\epsilon_1\|^2 - d_1^2) \epsilon_1$. It is evident that direction of the vector $\dot{p}_1$ is opposite to the direction of the vector $\epsilon_1 (= z_1)$ because $(\|\epsilon_1\|^2 - d_1^2)$ is negative by assumption. So agent 1 goes away from agent 2 when it is located around agent 2, not exactly same position. Without loss of generality, we can assume more general situation where some agents are co-located with their neighbor, and the others are achieving desired distances.

Consequently, we state the following lemma.

**Lemma 2:** If $p(0) \notin \mathcal{O}$, then the agents do not approach $\mathcal{O}$.

**C. Case 4**

In this subsection, we are going to show that agents do not form a single straight line if they are not on a single straight line initially. To handle it, we consider determinant that consists of two relative position vectors.

$$\det [z_i \ z_j], \ i \neq j.$$  \hspace{1cm} (15)

It is obvious that (15) is equal to zero if and only if $z_i$ and $z_j$ are parallel to each other. Note $\det [z_i \ z_j] = -\det [z_j \ z_i]$. Let us define a $\binom{n}{2}$ by 1 matrix $q$ as

$$q(z) = [D_{12} \ \cdots \ \ D_{1n} \ D_{23} \ \cdots \ \ D_{2n} \ \cdots \ \ D_{(n-1)n}]^T$$

where $D_{ij} = \det [z_i \ z_j]$. The time derivative of $D_{ij}$ can be found as

$$\frac{d}{dt} D_{ij} = -(e_i + e_j) D_{ij} + e_{i+1} D_{(i+1)j} + e_{j+1} D_{ij+1}.$$ \hspace{1cm} (16)

Since the right hand side of (16) is combination of elements of $q, \dot{q}$ can be represented as

$$\dot{q}(z) = M(e)q(z)$$

where $M(e)$ is an $\binom{n}{2}$ by $\binom{n}{2}$ matrix consisting of $e$. Bear in mind that $e$ is dependent on $z$ and $z$ is dependent on time $t$. In the view point of $t, \dot{q}$ can be written again as

$$\dot{q}(t) = M(t)q(t)$$

along the solution trajectory of (12). Note that $\|M\|$ is bounded because of boundedness of $e$ where $\|M\|$ denotes spectral norm of matrix $M$.

**Lemma 3:** It is true that $p(t) \in \mathcal{C}$ at $t = t_0$ if and only if $q(t_0) = 0$.

**Proof:** If $p(t_0) \in \mathcal{C}$, rank $Z < 2$ by definition. Also, if every matrix consisting of the element $q$ has rank less than 2, then the determinant becomes 0. Otherwise, rank $Z = 2$ so at least one determinant in $q$ is not equal to 0.

Now we are going to show that if $p(0) \notin \mathcal{C}$, then $p(t) \notin \mathcal{C}$ over the time interval of interest.

**Lemma 4:** Matrix $q(t) = 0$ if there exists at least one $t = t_a$ such that $q(t_a) = 0$, otherwise $q(t) \neq 0$ for all $t$ in the time interval of interest.

**Proof:** Assume $q(t) = 0$ is zero for some $t = t_a \geq 0$ and differentiable on interval of interest including $t_a$. Since $z$ and $e$ are bounded, there exists a constant $N$ such that

$$\|M(t)\| \leq N, \ N > 0, \ \forall t \in [t_a - 1, t_a + 1].$$

Take $\delta_a$ to meet

$$\delta_a < \min \left(1, \frac{1}{N^2}\right).$$

Then, for any $t \in (t_a - \delta_a, t_a + \delta_a)$,

$$\|q(t)\| = \|q(t) - q(t_a)\| = \left\| \int_{t_a}^{t} \dot{q}(s_1)ds_1 \right\|$$

$$= \left\| \int_{t_a}^{t} M(s_1)q(s_1)ds_1 \right\| \leq N \int_{t_a}^{t} \|q(s_1)\|ds_1$$

