Results for Integrals Involving $m$-th Power of the Gaussian $Q$-function Over Rayleigh Fading Channels with Applications

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Abstract—Exact-form results are presented for integrals involving higher-order power of the one dimensional Gaussian $Q$-function over identical and nonidentically distributed multi-channel Rayleigh fading with maximal ratio combining (MRC). The formulas known in the literature for the average of the 1st and 2nd powers are shown as special cases. The derived results herein are then used to obtain new expressions for the average error performance of differentially encoded quadrangle-phase shift-keying signals over identical as well as nonidentically distributed multi-channel Rayleigh fading.

I. INTRODUCTION

The average of the $m$th power of the one-dimensional Gaussian $Q$-function over statistics of fading amplitudes is of interest in evaluating the error performance of various digital modulation/detection schemes in wireless communication channels. The averaging integral under consideration is given by

$$I = \int_0^\infty Q^m(a\sqrt{\gamma}) p_\gamma(\gamma) \, d\gamma, \quad a > 0,$$

where $m \geq 1$ is a positive integer, $\gamma$ is the instantaneous received SNR having $p_\gamma(\gamma)$ probability density function (pdf), and $Q(x)$ is the Gaussian $Q$-function, which is defined as

$$Q(x) = 1 - Q(-x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-y^2/2} \, dy. \quad (2)$$

The integral in (1) for the case when $m = 1$ is needed, for example, in evaluating the error performance of coherent antipodal signals and $M$-ary pulse amplitude modulation ($M$-ary PAM) system. The results of (1) when $m = 1$ and $m = 2$ are involved in the error performance analysis of square $M$-ary quadrature amplitude modulation ($M$-ary QAM) and coherent quadrangle-phase shift-keying (QPSK) system. Also, the averaging results for $m = 1, m = 2, m = 3, m = 4$ are all important in evaluating the error performance of differentially encoded quadrangle-phase shift-keying (DE-QPSK) and coherently detected $\pi/4$-QPSK signals [1, pp. 258–261].

For the case when $m = 1$ and $m = 2$, the integral in (1) have been obtained in closed-form when the instantaneous SNR $\gamma$ has exponential distribution (when the fading envelope is Rayleigh distributed) in many places (see, for example, [2, Ch. 14], [1, Ch. 5] and references therein). Using the integral-based alternative presentations of the third and fourth powers of the $Q$-function [3], the averaging results when $m = 3$ and $m = 4$ over fading channels are known in integral forms, [1, pp. 144–145], and to the best of the authors’ knowledge, no closed-form solutions for these integral-based formulas are available. In addition, neither alternative representations nor averaging results for the case when $m > 4$ are known.

This paper derives exact formulas for the average of the $m$th power of $Q$-function in (1) over independent identically distributed (i.i.d.) and nonidentically distributed (i.n.d.) Rayleigh fading channel employing maximal-ratio combining (MRC). As special cases, closed-form expressions for the integral-based formulas in [1, (5.91) and (5.92)], which correspond to $m = 3$ and $m = 4$ in (1), are presented. In addition, the previously obtained closed-form results when $m = 1$ and $m = 2$ are also shown as special cases from the derived expressions herein. The derived formulas are then applied to obtain new expressions for the average error performance of DE-QPSK (and $\pi/4$-QPSK) over i.i.d. as well as i.n.d. multi-channel Rayleigh fading.

II. I.I.D. CHANNELS

For the case of i.i.d. Rayleigh fading channel with MRC, the function $p_\gamma(\gamma)$ in (1) denotes the pdf of Gamma random variable with parameters $L$ and $1/\gamma$ (i.e., $\gamma \sim G(L, 1/\gamma)$), and is given by

$$p_\gamma(\gamma) = \frac{1}{\Gamma(L)} \left( \frac{1}{\gamma} \right)^L \gamma^{L-1} e^{-\gamma/\gamma}, \quad (3)$$

where $L$ is the number of diversity branches and $\gamma$ is the average SNR per diversity branch. The integral $I$ in (1) can be expressed, using integration by parts technique, as

