THE DYADIC LIFTING SCHEMES AND THE DENOISING OF DIGITAL IMAGES

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The dyadic lifting schemes, which generalize Sweldens lifting schemes, have been proposed for custom-design of dyadic and bi-orthogonal wavelets and their duals. Starting with dyadic wavelets and exploiting the control provided in the form of free parameters, one can custom-design dyadic as well as bi-orthogonal wavelets adapted to a particular application. To validate the usefulness of the schemes, two construction methods have been proposed for designing dyadic wavelet filters with higher number of vanishing moments; using these design techniques, spline dyadic wavelet filters have been custom-designed for the denoising of digital images, which exhibit enhanced denoising effects.

Keywords: Dyadic wavelet; Lifting scheme; Image denoising.

AMS Subject Classification: 22E46, 53C35, 57S20

1. Introduction

Wavelets have been successfully employed in many signal processing and image processing tasks. Many constructions of wavelets have been proposed over the years, for a thorough treatise consult\textsuperscript{14,15}. Among the outstanding construction schemes are the Sweldens lifting schemes, which are employed for custom-design construction of bi-orthogonal wavelet filters. Though these schemes provide a very nice and
useful construction tool to construct wavelets adapted to some particular application starting with a trivial set of bi-orthogonal wavelet filters, their domain is limited only to bi-orthogonal wavelets, especially those bi-orthogonal wavelets that can be found using the machinery proposed by Cohen et al. \(^6\); these schemes are at a loss to custom-design dyadic wavelet filters adapted to a certain application. On the other hand, dyadic wavelet filters are crucial to discrete dyadic wavelet transform \(^11\), which is versed with better potential of certain image processing tasks like edge-detection, feature extraction, and noise reduction because of its shift-invariant nature as compared to discrete down-sampling type wavelet transform. Mallat \(^11\) has carried out edge detection by using discrete dyadic wavelet transform with quadratic spline dyadic wavelets.

To overcome the limitations of the Sweldens lifting scheme, we propose the dyadic lifting schemes for the construction of second generation wavelets starting with dyadic wavelets. These schemes generalize Sweldens lifting schemes and can be applied for designing not only dyadic wavelet filters adapted to certain application requirements but also for the custom-design construction of bi-orthogonal wavelet filters starting with dyadic wavelet filters, even those bi-orthogonal wavelet filters which can not be found using the machinery prosed by Cohen et al. \(^6\).

Exploiting the proposed schemes, we propose two construction methods for custom-design of dyadic wavelets having desirable number of vanishing moments. Employing these design techniques, we design spline dyadic wavelet filters with higher numbers of vanishing moments for denoising digital images. The designed wavelet filters together with a threshold masking scheme have been used for denoising digital images corrupted with Gaussian noise; this has been partially presented in \(^4\). Comparisons with some state-of-the-art techniques addressing the issue of denoising digital images reveal that the wavelet filters designed using the proposed method exhibit better denoising effects.

The rest of the paper is organized as follows. In Section 2, we briefly introduce dyadic wavelets and present a variation of à trous algorithm. The proposed dyadic lifting schemes have been presented in Section 3. Section 4 describes a method for designing dyadic wavelet filters with desirable numbers of vanishing moments. The application of this design method for the construction of spline dyadic wavelets with higher numbers of vanishing moments has been presented in Section 5. Section 6 is concerned with a noise reduction method, which exploits the designed spline dyadic wavelet filters together with a threshold masking technique. Simulation results are given in Section 7. Section 8 concludes the paper.

2. Fast dyadic wavelet transform and the à trous algorithm

In this section, we establish the terminology that will be used in onward discussion and present a modified version of à trous algorithm for fast calculation of dyadic wavelet transform.

