Whether and Where to Code in the Wireless Relay Channel

Xiaomeng Shi, Muriel Médard, Fellow IEEE, and Daniel E. Lucani

Abstract

The throughput benefits of random linear network codes have been studied extensively for wirelined and wireless erasure networks. It is often assumed that all nodes within a network perform coding operations, for minimal centralized control while maximizing the throughput. In energy-constrained systems, however, coding subgraphs should be chosen to minimize the number of coding nodes while maintaining the achievable throughput. In this paper, we explore the use of network coding in the wireless relay channel, where a single source communicates to a single sink through the aid of a relay in half-duplex mode. We characterize the throughput performance of this unicast connection when coding is performed at either the source or the relay. Markov chain models are used to describe the evolution of innovative packets at different nodes within the network, and numerical solutions are determined to find how much time the relay should participate in the transmission. We show that coding at the source alone can achieve a significant portion of the throughput attained by coding everywhere (at least 69% in the presented case), while coding at the relay alone is not throughput efficient. We also show that a limited amount of memory at the relay suffices when coding is performed at the source.

Index Terms

Random linear network coding, wireless relay channel

Xiaomeng Shi and Muriel Médard are with the Research Laboratory of Electronics, Massachusetts Institute of Technology, MA, USA, email: \{xshi, medard\}@mit.edu. Daniel E. Lucani is with the Instituto de Telecomunicações, DEEC Faculdade de Engenharia, Universidade do Porto, Portugal, email: dluca@fe.up.pt
I. INTRODUCTION

Network coding, although initially introduced as a theoretical tool in the field of network information theory [1], has been made practical by the use of random linear network codes (RLNC) [2], [3], and has been shown to offer throughput, energy, delay, as well as other advantages over classical store-and-forward strategies. To minimize the amount of centralized control, RLNC is often performed at the source and all intermediate nodes within a transmission subgraph. Decoding is conducted at a receiver once enough degrees of freedom (dof) are collected. Since coding operations come at a cost in terms of energy and computation, resource constrained networks, such as wireless body area networks, shall benefit significantly from strategies that allow a reduction in the number of coding nodes, while maintaining the benefits of network coding. In this paper, we study the strategic use of network coding in a wireless relay channel, as illustrated by Figure 1(a), and provide a performance characterization in terms of the average time to complete the transmission of a given set of data packets. Special emphasis is given to analyzing which nodes should code to attain the best performance. In particular, we determine if coding at the source or at the relay only is a reasonable approximation to the coding at both nodes case in terms of throughput. Although seemingly simple, this analysis and the insights drawn from it shall be important when finding the optimal coding nodes in general multi-relay and multi-hop networks. We use Markov chain models to describe the evolution of innovative packets in the network. By means of numerical evaluations, we show that performing RLNC at the source only, even with limited amount of memory at the relay, achieves much of the throughput gains of the full coding case.

[Fig. 1 about here.]

The focus of this paper is the wireless relay channel, as illustrated by the hypergraph in Fig 1(a). A hypergraph is a generalization of a graph. In a hypergraph, a broadcast link is represented by a hyperarc between a single start node and a set of end nodes, and a multiple access link is represented by a hyperarc between a set of start nodes and a single end node [4]. A wireless relay channel consists of a source node $s$, a relay node $r$, and a sink node $d$. Source $s$ has $n$ packets of the same length to transmit to $d$. It broadcasts to both the relay $r$ and the sink $d$, as represented by the hyperarc originating from $s$, while the relay $r$ assists the transmission
by either forwarding the original packet, or computing linear combinations of received packets before forwarding the ensuing mixtures.

This setup is similar to the three-node wireless networks described in [5]. However, we do not consider the multicast or two-way relay scenarios, but focus on the relay channel only. We consider packetized operations, independent of the physical layer implementation of the system. As such, erroneous packets are dropped, and channel losses are measured by a time-average erasure rate. The transmission success rates are assumed to be $p_{sr}$ between $s$ and $r$, $p_{rd}$ between $r$ and $d$, and $p_{sd}$ between $s$ and $d$. Nodes operate in half-duplex mode, where a node cannot transmit and receive at the same time. To avoid interference and collisions in a contention based scheme, we consider a time division framework, where $s$ and $r$ share the use of the wireless medium. A genie scheduler allocates the wireless medium to the source $\alpha$ fraction of the total time, $0 \leq \alpha \leq 1$, and allocates the wireless medium to the relay the remaining $1 - \alpha$ fraction of time. One possible implementation of such a genie-aided scheduler is to share the same randomness at $s$ and $r$. Figure 1(b) illustrates the maximum flow on each possible link in this network model, computed directly from the transmission success rates and the time-sharing constant $\alpha$.

In terms of memory, let both $s$ and $d$ contain $n$ units of memory, but assume $r$ contains $x$ units only, where $1 \leq x \leq n$. $r$ uses its memory as a queue: arriving packets are stored; if $r$ is already full, newly arrived packets are discarded. If $r$ does not perform coding, it sends to $d$ a packet from its memory directly; if $r$ performs RLNC before forwarding, it sends to $d$ a linear combination of stored packets, where each is multiplied by a random number chosen uniformly from a finite field $\mathbb{F}_q$ before being summed together. In this paper, $\mathbb{F}_q$ is assumed to be sufficiently large, such that with high probability, linear combinations resulting from the coding process are linearly independent from each other.

We assume transmissions occur in a rateless fashion, with minimalistic feedback: $s$ and $r$ take turns to transmit, until $d$ acknowledges that it has received enough dof to recover the original $n$ data packets from $s$. Such rateless operations are often desirable in systems where feedback can be costly in terms of energy or delay.