$$I = -m \int_0^\infty Q^m(a\sqrt{\gamma}) Q'(a\sqrt{\gamma}) P_\gamma(\gamma) \, d\gamma, \quad (4)$$

where $Q'(a\sqrt{\gamma})$ denotes the first-order derivative of $Q(a\sqrt{\gamma})$ with respect to $\gamma$, and $P_\gamma(\gamma)$ is the cumulative distribution function (CDF) of the instantaneous combined SNR $\gamma$, and is given by [2, Ch. 2]

$$P_\gamma(\gamma) = 1 - e^{-\gamma/\gamma} \sum_{k=0}^{L-1} \frac{1}{\Gamma(k+1)} \left( \frac{1}{\gamma} \right)^k \gamma^k. \quad (5)$$
In obtaining (4), the results \( Q(0) = 1/2, \) \( Q(\infty) = 0, \) \( P_\gamma(0) = 0, \) and \( P_\gamma(\infty) = 1 \) have been used. Substituting (5) into (4) gives

\[
I_{\text{i.i.d.}} = \left( \frac{1}{2} \right)^m + m \sum_{k=0}^{L-1} \frac{1}{\Gamma(k+1)} \left( \frac{1}{\gamma} \right)^k \times \int_0^\infty Q^{m-1} (a\sqrt{\gamma}) Q' (a\sqrt{\gamma}) e^{-\gamma/\gamma} \gamma^k d\gamma. \quad (6)
\]

The function \( Q(x) \) is related to \( F_1 \) through \([7, (7.1.21)]\)

\[
Q(x) = \frac{1}{2} \left[ 1 - \frac{2x}{\sqrt{2\pi}} e^{-x^2/2} F_1 \left( 1, 3; \frac{x^2}{2} \right) \right]. \quad (7)
\]

where \( F_1 \) is the confluent hypergeometric function. The \((m-1)\)-st power of \( Q(x) \) in (6) can now be re-expressed using (7) along with binomial expansion. Then (6) becomes

\[
I_{\text{i.i.d.}} = \left( \frac{1}{2} \right)^m - m \left( \frac{1}{2} \right)^m \frac{a}{\sqrt{2\pi}} \sum_{k=0}^{L-1} \frac{1}{\Gamma(k+1)} \left( \frac{1}{\gamma} \right)^k I_1
\]

\[
- m \left( \frac{1}{2} \right)^m \frac{a}{\sqrt{2\pi}} \sum_{k=0}^{L-1} \frac{1}{\Gamma(k+1)} \left( \frac{1}{\gamma} \right)^k \times \sum_{n=1}^{m-1} \left( \frac{m-1}{n} \right) (-1)^n \left( \frac{2a}{\sqrt{2\pi}} \right)^n I_2, \quad (8)
\]

where \([8, (3.381.5)]\)

\[
I_1 = \int_0^\infty \gamma^{-k-\frac{1}{2}} e^ {-(a^2/2+1/\gamma)} \gamma^k d\gamma
\]

\[
= \frac{\Gamma(k+\frac{1}{2})}{(a^2/2+1/\gamma)^{(k+1)/2}}, \quad (9)
\]

and \([6, (2.4.2)]\)

\[
I_2 = \int_0^\infty \gamma^{k+n/2-1/2} \left[ F_1 \left( 1, 3; \frac{a^2}{2} \right) \right]^n d\gamma
\]

\[
= \frac{\Gamma(k+n/2+1/2)}{(n+1)a^2/2+1/\gamma)^{k+n/2+1/2}}
\]

\[
\times F_A^{(n)} \left( k+n/2+1/2, \frac{1}{2}, \ldots, \frac{1}{2}, \frac{3}{2}, \ldots, \frac{3}{2}; \right)
\]

\[
\frac{a^2\gamma}{(n+1)a^2\gamma+2}, \ldots, \frac{a^2\gamma}{(n+1)a^2\gamma+2}, \quad (10)
\]

where \( F_A^{(n)}(a_1, b_1, \ldots, b_n; c_1, \ldots, c_n; x_1, \ldots, x_n) \) is the Lauricella hypergeometric function of \( n \) variables, which converges when \( \sum_{j=1}^n |x_j| < 1 \) \([6, \text{Ch. 2}].\) This convergence condition is satisfied in (10). A computer algorithm to calculate the function \( F_A^{(n)} \) is presented in \([6, \text{Appx. B}].\) Substituting (9) and (10) into (8) gives the final result. In the forthcoming section, we present some special cases of (8).