Let \(L^2(R)\) be the space of square integrable functions on real line \(R\) and \(\psi\) be
the Fourier transform of the function $\psi \in L^2(R)$ defined by

$$\hat{\psi}(\omega) = \int_{-\infty}^{+\infty} \psi(t)e^{-i\omega t}dt. \quad (2.1)$$

If there exist $A > 0$ and $B$ such that

$$A \leq \sum_{j=-\infty}^{+\infty} |\hat{\psi}(2^j\omega)|^2 \leq B, \quad (2.2)$$

then $\psi(t)$ is called a **dyadic wavelet function**. It follows from (2.2) that $\hat{\psi}(0) = 0$, i.e.

$$\int_{-\infty}^{+\infty} \psi(t)dt = 0. \quad (2.3)$$

**Dyadic wavelet transform** of $f(t)$ with the dyadic wavelet function $\psi(t)$ is defined by

$$Wf(u, 2^j) = \int_{-\infty}^{+\infty} f(t) \frac{1}{\sqrt{2^j}} \psi\left(\frac{t - u}{2^j}\right)dt = (f * \bar{\psi}_{2^j})(u), \quad (2.4)$$

where $u \in R$ is a translation parameter, and

$$\bar{\psi}_{2^j}(t) = \psi_{2^j}(-t) = \frac{1}{\sqrt{2^j}} \psi\left(\frac{-t}{2^j}\right).$$

The function $\phi(t)$ denotes a **scaling function** that satisfies the two-scale relation

$$\phi(t) = \sum_k h[k]\sqrt{2}\phi(2t - k). \quad (2.5)$$

It is usually normalized to unity i.e.

$$\int_{-\infty}^{+\infty} \phi(t)dt = 1.$$ From (2.5), it follows that the Fourier transform $\hat{\phi}$ of $\phi(t)$ must satisfy

$$\hat{\phi}(\omega) = \frac{1}{\sqrt{2}} \hat{h}(\frac{\omega}{2})\hat{\phi}(\frac{\omega}{2}), \quad (2.6)$$

where $\hat{h}(\omega)$ denotes the discrete Fourier transform

$$\hat{h}(\omega) = \sum_k h[k]e^{-i\omega k}. \quad (2.7)$$

Since $\hat{\phi}(0) = 1$, so (2.6) and (2.7) together yield $\hat{h}(0) = \sqrt{2}$ or $\sum_k h[k] = \sqrt{2}$. Using the scaling function $\phi(t)$ and the wavelet filter $g[k]$, a dyadic wavelet function is defined by

$$\psi(t) = \sum_k g[k]\sqrt{2}\phi(2t - k).$$
and its Fourier transform $\hat{\psi}$ satisfies
\[
\hat{\psi}(\omega) = \frac{1}{\sqrt{2}} \hat{g}(\omega/2) \hat{\phi}(\omega/2).
\] (2.8)

For the expansion of any $f(t) \in L^2$ by dyadic wavelet basis reconstruction condition must be satisfied \(^{11}\), which requires a dual scaling function and a dual wavelet function. The dual scaling function $\hat{\phi}(t)$ is defined by the two-scale relation
\[
\hat{\phi}(t) = \sum_k \hat{h}[k] \sqrt{2} \hat{\phi}(2t - k),
\]
and the dual wavelet function $\hat{\psi}(t)$ is given by
\[
\hat{\psi}(t) = \sum_k \hat{g}[k] \sqrt{2} \hat{\phi}(2t - k).
\]

The scaling functions and wavelets $\phi(t)$, $\psi(t)$, $\hat{\phi}(t)$ and $\hat{\psi}(t)$ are designed with the filters $h[k]$, $g[k]$, $\tilde{h}[k]$, and $\tilde{g}[k]$ respectively. Let $\hat{h}(\omega)$, $\hat{g}(\omega)$, $\tilde{h}(\omega)$, and $\tilde{g}(\omega)$ denote the Fourier transforms of $h[k]$, $g[k]$, $\tilde{h}[k]$, and $\tilde{g}[k]$ respectively, then they satisfy the following condition
\[
\hat{\tilde{h}}(\omega) \hat{h}^*(\omega) + \hat{\tilde{g}}(\omega) \hat{g}^*(\omega) = 2.
\] (2.9)

This condition is called Dyadic Perfect Reconstruction (DPR) condition. If this condition is satisfied then $f \in L^2(R)$ can be expanded as follows \(^{11}\)
\[
f(t) = \sum_{j=-\infty}^{+\infty} \frac{1}{2^j} (Wf(\cdot, 2^j) * \tilde{\psi}_{2^j})(t).
\]

Using condition (2.9) Mallat proposed à trous algorithm for fast calculation of dyadic wavelet transform. In the following proposition, we present a modified version of this algorithm, which involves filters directly and is easy to implement.