The remaining of the paper is organized as follows. Section II summarizes some of the previous work related to the wireless relay channel and the application of network coding in such settings. In particular, Section II-A restate the capacity results for the wireless relay channel by Traskov et
al., in the context and nomenclature of our work. This will serve as the baseline for comparison in later parts of the current paper. Section III describes Markov chain models to estimate the expected completion time when network coding is performed at the relay or the source only. Numerical evaluations and discussions are given in Section IV, while Section V concludes the paper with discussions on future work.

II. RELATED WORK

A relay channel models the problem where two nodes communicate through the help of one or more relays. This setup is common in multihop wireless networks such as sensor networks, where transmission power is limited, or in decentralized ad hoc networks, where nodes can only communicate with their immediate neighbors. Relays can overhear transmissions to the destination, owing to the broadcast advantage of the wireless medium. Recently there has been a renewed interest in the classical relay channel [6], [7], motivated by the potential to achieve cooperative diversity, thus better capacity bounds [5], [8]–[10]. Schemes such as amplify-and-forward, decode-and-forward, and compress-and-forward, have been proposed and studied extensively, in terms of capacity, outage, energy efficiency and optimal power allocation schemes [11], [12]. Much of the analysis has focused on the fundamental performance limits at the physical layer. Network coding, on the other hand, typically resides in higher layers of the network protocol stack, independent of physical layer implementations. The addition of network coding can relax the reliability requirements on the physical layer, by introducing additional redundancy at the network layer, leading to potential savings in energy and gains in throughput.

The use of RLNC in wireless erasure networks under packetized operations is first studied by Lun et al. [4]. Traskov et al. extend Lun’s flow-model to a scheduling framework using conflict graphs [13]. Traskov et al. have also analyzed the wireless relay channel explicitly [14]; their result will be restated later in this section. Other schemes that employ network coding in a relay setup includes the MORE protocol, which performs opportunistic routing when multiple relays assist the communication between two nodes [15]. Here RLNC is performed at the source only, and minimizes the amount of coordination required by the multiple relay nodes. The COPE protocol, on the other hand, employs network coding at the relay only in a 2-way relay channel to improve reliability, taking advantage of opportunistic listening and coding [16]. Fan et al. proposed a network coding based cooperative multicast scheme [17] to show that significant
throughput gains can be achieved when network coding is performed at the relay only. One assumption in this work is that feedback is available from both the destination and the relay to the source after each packet reception. In practical systems, feedback can be costly in terms of both throughput and energy, depending on the underlying hardware architecture [18]. In this paper, we explore rateless transmissions, where acknowledgement for successful reception is sent only once by the destination when transmission of all available data is completed.

A. Packet-Level Capacity Bound for Coding at Both the Source and the Relay

Lun et al. have previously studied the use of RLNC in wireless erasure networks under packetized operations [4]. They assume RLNC is performed at intermediate nodes, while transmissions occur in a rateless fashion, and determine optimal coding subgraphs by tracking the propagation of innovative packets. Such packets are innovative in the sense that they carry additional dof to a node. It is determined that the propagation of innovative packets through a network can be approximated by a fluid flow model. Given a particular schedule, RLNC at both the source and intermediate nodes can achieve the min-cut bound for single unicast and single multicast connections, in the limit of infinitely large payloads. The wireless relay channel is analyzed in detail as a particular example, with a contention-based slotted aloha medium access control scheme, assuming infinite or finite memory at the relay.

Traskov et al. [13], [14] put such flow-based analysis into a scheduling framework. For the wireless relay channel, it is assumed that the source injects innovative packets into the hyperarc \((s, \{r, d\})\) in \(\alpha\) fraction of the total transmission time, while the relay injects innovative packets into the arc \((r, d)\) in \(1 - \alpha\) fraction of the total transmission time. Packet transmissions form innovative flows in this setup because both the source \(s\) and the relay \(r\) perform RLNC over a large number of packets, and \(r\) is assumed to have infinite memory. Each mixture is an additional dof relative to the destination \(d\). The amount of innovative flow is limited by the capacity on each hyperarc, as illustrated by Figure 1(b). Assuming flow conservation at \(r\), the maximum achievable rate on the wireless relay channel can be derived. The optimal solution is divided into two cases, according to the channel conditions.

- **Case 1:** \(p_{sd} \leq p_{rd}\), then
  \[
  R^* = \frac{p_{rd}(p_{sr} + p_{sd} - p_{sd}p_{sr})}{p_{rd} + p_{sr}(1 - p_{sd})}, \quad \alpha^* = \frac{p_{rd}}{p_{rd} + p_{sr}(1 - p_{sd})}
  \]

  (1)
• Case 2: \( p_{sd} > p_{rd} \), then the relay is not used, and

\[
R^* = p_{sd}, \quad \alpha^* = 1
\]

Performing RLNC at both \( s \) and \( r \) maximizes the amount of innovation in each packet transmission through the relay channel. In this paper, we perform analytical and numerical studies on the wireless relay channel, to characterize the trade-offs among performance and available resources (e.g., coding nodes, memory available). Our ultimate goal is to provide insights into the best coding locations in larger, more complex wireless networks.

### III. Network Coding in the Wireless Relay Channel

In this section, we propose Markov chain models to characterize the expected completion time of transmitting \( n \) packets from the source \( s \) to the destination \( d \). We consider two separate cases: coding at the relay \( r \) only, and coding at the source \( s \) only. We also offer a brief discussion on the use of systematic codes, the performance of which will be studied in future works.

#### A. RLNC at the Relay Only

We first consider the scenario where \( s \) is limited to sending the original packets only, while \( r \) performs RLNC over all packets it has received in the past. Due to the rateless transmission with minimalistic feedback assumption, \( s \) is unaware of the knowledge (degrees of freedom) at \( r \) and \( d \). We assume that when given a transmission opportunity, \( s \) chooses one packet uniformly at random from the \( n \) available uncoded packets.