### III. Special Cases for I.I.D. Channels

The special cases under consideration are for \( m \leq 4 \) in (1).

#### A. When \( m = 1 \)

For the case when \( m = 1, \) (8) reduces to

\[
I_{\text{i.i.d.}, m=1} = \frac{1}{2} - \frac{\sqrt{a^2\gamma}}{2\sqrt{\pi}} \sum_{k=0}^{L-1} \frac{\Gamma(k+\frac{1}{2})}{\Gamma(k+1)} \left( \frac{2a^2\gamma}{a^2\gamma+2} \right)^k, \quad (11)
\]

which is equivalent to \([2, (14.4-15)].\) In addition, when \( L = 1, \) (11) becomes

\[
I_{\text{i.i.d.}, m=1, L=1} = \frac{1}{2} - \frac{1}{2} \sqrt{\frac{a^2\gamma}{a^2\gamma+2}}, \quad (12)
\]

which is identical to \([1, (5.6)].\)

#### B. When \( m = 2 \)

In the case when \( m = 2, \) (8) reduces to

\[
I_{\text{i.i.d.}, m=2} = \frac{1}{4} + I_{\text{i.i.d.}, m=1} + \frac{L-1}{2\pi} \sum_{k=0}^{L-1} \frac{1}{(a^2\gamma)^k}
\]

\[
\times \left( \frac{a^2\gamma}{a^2\gamma+1} \right)^{k+1} 2F_1 \left( k+1, 1; \frac{3}{2}; \frac{a^2\gamma}{2a^2\gamma+2} \right), \quad (13)
\]

where \( 2F_1 = F_A^{(1)} \) is the Gauss hypergeometric function of one variable. The result in (13) is analytically equivalent to \([1, (5A.21)]\) when their \( M = 2 \) (note that the index \( m \) in \([1, (5A.21)]\) is equivalent to \( L \) herein), and it gives the same numerical result. When \( L = 1, \) we have

\[
I_{\text{i.i.d.}, m=2, L=1} = \frac{1}{4} - \frac{1}{2} \sqrt{\frac{a^2\gamma}{a^2\gamma+2} + \frac{1}{2\pi} \left( \frac{a^2\gamma}{a^2\gamma+1} \right)}
\]

\[
\times 2F_1 \left( 1, 1; \frac{3}{2}; \frac{a^2\gamma}{2a^2\gamma+2} \right). \quad (14)
\]

Using \([7, (15.3.4)]\) and \([15.1.5])\) and then applying the transformation

\[
\tan^{-1}(x) = \frac{\pi}{2} - \tan^{-1} \left( \frac{1}{x} \right), \quad x > 0, \quad (15)
\]

the expression in (14) reduces to

\[
I_{\text{i.i.d.}, m=2, L=1} = \frac{1}{4} - \frac{1}{\pi} \sqrt{\frac{a^2\gamma}{a^2\gamma+2} \tan^{-1} \left( \frac{a^2\gamma+2}{a^2\gamma} \right)}, \quad (16)
\]

which is identical to \([1, (5.29)].\)