**Proposition**. If the filters satisfy the condition (2.9), then
\[
a_{j+1}[n] = \sum_k h[k] a_j[n + 2^j k], \quad j = 0, 1, ..., \quad (2.10)
\]
\[
d_{j+1}[n] = \sum_k g[k] a_j[n + 2^j k], \quad j = 0, 1, ..., \quad (2.11)
\]
and
\[
a_j[n] = \frac{1}{2} \sum_k (\tilde{h}[k] a_{j+1}[n - 2^j k] \\
+ \tilde{g}[k] d_{j+1}[n - 2^j k]), \quad j = 0, 1, ..., \quad (2.12)
\]

Note that here $a_0[n]$ is given by $a_0[n] = \int_{-\infty}^{+\infty} f(t) \phi(t-n)dt$. Formulae (2.10) and (2.11) decompose a 1-D signal and so are called decomposition formulae. Formula (2.12) reconstructs the 1-D signal and is referred to as reconstruction formula. In case of images, which are 2-D signals, first these formulae are applied in the horizontal direction, and then in the vertical direction.
3. Dyadic lifting schemes

After having established the terminology and the necessary machinery for dyadic wavelets in the previous section, we now present dyadic lifting schemes for the custom-design of dyadic and bi-orthogonal wavelets starting with dyadic wavelets. Wavelet filters that satisfy DPR condition (2.9) are termed as dyadic wavelet filters, so this condition plays a very important role in designing lifting dyadic wavelet filters.

**Theorem 1.** If the filters \( h^o, g^o, \hat{h}^o \) and \( \hat{g}^o \) are dyadic wavelet filters then, the lifted filters \( h, g, \hat{h} \) and \( \hat{g} \) defined by

\[
\begin{align*}
  h[k] &= h^o[k], \\
  g[k] &= g^o[k] - \sum_m s[m]h^o[k - m], \\
  \hat{h}[k] &= \hat{h}^o[k] + \sum_m s[m]\hat{g}^o[k - m], \\
  \hat{g}[k] &= \hat{g}^o[k],
\end{align*}
\]

(3.1)

are also dyadic wavelet filters.

Note that here \( s[m] \) are free parameters which provide control for custom-design construction of dyadic and bi-orthogonal wavelet filters. The suffix ‘\( o \)’ on filters stands for ‘old’ and indicates that the respective filters are old filters which are used as starting filters. The expressions given by (3.1) are called dyadic lifting scheme.

**Proof.** If the old filters are dyadic wavelet filters then their Fourier transforms \( \hat{h}^o(\omega), \hat{g}^o(\omega), \tilde{h}^o(\omega) \) and \( \tilde{g}^o(\omega) \) satisfy (2.9), which can be written in matrix form as follows

\[
\begin{pmatrix}
  \hat{h}^o(\omega) & \hat{g}^o(\omega)
\end{pmatrix}
\begin{pmatrix}
  \hat{h}^o(\omega) \\
  \hat{g}^o(\omega)
\end{pmatrix} = 2.
\]

(3.2)

The Fourier transforms of the expressions in (3.1) are given by

\[
\begin{align*}
  \hat{h}(\omega) &= \hat{h}^o(\omega), \\
  \hat{g}(\omega) &= \hat{g}^o(\omega) - \hat{s}(\omega)\hat{h}^o(\omega), \\
  \tilde{h}(\omega) &= \tilde{h}^o(\omega) + \hat{s}(\omega)\tilde{g}^o(\omega), \\
  \tilde{g}(\omega) &= \tilde{g}^o(\omega).
\end{align*}
\]

(3.3)