One remark here is that more exhaustive or frequent feedbacks would allow retransmissions from \( s \) to be more intelligent. For example, if per-packet acknowledgement is available, \( s \) can choose to repeat only those that have not been successfully received by \( r \) or \( d \). On the other hand, if \( d \) can acknowledge the exact dof it has received, \( r \) can adjust the number of coded packets it sends out to maximize throughput to \( d \) while minimizing its own energy consumption. We consider transmission with minimal feedback here because we want to compare the throughput behavior of the system with the full coding case (i.e., coding at both \( s \) and \( r \)), which uses a feed-forward, rateless scheme.

For transmissions to be free of collisions, recall from the introduction that we assume there is a genie-aided scheduler, such that \( s \) and \( r \) do not access the wireless medium at the same time.
In each time slot, \( s \) transmits with probability \( \alpha \), while \( r \) transmits with probability \( 1 - \alpha \). By randomizing the transmitter at each time slot, the transmission process becomes memoryless, and the numbers of dof at each node can be tracked through a Markov chain. When \( s \) is given the opportunity to transmit, it chooses one packet uniformly at random from its \( n \) uncoded packets. When \( r \) is given the opportunity to transmit, it computes a linear combination of the content of its memory, and sends the mixture to \( d \). Here we assume \( r \) has enough memory to store all \( n \) distinct packets.

We represent states in the Markov chain by a three-tuple \( (m, k, l) \): \( m \) is the number of unique dof at \( d \), \( k \) is the number of dof shared by \( d \) and \( r \), while \( l \) is the number of dof at \( r \) only. Since source \( s \) does not code and relay \( r \) stores only uncoded packets, \( m \) and \( l \) here represent the number of distinct packets at \( r \) and \( d \). Once a packet at \( r \) has been mixed into a coded packet, if it is received at \( d \), it becomes part of the shared dof between \( r \) and \( d \). With such state definitions, any three-tuple satisfying \( m + k + l \leq n \) is a valid state. Transmission initiates in state \( (0, 0, 0) \), and terminates in states \( \{(m^*, k^*, l^*)| m^* + k^* = n\} \). Figure 2 gives a sample Markov chain when \( n = 2 \). We have drawn the Markov chain as a tetrahedron, with the starting state on top, and the terminating states on the left vertical edge.

An alternative to genie-aided randomized transmissions is a collision-free, deterministic schedule, where \( s \) and \( r \) take turns to transmit for a fixed amount of time, determined by the value of \( n \), \( \alpha \), and the channel conditions. However, with this deterministic schedule, since packets from \( s \) are not coded, counting the numbers of dof at different nodes requires knowledge of the exact packets at \( r \) and \( d \). Even with \( n \) being small, it is hard to track the evolution of packets at different nodes in the system.

In the Markov chain described above, state transitions occur after the transmission of a single packet, either from \( s \) or from \( r \). In the case where \( r \) has not received any packet successfully, but is chosen to transmit, we assume the slot is wasted. At state \( (m, k, l) \), the state transition probabilities can be computed by considering all possible outcomes of the transmission, assuming independent packet losses. First, when \( m + k < n \), \( s \) broadcasts with probability \( \alpha \), while \( r \) transmits with probability \( 1 - \alpha \). Let the indicator function \( I_l \) be 1 when \( l \neq 0 \), and 0 otherwise, then
\[
P_{\{m,k,l\} \rightarrow \{m+1,k,l\}} = \frac{n-m-k-l}{n} p_{sd}(1 - p_{sr})^\alpha
\]

\[
P_{\{m,k,l\} \rightarrow \{m,k+1,l\}} = \frac{n-m-k-l}{n} p_{sd}p_{sr}^\alpha
\]

\[
P_{\{m,k,l\} \rightarrow \{m,k,l+1\}} = \frac{n-m-k-l}{n} p_{sr}(1 - p_{sd})^\alpha
\]

\[
P_{\{m,k,l\} \rightarrow \{m-1,k+1,l\}} = \frac{m}{n} p_{sr}^\alpha
\]

\[
P_{\{m,k,l\} \rightarrow \{m,k+1,l-1\}} = \frac{l}{n} p_{sd}^\alpha + I_l p_{rd}(1 - \alpha)
\]

\[
P_{\{m,k,l\} \rightarrow \{m,k,l\}} = \left[ \frac{m}{n}(1 - p_{sr}) + \frac{l}{n}(1 - p_{sd}) + \frac{k}{n} + \frac{n-m-k-l}{n}(1 - p_{sd})(1 - p_{sr}) \right] \alpha
\]

\[
+ (1 - I_l p_{rd})(1 - \alpha)
\]

When \(s\) broadcasts, this transmission may succeed in any of the three hyperarcs originating from \(s\). The different hyperarcs are shown in Figure 1(b). Depending on which packets \(r\) and \(d\) already have, a successful transmission may or may not lead to a transition to a different state. For example, if the transmitted packet has already been received by both \(d\) and \(r\), a state transition does not occur regardless if the transmission is successful. In particular, the following state transitions are possible.

1) If \(d\) receives one more dof, while \(r\) does not, \(P_{\{m,k,l\} \rightarrow \{m+1,k,l\}} = \frac{n-m-k-l}{n} p_{sd}(1 - p_{sr})^\alpha\).

2) if both \(d\) and \(r\) receive one more dof, \(P_{\{m,k,l\} \rightarrow \{m,k+1,l\}} = \frac{n-m-k-l}{n} p_{sd}p_{sr}^\alpha\);

3) if \(r\) receives one more dof, while \(d\) does not, \(P_{\{m,k,l\} \rightarrow \{m,k,l+1\}} = \frac{n-m-k-l}{n} p_{sr}(1 - p_{sd})^\alpha\).

4) if the broadcasted packet has been previously received by \(d\), and is now received at \(r\), then \(P_{\{m,k,l\} \rightarrow \{m-1,k+1,l\}} = \frac{m}{n} p_{sr}^\alpha\). Note that if \(m = 0\), this transition probability is 0.