#### C. When \( m = 3 \)

In the case when \( m = 3, \) (8) reduces, after some manipulations, to

\[
I_{\text{i.i.d.}, m=3} = \frac{1}{8} + \frac{3}{2} I_{\text{i.i.d.}, m=2} - \frac{3}{4} I_{\text{i.i.d.}, m=1}
\]

\[
- \frac{3}{4\pi} \sqrt{\frac{2a^2\gamma}{2\pi}} \sum_{k=0}^{L-1} \left( \frac{a^2\gamma}{\Gamma(k+1)} \right)^k \left( \frac{2a^2\gamma}{a^2\gamma+2} \right)^{k+1/2}
\]

\[
\times F_2 \left( \frac{3}{2}, 1, 1; \frac{3}{2}; \frac{3}{2a^2\gamma+2}, \frac{a^2\gamma}{a^2\gamma+2} \right), \quad (17)
\]
where \( F_2 = F_A^{(2)} \) is the hypergeometric function of two variables. For the case when \( L = 1 \), we have

\[
\mathcal{I}_{\text{i.i.d.}}, m = 3, L = 1 = \frac{1}{8} + \frac{3}{2} \mathcal{I}_{\text{i.i.d.}}, m = 2, L = 1 - \frac{3}{4} \mathcal{I}_{\text{i.i.d.}}, m = 1, L = 1
\]

\[
\times F_2 \left( \frac{3}{2}, 1, 1; \frac{3}{2}, \frac{3}{2}; \frac{a^2\gamma}{3a^2\gamma + 2}, \frac{a^2\gamma}{3a^2\gamma + 2} \right).
\]

Using the transformation \([8, (9.182.3)]\)

\[
F_2(a, b, b'; a; x, y) = (1 - x)^{-b}(1 - y)^{-b'}
\times 2F_1 \left( b, b'; a; \frac{xy}{(1 - x)(1 - y)} \right),
\]

the function \( F_2 \) in (18) can be expressed in terms of \( 2F_1 \). Then, using \([7, (15.3.4) \text{ and } (15.1.5)\)], the result in (18) reduces to

\[
\mathcal{I}_{\text{i.i.d.}}, m = 3, L = 1 = \frac{1}{8} + \frac{3}{2} \mathcal{I}_{\text{i.i.d.}}, m = 2 + \frac{1}{2} \mathcal{I}_{\text{i.i.d.}}, m = 1
\]

\[
- \frac{3}{2\pi} \left[ \sqrt{\frac{a^2\gamma + 2}{a^2\gamma}} \tan^{-1} \left( \sqrt{\frac{a^2\gamma + 2}{2a^2\gamma}} \right) \right] + \frac{1}{2} \tan^{-1} \left( \frac{a^2\gamma}{(a^2\gamma + 2)(a^2\gamma + 2)} \right). \tag{20}
\]

To the best of the authors’ knowledge, the results in (17) and (20) are new and have not been presented elsewhere.

**D. When \( m = 4 \)**

When \( m = 4 \), one has, after some simplifications, the following:

\[
\mathcal{I}_{\text{i.i.d.}}, m = 4 = \frac{1}{16} + 2\mathcal{I}_{\text{i.i.d.}}, m = 3 - \frac{3}{2} \mathcal{I}_{\text{i.i.d.}}, m = 2 + \frac{1}{2} \mathcal{I}_{\text{i.i.d.}}, m = 1
\]

\[
+ \frac{1}{2\pi^2} \sum_{k=0}^{L-1} \left( \frac{a^2\gamma}{a^2\gamma} \right)^k \Gamma(k + 2) \Gamma(k + 1) \left( \frac{a^2\gamma}{2a^2\gamma + 1} \right)^{k+2}
\]

\[
\times F_A^{(3)} \left( k + 2, 1, 1, 1; \frac{3}{2}, \frac{3}{2}; \frac{a^2\gamma}{4a^2\gamma + 2}, \frac{a^2\gamma}{4a^2\gamma + 2}, \frac{a^2\gamma}{4a^2\gamma + 2} \right). \tag{21}
\]

For the special case of \( L = 1 \) when \( m = 4 \), (21) becomes

\[
\mathcal{I}_{\text{i.i.d.}}, m = 4, L = 1 = \frac{1}{16} + 2\mathcal{I}_{\text{i.i.d.}}, m = 3, L = 1 - \frac{3}{2} \mathcal{I}_{\text{i.i.d.}}, m = 2, L = 1
\]

\[
+ \frac{1}{2\pi^2} \left( \frac{a^2\gamma}{2a^2\gamma + 1} \right)^2
\]

\[
\times F_A^{(3)} \left( 2, 1, 1, 1; \frac{3}{2}, \frac{3}{2}; \frac{a^2\gamma}{4a^2\gamma + 2}, \frac{a^2\gamma}{4a^2\gamma + 2}, \frac{a^2\gamma}{4a^2\gamma + 2} \right). \tag{22}
\]