$$\leq N \int_{t_a}^{t} \int_{t_a}^{s_1} \|q(s_2)\|ds_2 \|ds_1\|$$

$$\leq N^2 \int_{t_a}^{t} \int_{t_a}^{s_1} \|q(s_2)\|ds_2 |ds_1|$$

$$\leq N^k \int_{t_a}^{t} |s - t_a| ds \cdots |s_{k-1} - t_a| \leq (N\delta_a)^k L$$

where

$$L = \sup \{\|q(t)\| : t \in (t_a - \delta_a, t_a + \delta_a)\}.$$

Since $N\delta_a < 1$, $(N\delta_a)^k L \to 0$ as $k \to \infty$. Therefore, if $q$ is 0 at $t = t_a$, it is also 0 for all $t$ on the interval $(t_a - \delta_a, t_a + \delta_a)$. By applying the same procedure of the above arguments to the arbitrary $t \in (t_a - \delta_a, t_a + \delta_a)$, $(t_a - \delta_a, t_a + \delta_a)$ is extended to larger interval of our interest.

**Theorem 1:** If $p(0) \notin \mathcal{C}$, then $p(t) \notin \mathcal{C}$ over the time interval of interest, which means that the agents do not form formations of case 4.

**Proof:** If $p(0) \notin \mathcal{C}$, $q(0) \neq 0$ by Lemma 3. Assume that $q(t) = 0$ at some $t = t_b \neq 0$. Then, $q(t)$ must be 0 for all $t$ over the interval of interest by Lemma 4, but it contradicts to the assumption that $q(0) \neq 0$ so there does not exist a moment that $q(t)$ can be 0, which means that $p(t) \notin \mathcal{C}$.

**Corollary 1:** $\mathcal{C}$ is an invariant subset.

**Proof:** If $p(0) \in \mathcal{C}$, $q(0) = 0$ by Lemma 3. Then, $q(t) = 0$ on $[0, \infty)$ by Lemma 4, which means that $p(t) \in \mathcal{C}$ again by Lemma 3. Therefore, $\mathcal{C}$ is invariant.
From the preceding discussions, we can say that agents initially located in the outside of $\mathcal{O} \cup \mathcal{C}$ do not move toward case 3) and 4), automatically also case 2).

**Theorem 2:** If the initial positions of agents meet $p(0) \notin \mathcal{O} \cup \mathcal{C}$, $e$ converges to 0 along the solution trajectory of the overall system.

**Proof:** From Lemma 2 and Theorem 1, $p(t)$ does not form formations of case 3) and 4) if $p(0) \notin \mathcal{O} \cup \mathcal{C}$. Thus, $V$ is always negative except for the case 1), $e = 0$, which leads us to the result that $V \to 0$ so that $e \to 0$.

### IV. Simulations

We carried out some simulations supporting our result. In this section, we use notations $I_k$ and $F_k$ to represent the initial and final locations of agent $k$, respectively in figures.

Fig. 3 represents an example of case 1). Initial positions of the agents meet $p(0) \notin \mathcal{O} \cup \mathcal{C}$. Therefore, relative distances converge to the desired values$^2$ that the agents are required to maintain.

Fig. 4 shows an example of case 3) with perturbations. Two pairs of agents consisting of agent 1,2 and 4,5 are located almost at the same position, respectively. Due to the perturbations, the agents do not approach the case 3).

### V. Conclusion and Future Work

#### A. Conclusion

This paper has dealt with control problem of inter-agent distances in cyclic polygon formations formed by $n$ agents. Although the formations are previously studied by Smith et al., their analysis is valid only for three agents. We have extended the analysis to more general case. We have shown that there exist equilibrium state and invariant subset which make the agents do not achieve the desired distances. However, if the initial formation is not in those invariant subsets, the agents do not approach them, and the inter-agent distances converge to the desired values. Note that there exist some possibilities that agents achieve desired distances even when they are in the invariant subsets depending on the specific desired distance values.

#### B. Future Work

The proposed control laws in (5) are proposed without considering performance. In the case of undirected for-
mations, there are some results proposing another control strategy which is more effective than previous strategies [15]. Finding more effective control strategies would be worth considering in directed formation control problems as well as undirected formation control problems. Also, extending cyclic polygon formations to arbitrary persistent formations including cycle structures would be important work.

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