5) if the broadcasted packet has been received by \(r\) previously, and is now received at \(d\), then \(P_{\{m,k,l\} \rightarrow \{m,k+1,l-1\}} = \frac{l}{n} p_{sd}^\alpha\). If \(l = 0\), this transition probability is 0.

6) a self transition occurs if the transmitted dof is initially at \(d\) but is not received at \(r\), if the transmitted dof is initially at \(r\) but is not received at \(d\), if the transmitted dof is already shared, or if the transmitted dof is neither shared nor received. Correspondingly, \(P_{\{m,k,l\} \rightarrow \{m,k,l\}}\) is the sum of four terms: \(\frac{m}{n}(1 - p_{sr})\), \(\frac{l}{n}(1 - p_{sd})\), \(\frac{k}{n}\), and \(\frac{n-m-k-l}{n}(1 - p_{sd})(1 - p_{sr})\).

Relay \(r\) transmits coded packets with probability \(1 - \alpha\). Observe that when \(r\) has received only
a small number of packets, a mixture it generates for transmission may not be innovative with respect to \( d \), because the packets \( r \) has could also have been received by \( d \) itself. For example, if \( n \) is equal to 3, and \( d \) has already received packets 2 and 3, then a coded packet from \( r \) containing the sum of packets 2 and 3 is not innovative even if it is successfully received at \( d \), but a coded packet containing the sum of packets 1 and 2 is innovative. Again, explicitly tracking the contents of coded packets is a difficult task. Instead, we can assume that all packets transmitted from the relay to the sink are innovative. The computed expected completion time under this assumption is an lower bound on the actual expected completion time, and the corresponding throughput is an upper bound on the actual system throughput. With such assumptions, the following state transitions can occur

1) if \( r \) has no unique dof to share, \( l = 0 \), \( P_{\{m,k,0\} \rightarrow \{m,k,0\}} = 1 \);
2) if \( r \) has a unique dof to share, \( l > 0 \), and \( d \) receives successfully, \( P_{\{m,k,l\} \rightarrow \{m,k+1,l-1\}} = p_{rd} \);
3) if \( r \) has a unique dof to share, \( l > 0 \), but \( d \) does not receive successfully, \( P_{\{m,k,l\} \rightarrow \{m,k,l\}} = 1 - p_{rd} \)

The transmission process terminates when \( m + k = n \), and \( P_{\{m,n-m,0\} \rightarrow \{m,n-m,0\}} = 1 \). Note that all states are transient except the absorbing states, each of which is a recurrent class. When \( n > 1 \), multiple recurrent classes exist. Since there is a single starting state, there exists a unique steady state distribution. To simplify the computation of the absorbing time, we append a virtual terminating state \( S_T \), such that \( P_{\{m,n-m,0\} \rightarrow \{S_T\}} = 1 \), and \( P_{\{S_T\} \rightarrow \{S_T\}} = 1 \). Thus, the new Markov chain has one recurrent state only.

To explicitly compute the transition state matrix, we can index the states linearly starting from \( S_T \) to \((0,0,0)\). Let \( S_T \) be state 0 under this counting notation. In Figure 2 we give one possible set of linear indices, starting from the bottom to the top of the tetrahedron.

Let \( T_i \) be the expected first passage time to state 0, i.e., the expected number of steps to reach state 0, starting in state \( i \). The expected completion time therefore equals to \( T_{10} \) in this example. We represent this expected transmission completion time by \( T \), without any subscripts. Since there are no cycles in this Markov chain, \( T_i \) can be solved recursively using the equation

\[
T_i = \frac{1}{1 - P_{ii}} \left\{ 1 + \sum_{j \neq i} P_{ij} T_j \right\}, \quad T_0 = 0, i \neq 0.
\]

Alternatively, if we express the expected first passage times to state 0 in the vector form \( \bar{T} = \left[ T_1 \ T_2 \ \ldots \ \ T_{10} \right]^T \), then \( \bar{T} = 1 + P_{\{0\} \rightarrow \{0\}} \bar{T} \), where \( 1 \) is the vector of ones, and \( P_{\{0\} \rightarrow \{0\}} \) is the
submatrix of $P$ with the first row and first column removed. Solving this linear system of equations gives

$$\bar{T} = (I - P_{(0)})^{-1}1.$$  

Note that since the original Markov chain does not contain cycles, the states can always be ordered according to an ancestral relationship, such that the resulting $P$ is a lower-triangular matrix. In addition, since self-transition probabilities are non-zero for all states except those next to $S_T$, $I - P_{(0)}$ is strictly lower-triangular, thus invertible in the real field.

One last observation in this scenario is that it is possible to constrain the amount of memory at $r$ to less than $n$, and have $r$ function as an accumulator, such that whenever an innovative packet is received, it is multiplied by different random coefficients and added to each of the memory units. When given a transmission opportunity, $r$ uniformly randomly chooses one mixture from its memory. The achievable rate region of the limited memory case should be outer-bounded by the full memory case. We will show in the numerical results section that even with full memory, coding at the relay alone is not throughput efficient.

**B. RLNC at the Source Only**

When RLNC is performed at $s$ only, the analysis of the system performance is similar to the previous case, where RLNC is performed at $r$ only. We can again use a Markov chain to track the evolution of dof at different nodes. Let state $(m, k, l)$ represent $m$ unique mixtures at $d$, $k$ mixtures shared by $r$ and $d$, and $l$ unique mixtures at $r$. Assume $r$ acts as a queue with $x$ finite units of memory, where $1 \leq x \leq n$. Here $r$ is allowed to have less than $n$ units of memory, since it does not need to store distinct packets for explicit coding operations. Instead, it receives linear mixtures directly from $s$, functions as a queue, and drops any mixtures received after it is full.