The results in (21) and (22) also appear to be new. Unfortunately, further mathematical simplification on the function \( F_A^{(3)} \) is not available. However, the function \( F_A^{(3)} \) can be easily evaluated numerically, and a computer algorithm for this purpose is given in \([6, \text{App. B}]\).

**IV. I.N.D. CHANNELS**

For the case of i.n.d. Rayleigh fading channels, in which the average received SNRs over diversity branches are distinct, the pdf of the combined instantaneous SNR \( \gamma = \sum_{\ell=1}^{L} \gamma_{\ell} \) can be expressed as a weighted sum of Gamma distributed random variables having parameters of \( \gamma_{\ell} \sim \mathcal{G}(1, 1/\tau_{\ell}) \). That is \([2, \text{Ch. 14}]\)

\[
p_{\gamma}(\gamma) = \sum_{\ell=1}^{L} A_{\ell} e^{-\gamma/\tau_{\ell}}, \tag{23}
\]

where \( \tau_{\ell} \) is the average received SNR over the \( \ell \)th diversity branch, and \( \{A_{\ell}\}_{\ell=1}^{L} \) are defined as

\[
A_{\ell} = \prod_{k=1, k \neq \ell}^{L} (1 - \frac{\gamma_{k}}{\tau_{k}})^{-1}, \quad L > 1;
\]

\[
1, \quad L = 1. \tag{24}
\]

The CDF of \( \gamma \) is now given by

\[
P_{\gamma}(\gamma) = 1 - \sum_{\ell=1}^{L} A_{\ell} e^{-\gamma/\tau_{\ell}}. \tag{25}
\]

Substituting (25) into (4) results in

\[
\mathcal{I}_{\text{i.n.d.}} = \left( \frac{1}{2} \right)^m - m \sum_{\ell=1}^{L} A_{\ell} \sqrt{\frac{a^2\gamma_{\ell}}{a^2\gamma_{\ell} + 2}}
\]

\[
- \left( \frac{1}{2} \right)^m \frac{m}{\sqrt{2\pi}} \sum_{\ell=1}^{L} A_{\ell} \sum_{n=1}^{m-1} \Gamma \left( \frac{n + 1}{2} \right)
\]

\[
\times \left( \frac{m - 1}{n} \right)^{(n-1)} \left( \frac{2}{\sqrt{2\pi}} \right)^{n} \left( \frac{2a^2\gamma_{\ell}}{(n+1)a^2\gamma_{\ell} + 2} \right)^{\frac{n+1}{2}}
\]

\[
\times F_A^{(n)} \left( \frac{n + 1}{2}, 1, \ldots, 1; \frac{3}{2}, \ldots, \frac{3}{2}; \frac{a^2\gamma_{\ell}}{(n+1)a^2\gamma_{\ell} + 2}, \ldots, \frac{a^2\gamma_{\ell}}{(n+1)a^2\gamma_{\ell} + 2} \right). \tag{26}
\]

Following the analysis in section II, the integral in (26) can be obtained, after some manipulations, in closed-form, and then \( \mathcal{I}_{\text{i.n.d.}} \) becomes

\[
\mathcal{I}_{\text{i.n.d.}} = \left( \frac{1}{2} \right)^m - m \sum_{\ell=1}^{L} A_{\ell} \sqrt{\frac{a^2\gamma_{\ell}}{a^2\gamma_{\ell} + 2}}
\]

\[
- \left( \frac{1}{2} \right)^m \frac{m}{\sqrt{2\pi}} \sum_{\ell=1}^{L} A_{\ell} \sum_{n=1}^{m-1} \Gamma \left( \frac{n + 1}{2} \right)
\]