Writing (3.3) in matrix form and factorizing it, we obtain

\[
\begin{pmatrix}
  \hat{h}(\omega) & \tilde{h}(\omega)
\end{pmatrix}
\begin{pmatrix}
  \hat{h}^o(\omega) \\
  \hat{g}^o(\omega)
\end{pmatrix} = \begin{pmatrix}
  \hat{h}^o(\omega) + \hat{s}(\omega)\tilde{g}^o(\omega) & \hat{g}^o(\omega)
\end{pmatrix}
\begin{pmatrix}
  \hat{h}^o(\omega) \\
  \tilde{g}^o(\omega)
\end{pmatrix} = \begin{pmatrix}
  \hat{h}^o(\omega) & \tilde{g}^o(\omega)
\end{pmatrix}
\begin{pmatrix}
  1 & 0 \\
  \hat{s}(\omega) & 1
\end{pmatrix}
\begin{pmatrix}
  \hat{h}^o(\omega) \\
  \hat{g}^o(\omega)
\end{pmatrix} = \begin{pmatrix}
  \hat{h}^o(\omega) & \tilde{g}^o(\omega)
\end{pmatrix}
\begin{pmatrix}
  \hat{h}^o(\omega) \\
  \hat{g}^o(\omega)
\end{pmatrix}.
\]
In view of (3.2) it is obvious that (3.3) satisfies the DPR condition (2.9) i.e. the lifted filters are also dyadic wavelet filters.

The following proposition proves that the lifting scheme introduced by Sweldens \cite{sweldens97} is a special case of the dyadic lifting scheme introduced in Theorem 1. Note that the space of bi-orthogonal wavelets is a subspace of that of the dyadic wavelets.

**Proposition 2.** If the old filters \( h^o, g^o, \tilde{h}^o \) and \( \tilde{g}^o \) are bi-orthogonal wavelet filters, then the scheme (3.1) with \( s[2m + 1] = 0 \) becomes Sweldens lifting scheme \cite{sweldens97}.

**Proof.** Putting \( s[2m + 1] = 0 \) in (3.1) and replacing \( s[2m] \) with \( s[m] \), we get

\[
\begin{align*}
    h[k] &= h^o[k], \\
    g[k] &= g^o[k] - \sum m s[m] h^o[k - 2m], \\
    \tilde{h}[k] &= \tilde{h}^o[k] + \sum m s[-m] \tilde{g}^o[k - 2m], \\
    \tilde{g}[k] &= \tilde{g}^o[k].
\end{align*}
\]

This is Sweldens lifting scheme \cite{sweldens97}.

It is obvious from Theorm 1 that dyadic wavelet scheme changes only wavelet and dual scaling filters and leaves scaling and dual wavelet filters untouched. In the following theorem we propose dyadic dual lifting scheme for custom-design of dual wavelets.

**Theorem 2.** If the old filters \( h^o, g^o, \tilde{h}^o \) and \( \tilde{g}^o \) are dyadic wavelet filters, then the lifted filters \( h, g, \tilde{h} \) and \( \tilde{g} \) defined by

\[
\begin{align*}
    h[k] &= h^o[k] + \sum m r[-m] g^o[k - m], \\
    g[k] &= g^o[k], \\
    \tilde{h}[k] &= \tilde{h}^o[k], \\
    \tilde{g}[k] &= \tilde{g}^o[k] - \sum m r[m] \tilde{h}^o[k - m]
\end{align*}
\]

are also dyadic wavelet filters.

Note that here \( r[m] \) are free parameters. The expression given by (3.4) altogether are termed as the dyadic dual lifting scheme.

**Proof.** The proof is similar to that of Theorem 1.

The following proposition proves that sweldens dual lifting scheme is a special case of dyadic dual lifting scheme.

**Proposition 3.** If the old filters \( h^o, g^o, \tilde{h}^o \) and \( \tilde{g}^o \) are bi-orthogonal wavelet filters, then (3.4) with \( r[2m + 1] = 0 \) becomes Sweldens dual lifting scheme \cite{sweldens97}.

**Proof.** Putting \( r[2m + 1] = 0 \) in (3.4) and replacing \( r[2m] \) by \( r[m] \) yield

\[
\begin{align*}
    h[k] &= h^o[k] + \sum m r[-m] g^o[k - 2m], \\
    g[k] &= g^o[k], \\
    \tilde{h}[k] &= \tilde{h}^o[k], \\
    \tilde{g}[k] &= \tilde{g}^o[k] - \sum m r[m] \tilde{h}^o[k - 2m],
\end{align*}
\]
which is Sweldens dual lifting scheme 14.