In terms of state definitions, compared with the coding at $r$ only case, since every mixture sent by $s$ can be innovative with respect to $r$ and $d$, a state $(m, k, l)$ is valid as long as $m + k \leq n$ and $k + l \leq x$. Again, $s$ transmits $\alpha$ fraction of the time. We assume that $P_{(m,k,l) \rightarrow (m',k',l')}$ is non-zero only if both $(m, k, l)$ and $(m', k', l')$ are valid states. Let the indicator function $I_{f(\cdot)}$ be 1 if the logic function $f(\cdot)$ is true, and 0 otherwise. The state transition probabilities is as follows,
\[
\begin{align*}
P\{m,k,l\} &\rightarrow \{m+1,k,l\} = \alpha p_{sd}(1 - p_{sr}) + \alpha p_{sd}(1 - p_{sr})I_{k+l=n} \\
P\{m,k,l\} &\rightarrow \{m,k+1,l\} = \alpha p_{sd}p_{sr}I_{k+l=x} \\
P\{m,k,l\} &\rightarrow \{m,k,l+1\} = \alpha p_{sr}(1 - p_{sd})I_{k+l=x} + \alpha(1 - p_{sr})(1 - p_{sd}) + (1 - \alpha)I_{k+l=x} \\
P\{m,k,l\} &\rightarrow \{m+1,k,l\} = \alpha p_{sd}(1 - p_{sr})I_{k+l=x} + \alpha(1 - p_{sr})(1 - p_{sd}) + (1 - \alpha)I_{k+l=x} \\
P\{m,k,l\} &\rightarrow \{m,k,l+1\} = \alpha p_{sr}(1 - p_{sd})I_{k+l=x} + \alpha(1 - p_{sr})(1 - p_{sd}) + (1 - \alpha)I_{k+l=x} \\
P\{m,k,l\} &\rightarrow \{m+1,k,l\} = (1 - \alpha)\frac{k}{l+k}I_{k>0} \\
P\{m,k,l\} &\rightarrow \{m+1,k,l-1\} = (1 - \alpha)\frac{l}{l+k}p_{rd}I_{l>0} \\
P\{m,k,l\} &\rightarrow \{m+1,k,l-1\} = (1 - \alpha)\frac{l}{l+k}(1 - p_{rd})I_{l>0}
\end{align*}
\]

State transitions occur after the transmission of a single packet, either from \(s\) or from \(r\). In the case where \(r\) has not received any packet successfully but is chosen to transmit, we assume the slot is wasted. At state \((m, k, l)\), the state transition probabilities can be computed by considering all possible outcomes of the transmission, assuming independent packet losses. First, when \(m + k < n\), \(s\) broadcasts with probability \(\alpha\), and the following state transitions can occur.

1) if \(d\) receives the transmitted mixture, while \(r\) does not, \(P\{m,k,l\} \rightarrow \{m+1,k,l\} = p_{sd}(1 - p_{sr})\).
2) if both \(d\) and \(r\) receive the transmitted mixture, and \(k + l < x\), \(P\{m,k,l\} \rightarrow \{m,k+1,l\} = p_{sd}p_{sr}\).
3) if both \(d\) and \(r\) receive the transmitted mixture, and \(k + l = x\), \(P\{m,k,l\} \rightarrow \{m+1,k,l\} = p_{sd}p_{sr}\).
4) if \(r\) receives the transmitted mixture, while \(d\) does not, and \(k + l < x\), then the mixture is stored in memory, and \(P\{m,k,l\} \rightarrow \{m+1,k,l\} = p_{sr}(1 - p_{sd})\).
5) if \(r\) receives the transmitted mixture, while \(d\) does not, and \(k + l = x\), then the mixture is dropped, and \(P\{m,k,l\} \rightarrow \{m,k\} = p_{sr}(1 - p_{sd})\).
6) if neither \(r\) nor \(d\) receives the packet, \(P\{m,k,l\} \rightarrow \{m,k\} = (1 - p_{sr})(1 - p_{sd})\).

The relay \(r\) transmits coded packets with probability \(1 - \alpha\), and the following state transitions may occur.

1) if \(r\) has no unique mixture to share, \(l = 0\), and \(k = 0\), \(P\{m,k,l\} \rightarrow \{m,k\} = 1\);
2) if \(r\) has no unique mixture to share, \(l = 0\), and \(k > 0\), \(P\{m,k,l\} \rightarrow \{m+1,k-1\} = 1\);
3) if \(r\) has a unique mixture to share, \(l > 0\), and
   - a unique mixture is sent, \(d\) receives successfully, \(P\{m,k,l\} \rightarrow \{m+1,k,l-1\} = \frac{l}{l+k}p_{rd}\).
• a unique mixture is sent, transmission is unsuccessful, \( P_{\{m,k,l\} \rightarrow \{m,k,l-1\}} = \frac{r}{l+k}(1-p_{rd}) \);
• \( k > 0 \), a shared mixture is sent, \( d \) receives successfully, \( P_{\{m,k,l\} \rightarrow \{m+1,k-1,l\}} = \frac{k}{l+k}p_{rd} \);
• \( k > 0 \), a shared mixture is sent, transmission is unsuccessful, \( P_{\{m,k,l\} \rightarrow \{m+1,k-1,l\}} = \frac{r}{l+k}(1-p_{rd}) \);

Transmission terminates when \( m+k = n \), and \( P_{\{m,n-m,0\} \rightarrow \{m,n-m,0\}} = 1 \). Again, we append a virtual terminating state \( S_T \), such that \( P_{\{m,n-m,l\} \rightarrow \{S_T\}} = 1 \), and \( P_{\{S_T\} \rightarrow \{S_T\}} = 1 \). With the added state, the Markov chain has one recurrent state only.

Figure 3 gives a sample Markov chain when \( n = 2 \). The states can be indexed linearly starting from \( S_T \) as state 0, to \((0,0,0)\) as the last state, which in this example is state number 14.