\[
\times \left( \frac{m - 1}{n} \right)^{(n-1)} \left( \frac{2}{\sqrt{2\pi}} \right)^{n} \left( \frac{2a^2\gamma_{\ell}}{(n+1)a^2\gamma_{\ell} + 2} \right)^{\frac{n+1}{2}}
\]

\[
\times F_A^{(n)} \left( \frac{n + 1}{2}, 1, \ldots, 1; \frac{3}{2}, \ldots, \frac{3}{2}; \frac{a^2\gamma_{\ell}}{(n+1)a^2\gamma_{\ell} + 2}, \ldots, \frac{a^2\gamma_{\ell}}{(n+1)a^2\gamma_{\ell} + 2} \right). \tag{27}
\]

In the forthcoming section, special cases of the result in (27) are considered for \( m \leq 4 \).

**V. SPECIAL CASES OF I.N.D. CHANNELS**

**A. When \( m = 1 \)**

When \( m = 1 \), (28) becomes

\[
\mathcal{I}_{\text{i.n.d.}, m = 1} = \frac{1}{2} - \frac{1}{2} \sum_{\ell=1}^{L} A_{\ell} \sqrt{\frac{a^2\gamma_{\ell}}{a^2\gamma_{\ell} + 2}}. \tag{28}
\]

In addition, when \( L = 1 \), (28) reduces to (12). Note that (28) is similar to \([2, (14-5-28)]\).
B. When \( m = 2 \)

For the special case when \( m = 2 \), using \([7, (15.3.4)\) and \((15.1.5)\), and \((15)\), \((27)\) reduces to

\[
I_{\text{i.n.d.}, m=2} = \frac{1}{4} \frac{1}{\pi} \sum_{\ell=1}^{L} A_{\ell} \sqrt{\frac{a^{2\gamma_{\ell}}}{a^{2\gamma_{\ell}} + 2}} \tan^{-1} \left( \sqrt{\frac{a^{2\gamma_{\ell}} + 2}{a^{2\gamma_{\ell}}} - \gamma_{\ell}} \right),
\]

which is a generalization of \([1, (5.29)]\) to the case of nonidentical multichannel Rayleigh fading.

C. When \( m = 3 \)

When \( m = 3 \), one obtains the following result from \((27)\):

\[
I_{\text{i.n.d.}, m=3} = \frac{1}{8} + \frac{3}{2} I_{\text{i.n.d.}, m=2} - \frac{3}{4} I_{\text{i.n.d.}, m=1} - \frac{3}{4\pi} \sum_{\ell=1}^{L} A_{\ell} \left( \frac{a^{2\gamma_{\ell}}}{3a^{2\gamma_{\ell}} + 2} \right)^{2} \times F_{2} \left( \frac{3}{2}; 1, 1; \frac{3}{2}; \frac{a^{2\gamma_{\ell}}}{3a^{2\gamma_{\ell}} + 2}, \frac{a^{2\gamma_{\ell}}}{3a^{2\gamma_{\ell}} + 2} \right).
\]

The function \( F_{2} \) in \((30)\) can be re-expressed using \([7, (15.3.4)\) and \((15.1.5)\), and \((19)\). Then \((30)\) becomes

\[
I_{\text{i.n.d.}, m=3} = \frac{1}{8} + \frac{3}{2} I_{\text{i.n.d.}, m=2} - \frac{3}{4} I_{\text{i.n.d.}, m=1} - \frac{3}{2\pi} \sum_{\ell=1}^{L} A_{\ell} \left[ \frac{a^{2\gamma_{\ell}}}{a^{2\gamma_{\ell}} + 2} \left( \frac{a^{2\gamma_{\ell}} + 2}{a^{2\gamma_{\ell}}} \right) \right] + \frac{a^{2\gamma_{\ell}}}{\sqrt{a^{2\gamma_{\ell}} + 2}} \tan^{-1} \left( \frac{a^{2\gamma_{\ell}}}{\sqrt{a^{2\gamma_{\ell}} + 2}} \right).
\]

When \( L = 1 \), \((31)\) reduces to \((20)\), as would be expected.

D. When \( m = 4 \)

For the case when \( m = 4 \), \((27)\) reduces to

\[
I_{\text{i.n.d.}, m=4} = -\frac{1}{16} + 2I_{\text{i.n.d.}, m=3} - \frac{3}{2} I_{\text{i.n.d.}, m=2} + \frac{1}{2} I_{\text{i.n.d.}, m=1} + \frac{1}{2\pi} \sum_{\ell=1}^{L} A_{\ell} \left( a^{2\gamma_{\ell}} \right) \times F_{2}^{(3)} \left( \frac{2, 1, 1, 1; 3}{2}; \frac{2\gamma_{\ell}}{4a^{2\gamma_{\ell}} + 2}, \frac{a^{2\gamma_{\ell}}}{4a^{2\gamma_{\ell}} + 2} \right).
\]

It is worth mentioning that, while the results in the present and previous sections are for independent channels, they can be directly applied to the case of correlated channels after replacing \( \{\gamma_{\ell}\}_{\ell=1}^{L} \) by \( \{\lambda_{\ell}\}_{\ell=1}^{L} \), where \( \lambda_{\ell} \) is the \( \ell \)-th eigenvalue of the \( L \times L \) positive definite matrix \( CR \). Here \( C = \text{diag} [\gamma_{1}, \gamma_{2}, \ldots, \gamma_{L}] \), and \( R \) is the correlation matrix whose \((i, j)\) entry is \( R_{i,j} = \rho_{ij} \), where \( \rho_{ij} \) is the power correlation coefficient between the received SNRs on the \( i \)-th and \( j \)-th diversity branches (note that \( \rho_{ii} = 1 \) when \( i = j \), for \( i, j = 1, 2, \ldots, L \)).

VI. APPLICATION TO THE AVERAGE ERROR PERFORMANCE ANALYSIS

The DE-QPSK modulation can be used to resolve the phase ambiguity that may be associated with the ideal coherent QPSK modulation scheme due to unsynchronized reference carrier \([9, \text{Ch. 4}]\). The idea is to employ differential phase encoding at the transmitter and differential phase decoding at the receiver side following the coherent detection stage. In the DE-QPSK, the maximum change in phase from transmission to transmission is \( \pi \), which results in maximized instantaneous amplitude fluctuation of the transmitted signal. The \( \pi/4\)-QPSK modulation was proposed to reduce the maximum change in phase (which is \( 3\pi/4 \) instead of \( \pi \)), and hence, decreasing the instantaneous amplitude fluctuations. In additive white Gaussian noise (AWGN) channel employing coherent detection, both the DE-QPSK and \( \pi/4\)-QPSK signals have identical conditional symbol error performance (conditioned over fading statistics), which is given by \([1, (8.39)]\)

\[
P_{e} (e \mid \gamma) = 4Q (\sqrt{\gamma}) - 8Q^{2} (\sqrt{\gamma}) + 8Q^{3} (\sqrt{\gamma}) - 4Q^{4} (\sqrt{\gamma}).
\]

For AWGN channel, the first two terms can be considered as a good approximation for \((33)\), especially at high-SNR applications. However, as pointed out in \([1, \text{p. 259}]\), all four terms in \((33)\) are important because of the need to average \( \gamma \) over the interval \([0, \infty)\).

A. i.i.d. Channels

Averaging \((33)\) over the pdf of \( \gamma \) in \((3)\) gives the following result (note that \( a = 1 \) herein):

\[
P_{e} (e \mid \gamma) = 4 \mathcal{I}_{\text{i.i.d.}, m=1} - 8 \mathcal{I}_{\text{i.i.d.}, m=2} + 8 \mathcal{I}_{\text{i.i.d.}, m=3} - 4 \mathcal{I}_{\text{i.i.d.}, m=4}
\]

\[
= \frac{3}{4} - \frac{1}{\pi} \sum_{k=0}^{L-1} \left( \frac{1}{\gamma} \right)^{k} \times \left[ \left( \frac{\gamma}{2\gamma + 1} \right)^{k+1} 2 F_{1} \left( k + 1, 1; \frac{3}{2}; \frac{\gamma}{2\gamma + 1} \right) + \frac{2(k + 1)}{\pi} \left( \frac{\gamma}{2\gamma + 1} \right)^{k+2} \times F_{A}^{(3)} \left( k, 2, 1, 1; \gamma \right) \right].
\]