Our goal is to find the value of \( \alpha \) that minimizes the expected completion time \( T = T_{14} \) of the overall transmission process. Unlike the coding at \( r \) only case, the Markov chain now contains cyclic paths in addition to loops. The expected first passage time starting from different states is \( \bar{T} = (I - P_{\emptyset})^{-1}1 \). Again, the invertibility of \( I - P_{\emptyset} \) is guaranteed because \( P_{\emptyset} \) has entries less than 1 on the main diagonal, and \( I - P_{\emptyset} \) is a lower Hessenberg matrix with non-zero entries on the main diagonal.

C. Use of Systematic Codes

Systematic network codes are an attractive alternative to non-systematic random linear network codes, since they often reduce computation complexity and energy use, while maintaining the innovation of independent flows [19].

With a systematic code at \( s \) only, \( s \) can first broadcasts the uncoded packets one by one in order, it then computes random linear mixtures for all remaining packets transmitted from \( s \), once all \( n \) packets have been broadcasted once. \( r \) again acts as a size \( n \) queue, and forwards packets in its memory without checking if the packet is an actual data packet, or a linear mixture. Since every packet sent by \( s \) can be innovative with respect to \( r \) and \( d \), if we view each uncoded packet as an innovative mixture, the state evolution under this setup should be the same as the case where full RLNC is performed at \( s \) only.

If systematic coding is performed at \( s \), and RLNC is performed at \( r \), the system should give the same performance as the case where RLNC is performed at both nodes.
Another possibility is to perform systematic coding at both $s$ and $r$. $s$ first broadcasts uncoded packets one by one in order. It then computes a random linear mixture of all $n$ packets whenever a transmission opportunity becomes available. $r$ acts as a size $n$ queue. When it has the opportunity to transmit, it examines the next packet in the queue. If this packet is uncoded, $r$ transmits the uncoded packet directly. If this packet is coded, $r$ linearly combines all data it has in memory before sending out the mixture to $d$. The system performance under this setup should be upper-bounded by the full coding case, and lower-bounded by the coding at $s$ only case. We leave detailed analysis for future studies.

IV. Numerical Results

A. RLNC at the Relay Only

Figure 4 plots the expected completion time per transmitted data packet as a function of $\alpha$ for different values of $n$, while $p_{sd} = 0.5$, $p_{sr} = 0.8$, and $p_{rd} = 0.8$. The optimal $\alpha^*$ value that achieves the lowest $T^*/n$ is indicated by a large dot on each curve.

In this figure, when $\alpha = 1$, $r$ listens but does not transmit. Here if $n = 1$, the expected number of transmissions per data packet is 2. This is the solution to the ARQ scheme when $p_{sd} = 0.5$, where each packet is retransmitted until successfully received at $d$. When $n = 2$, the expected number of transmissions per data packet is 3. Observe that since $r$ is unused and $s$ does not code, $s$ simply retransmits one of the 2 uncoded data packets each round, until both are received at $d$. This scenario is similar to the coupon collector’s problem when there are 2 coupons available, except packet erasures need to be taken into account. When 2 coupons are to be collected, the expected number of trials until success is $2 \times (1 + \frac{1}{2}) = 3$. When divided by $p_{sd}$ and normalized by the number of packets, this solution leads to the value of 3, which is the value on the curve $n = 2$, at $\alpha = 1$, in Figure 4.

Another observation from this figure is that as $n$ increases, the expected completion time $T/n$ increases as well, i.e., the curves move upwards. The increase in $T/n$ comes from repeated transmissions at $s$. Since transmissions occur in a rateless fashion, with $s$ randomly choosing one from $n$ packets to sent, a packet to be retransmitted would have been received by $r$ or $d$ already with non-zero probability. This effect is especially significant towards the end of the transmission, when $d$ has collected most of the dof. In addition, the optimal $\alpha$ values, which correspond to the horizontal coordinates of the large dots, first decrease in value as $n$ goes from
1 to 5, then increases in value as $n$ increases to 20. This effect indicates that a tradeoff exists between the use of the relay and the amount of wasted repetitions by the source.

Figure 5 compares the expected completion time per data packet when the channel between $s$ and $d$ varies: $p_{sd}$ represents the packet transmission success probability. Here $n$ is set to be 5. As $p_{sd}$ increases, $s$ is used a larger fraction of the time. When we zoom into the region where $p_{sd}$ increases from 0.7 to 1, it can be observed that $r$ is not used as long as $p_{sd}$ is larger than 0.8. Note that this is similar to the condition $p_{sd} > p_{rd}$ as discussed Section II-A.

From Figures 4 and 5 we can conclude that RLNC at the relay $r$ only while operating in a rateless fashion is not an efficient transmission scheme. Figure 5 shows that given $n > 1$, the decision of whether to code depends on the channel condition. In the case where $r$ is used, Figure 4 shows that using ARQ without coding ($n = 1$, $\alpha = 1$) achieves the best expected completion time, or the best throughput. One issue with the $n = 1$ case is that each data packet, when transmitted successfully, requires an acknowledgement from $d$. Such frequent feedbacks are not energy efficient, neither is coding at $r$ with repetitions at $s$.

B. RLNC at the Source Only

Figure 6 plots the expected completion time per data packet as a function of $\alpha$, when the number of data packets to transmit by $s$ is varied. $r$ is assumed to have $x = n$ units of memory to store mixtures received from $s$. Unlike the coding at $r$ case, here $T/n$ decreases as $n$ becomes larger. This is because each packet sent by $s$ is innovative relative to $r$ and $d$. As more packets are combined, the probability that a mixture sent by $r$ is innovative becomes larger. In addition to reducing $T/n$, another advantage of coding $n$ number of packets together at $s$ is that the cost for feedback can be amortized over a large number of data packets.

Figure 7 plots the expected completion time per data packets as a function of $\alpha$, when $p_{sd}$ is varied between 0.1 to 1. The top curve corresponds to $p_{sd} = 0.1$, while the bottom one...
corresponds to $p_{sd} = 1$. When zooming into the region where $\alpha^*$ first approaches 1, $p_{sd}$ varies between 0.435 to 0.443. The optimal transmission scheme uses $r$ only when the transmission success probability $p_{sd}$ is less than 0.442. Also observe that the optimal $\alpha$ value lies in between 0.55 and 1. Since the curves are grouped close together, if $\alpha$ has been chosen to be in the range of 0.6 and 0.8, the loss in performance is approximately less than 30%.

So far we have shown only numerical results when the channel between $s$ and $d$ is varied. Figure 8 plots the expected completion time per data packet as a function of $\alpha$, for different channel values of $p_{sd}$, $p_{sr}$, and $p_{rd}$. Comparison among curves (1), (4) and (5) show that $r$ should be given more time to transmit when the tandem links from $s$ to $d$ through $r$ is more reliable than the direct link between $s$ and $d$. Comparison between (3) and (4), however, show that $r$ should not be used if the channel between $r$ and $d$ sees large packet erasure probabilities, even if the channel between $s$ and $r$ is relatively reliable. This observation echoes the decision in the full coding case, as discussed in Section II-A. Moreover, comparison among curves (2), (3), and (4) show that the optimal value of $\alpha$ is a function of transmission success probabilities on all links in the relay channel.

[Fig. 8 about here.]

Figure 9 plots the expected completion time per data packet as a function of $\alpha$, when the amount of memory at $r$ is varied. Here we assume $p_{sd} = 0.25$, $p_{sr} = 0.8$, $p_{rd} = 0.8$, and $n = 10$. Observe that as low as $x = 3$ units of memory suffices to achieve the expected completion time of the full memory $x = n$ case.

[Fig. 9 about here.]

C. Comparisons

Figure 10 compares the achievable rates of three cases: coding at $r$ only as discussed in Section III-A, coding at $s$ only as discussed in Section III-A, and coding at both $s$ and $r$, as discussed in Section II-A. Figure 11 plots the corresponding $\alpha^*$ values that achieve these rates. For the coding at $r$ and coding at $s$ cases, the metric being plotted is the inverse of the optimal expected transmission completion time per data packet ($T^*/n$). This inverse corresponds to the throughput of the systems under discussion. For the case where coding is performed at both $s$ and $r$, the achievable rate is computed using Equations (1) and (2).
When RLNC is performed at \( r \) only, as previous discussions have suggested, it is more desirable to combine fewer number of data packets. Since packets retransmitted from \( s \) are uncoded, a larger fraction of the repetitions are wasted, as in the coupon collector’s problem. In Figure 10, we have given the achievable rates for two different values of \( n \). When \( n = 1 \), network coding is not used, and the transmission degenerates into a routing scheme: \( s \) and \( r \) repeatedly send a single data packet until an acknowledgement is received from \( d \). Observe from Figure 11 that when the channel between \( s \) and \( d \) is poor (eg. \( p_{sd} = 0.2 \)), the route through \( r \) is preferred (\( \alpha^* \sim 0.65 \)), otherwise \( r \) is not used (\( \alpha^* = 1 \)). When \( n = 2 \), \( r \) still performs network coding, but only as the sum of two packets. The second curve (‘\( r, n = 2, x = 2 \)’) in Figure 10 reconfirms the conclusion that coding at \( r \) only is not throughput efficient.

One important remark for the coding at \( r \) only case is that the performance curve here is a result of the randomized transmission scheme, which was discussed in Section III-A. Another possibility is to start with a deterministic schedule, or a ‘systematic code,’ where uncoded packets are first sent in order, before \( s \) chooses packets at random for retransmissions. Since the first \( n \) packets are guaranteed to be innovative with respect to both \( r \) and \( d \), the behavior of the system should be similar to the coding everywhere case for the first \( n \) transmissions. Furthermore, as \( p_{sd} \) approaches 1, the achievable rate should also approach 1. We leave the detailed study of the systematic network codes in relay networks for future work.

When RLNC is performed at \( s \) only, Figure 10 shows that more than 69\% of the rate attained by the coding at both nodes scheme can be achieved. Here we have plotted the achievable rates for only one set of channel realizations, where \( p_{sr} = 0.8 \) and \( p_{rd} = 0.8 \). The exact amount of coding gain achievable by RLNC at the source should depend on the reliability of all three links in the relay channel. Also observe that the performance gap decreases as the channel between \( s \) and \( d \) becomes more reliable. Moreover, Figure 11 shows that when RLNC is performed at \( s \), transmissions from \( r \) are no longer required after \( p_{sd} \) becomes reasonably good (eg. \( p_{sd} > 0.442 \)). This is because transmissions from \( r \) also follow a randomized scheme, which leads to redundant repetitions that do not contribute additional dof to \( d \).

[Fig. 10 about here.]

[Fig. 11 about here.]
V. Conclusion

We analyzed the expected completion time, or equivalently, the throughput performance of network coding strategies in the wireless relay channel. Markov chain models were established to track the evolution of innovative packets when either or both the source and the relay perform random linear network coding. We showed through numerical evaluations that using a random code at the relay alone is not throughput efficient. Unless the channel between the source and the destination is very lossy, it is preferable to not use the relay at all. On the other hand, coding at the source alone can achieve a significant portion of the throughput gain attained when coding is performed at both the source and the relay. In addition, only a very small amount of memory is required at the relay to achieve good throughput performances when RLNC is performed at the source only.

Future work will consider the use of systematic codes, which have been mentioned in this paper but not studied in detail. Tradeoffs among energy consumption for coding, transmission, and reception, can be analyzed to find the most energy-efficient coding strategies. The energy-throughput tradeoff is also an important design question to answer. For example, in wireless body area networks, the direct link between a sensor and the data-collecting base station can be very lossy. Our analysis has shown that the use of the relay should be preferred in terms of throughput in this case, but it is unclear if such a strategy should also be preferred in terms of energy depletion.