When \( L = 1 \), we have

\[
P_{e} (e \mid \gamma, L=1) = \frac{3}{4} - \frac{1}{\pi} \left[ \sqrt{\frac{\gamma}{2\gamma + 1}} \tan^{-1} \left( \sqrt{\frac{\gamma}{2\gamma + 1}} \right) \right. \]

\[
+ \frac{1}{\pi} \left( \frac{\gamma}{2\gamma + 1} \right)^{2} \times F_{A}^{(3)} \left( 2, 1, 1, 1; \gamma \right)
\]

\[
= \frac{\pi}{4} \left[ \sqrt{\frac{\gamma}{2\gamma + 1}} \tan^{-1} \left( \sqrt{\frac{\gamma}{2\gamma + 1}} \right) \right. \]

\[
+ \frac{1}{\pi} \left( \frac{\gamma}{2\gamma + 1} \right)^{2} \times F_{A}^{(3)} \left( 2, 1, 1, 1; \gamma \right).\]

B. i.n.d. Channels

In the case of i.n.d Rayleigh fading channels, $P_s(e)$ is given by

$$
P_s(e)_{\text{i.n.d.}} = 4I_{\text{i.n.d.},m=1} - 8I_{\text{i.n.d.},m=2},
+8I_{\text{i.n.d.},m=3} - 4I_{\text{i.n.d.},m=4}
= \frac{3}{4} - 2\pi^2 \sum_{\ell=1}^{L} \left[ \frac{\tau_{\ell}}{\gamma_{\ell} + 2} \tan^{-1} \left( \frac{\tau_{\ell}}{\gamma_{\ell} + 2} \right) \right]
+ \frac{1}{\pi} \left( \frac{\tau_{\ell}}{2\gamma_{\ell} + 1} \right)^2
\times F_A^{(3)} \left( 2, 1, 1; \frac{3}{2}, \frac{3}{2}; \frac{\tau_{\ell}}{4\gamma_{\ell} + 2}, \frac{\tau_{\ell}}{4\gamma_{\ell} + 2} \right).
$$  

(36)

Note that (35) follows from (36) when $L = 1$.

Two numerical examples for the average error performance are shown in Figs. 1 and 2 for the cases of i.i.d. and i.n.d. channels respectively. In addition, results for the average error performance of the coherent QPSK signals are presented in the figures for the purpose of comparison. The solid line curves in the figures have been obtained using (34) and (36). In Fig. 2, the received average SNRs are assumed to be exponentially decaying following the model: $\gamma_{\ell} = \gamma_1 e^{-\sigma(\ell-1)}$, where $\sigma$ denotes the power decaying factor. Note that there is a slight difference in performance between the ideal coherent QPSK and DE-QPSK due to having differential phase encoding/decoding stages in the DE-QPSK modulation.

VII. CONCLUSION

Exact-form results for integrals involving $m$-th power of the Gaussian $Q$-function have been presented for the i.i.d. and i.n.d. Rayleigh fading channels employing maximal-ratio diversity combining at the receiver. The results are valid for integer $m \geq 1$, and reduce the previously obtained formulas when $m = 1$ and $m = 2$ as special cases. In addition, the derived formulas have been used to obtain new exact-form averaging results of $Q^m(\cdot)$ when $m = 3$ and $m = 4$. Such results have then been employed to obtain expressions for the average error performance of DE-QPSK signals in i.i.d. and i.n.d. multi-channel Rayleigh fading. The derived results in this paper have several applications in evaluating the average error performance of digital signaling systems (coded and uncoded) over Rayleigh fading channels, wherein the average of higher-order powers of the Gaussian $Q$-function is needed.

REFERENCES