References


List of Figures

1 Single relay unicast network, with corresponding flow hypergraph. ........................................ 20
2 Markov chain model, n = 2, with added terminating state $S_T$. Any three-tuple $(m, k, l)$ satisfying $m + k + l \leq n$ is a valid state. .................................................. 21
3 Markov Chain Model, coding at the source only, with added terminating state. The number of packets to send at the source is $n = 2$; the amount of memory at the relay $r$ is $x = 2$. .......................................................... 22
4 Coding at the relay $r$ only, normalized expected completion time $T/n$ vs. $\alpha$, as $n$ changes in value; $p_{sd} = 0.5$, $p_{sr} = 0.8$, $p_{rd} = 0.8$. The optimal $T^*/n$ is labeled with a large dot on each curve. .......................................................... 23
5 Coding at the relay $r$ only, normalized expected completion time $T/n$ vs. $\alpha$, as $p_{sd}$ changes in value; $p_{sr} = 0.8$, $p_{rd} = 0.8$, $n = 5$. .......................................................... 24
6 Coding at the source $s$ only, normalized expected completion time $T/n$ vs. $\alpha$, as $n$ changes in value; $p_{sd} = 0.5$, $p_{sr} = 0.8$, $p_{rd} = 0.8$. The optimal $T^*/n$ is labeled with a large dot on each curve. .......................................................... 25
7 Coding at the source $s$ only, normalized expected completion time $T/n$ vs. $\alpha$, as $p_{sd}$ varies between 0.1 to 1. $p_{sr} = 0.8$, $p_{rd} = 0.8$, $n = 10$, $x = n$. The top curve corresponds to $p_{sd} = 0.1$, while the bottom one corresponds to $p_{sd} = 1$. When zooming into the region where $\alpha^*$ first approaches 1, $p_{sd}$ is varied between 0.435 to 0.443. .......................................................... 26
8 Coding at the source $s$ only, normalized expected completion time $E/n$ vs. $\alpha$, as $p_{sr}$ and $p_{rd}$ change; $p_{sd} = 0.5$, $n = 10$, $x = n$. .......................................................... 27
9 Coding at the source $s$ only, normalized expected completion time $E/n$ vs. $\alpha$, as $x$ changes; $p_{sd} = 0.25$, $p_{sr} = 0.8$, $p_{rd} = 0.8$, $n = 10$. .......................................................... 28
10 Achievable throughput as a function of $p_{sd}$, $\frac{1}{E^*/n}$; $p_{sr} = 0.8$, $p_{rd} = 0.8$. .................. 29
11 Optimal $\alpha^*$ corresponding to throughput values in Figure 10; $p_{sr} = 0.8$, $p_{rd} = 0.8$. .... 30
Fig. 1. Single relay unicast network, with corresponding flow hypergraph.
Fig. 2. Markov chain model, n = 2, with added terminating state $S_T$. Any three-tuple $(m, k, l)$ satisfying $m + k + l \leq n$ is a valid state.
Fig. 3. Markov Chain Model, coding at the source only, with added terminating state. The number of packets to send at the source is $n = 2$; the amount of memory at the relay $r$ is $x = 2$. 
Fig. 4. Coding at the relay r only, normalized expected completion time $T/n$ vs. $\alpha$, as $n$ changes in value: $p_{sd} = 0.5$, $p_{sr} = 0.8$, $p_{rd} = 0.8$. The optimal $T^*/n$ is labeled with a large dot on each curve.
Fig. 5. Coding at the relay $r$ only, normalized expected completion time $T/n$ vs. $\alpha$, as $p_{sd}$ changes in value; $p_{sr} = 0.8$, $p_{rd} = 0.8$, $n = 5$. 
Fig. 6. Coding at the source $s$ only, normalized expected completion time $T/n$ vs. $\alpha$, as $n$ changes in value; $p_{sd} = 0.5$, $p_{sr} = 0.8$, $p_{rd} = 0.8$. The optimal $T^*/n$ is labeled with a large dot on each curve.
Fig. 7. Coding at the source \( s \) only, normalized expected completion time \( T/n \) vs. \( \alpha \), as \( p_{sd} \) varies between 0.1 to 1. \( p_{sr} = 0.8, p_{rd} = 0.8, n = 10, x = n \). The top curve corresponds to \( p_{sd} = 0.1 \), while the bottom one corresponds to \( p_{sd} = 1 \). When zooming into the region where \( \alpha^* \) first approaches 1, \( p_{sd} \) is varied between 0.435 to 0.443.
Fig. 8. Coding at the source $s$ only, normalized expected completion time $E/n$ vs. $\alpha$, as $p_{sr}$ and $p_{rd}$ change; $p_{sd} = 0.5$, $n = 10$, $x = n$. 

\begin{itemize}
    \item (1) $p_{sd}=0.25$, $p_{sr}=0.1$, $p_{rd}=0.1$
    \item (2) $p_{sd}=0.25$, $p_{sr}=0.25$, $p_{rd}=0.8$
    \item (3) $p_{sd}=0.25$, $p_{sr}=0.8$, $p_{rd}=0.25$
    \item (4) $p_{sd}=0.25$, $p_{sr}=0.8$, $p_{rd}=0.8$
    \item (5) $p_{sd}=0.1$, $p_{sr}=1$, $p_{rd}=0.8$
\end{itemize}
Fig. 9. Coding at the source \( s \) only, normalized expected completion time \( E/n \) vs. \( \alpha \), as \( x \) changes; \( p_{sd} = 0.25, p_{sr} = 0.8, p_{rd} = 0.8, n = 10 \).
Fig. 10. Achievable throughput as a function of $p_{sd} \cdot \frac{1}{T^*/n}$; $p_{sr} = 0.8$, $p_{rd} = 0.8$. 
Fig. 11. Optimal $\alpha^*$ corresponding to throughput values in Figure 10: $p_{se} = 0.8$, $p_{rd} = 0.8$. 