Alternating between dyadic lifting scheme and dyadic dual lifting scheme leads to cakewalk construction of wavelets with desired properties on primal and dual wavelets. Note that Sweldens lifting schemes involve only even indexed free parameters, whereas dyadic lifting schemes include both even and odd indexed free parameters. Even and odd indexed parameters can be manipulated separately to have desirable properties on primal and dual wavelets. For example, even indexed parameters can be determined so as to ensure that the lifted filters are bi-orthogonal and odd index parameters can be fixed for gain control 2. It is interesting to note that with Sweldens lifting scheme only those bi-orthogonal wavelet filters can be constructed which satisfy the bi-orthogonal perfect reconstruction (BPR) condition 15. Since BPR is stronger than DPR, so dyadic lifting schemes can construct bi-orthogonal wavelet filters which cannot be constructed with Sweldens lifting schemes 5.

The following two theorems deal with the implementation of the fast dyadic wavelet transform; they show how the implementation of fast dyadic wavelet transforms can be facilitated and accelerated without explicitly computing the new filters.

**Theorem 3** The forward fast lifting dyadic wavelet transform is given by

\[
a_{j+1}[n] = \sum_k h^o[k] a_j[n + 2^j k], \quad j = 0, 1, 2, \cdots \tag{3.5}
\]

\[
d_{j+1}[n] = \sum_k \left( g^o[k] - \sum_m s[k - m] h^o[m] \right) a_j[n + 2^j k], \quad j = 0, 1, 2, \cdots \tag{3.6}
\]

The inverse transform is

\[
a_j[n] = \frac{1}{2} \sum_k \left( \tilde{h}^o[k] + \sum_m s[m - k] \tilde{g}^o[m] \right) a_{j+1}[n - 2^j k]
\]

\[+ \frac{1}{2} \sum_k \tilde{g}^o[k] d_{j+1}[n - 2^j k], \quad j = 0, 1, 2, \cdots \tag{3.7}
\]

**Theorem 4** The forward fast dual lifting dyadic wavelet transform is given by

\[
a_{j+1}[n] = \sum_k h^o[k] + \sum_m r[m - k] g^o[m] a_j[n + 2^j k], \quad j = 0, 1, 2, \cdots \tag{3.9}
\]

\[
d_{j+1}[n] = \sum_k g^o[k] a_j[n + 2^j k], \quad j = 0, 1, 2, \cdots \tag{3.10}
\]

The inverse transform is

\[
a_j[n] = \frac{1}{2} \sum_k \tilde{h}^o[k] a_{j+1}[n - 2^j k]
\]

\[+ \frac{1}{2} \sum_k \tilde{g}^o[k] + \sum_m s[m - k] \tilde{g}^o[m] d_{j+1}[n - 2^j k], \quad j = 0, 1, 2, \cdots \tag{3.11}
\]
4. Dyadic wavelet filters with vanishing moments

In this section, exploiting dyadic lifting schemes we present two constructive results with which one can design dyadic wavelets having desirable numbers of vanishing moments. A dyadic wavelet $\psi$ has $p$ vanishing moments if

$$M_k = \int_{-\infty}^{+\infty} t^k \psi(t) dt = 0 \quad \text{for} \quad 0 \leq k < p. \quad (4.1)$$

It is easy to show that (4.1) is equivalent to each of the conditions

$$\frac{d^k \hat{\psi}(\omega)}{d\omega^k} \bigg|_{\omega=0} = 0 \quad \text{for} \quad 0 \leq k < p,$$

and

$$\frac{d^k \hat{g}(\omega)}{d\omega^k} \bigg|_{\omega=0} = 0, \quad \text{for} \quad 0 \leq k < p. \quad (4.2)$$

The following theorem shows how can we increase the number of vanishing moments of a lifted wavelet by fixing the degrees of freedom provided by the free parameters involved in the dyadic lifting scheme.

**Theorem 5.**

(1) If $\hat{g}^o$ in Theorem 1 satisfies $\hat{g}^o(0) \neq 0$, then a sufficient condition for $\hat{g}(0) = 0$ is

$$\hat{s}(0) = \frac{\hat{g}^o(0)}{\hat{h}^o(0)}.$$

(2) If a dyadic wavelet $\psi^o$ corresponding to the wavelet filter $g^o$ has $p$ vanishing moments, then a sufficient condition for the lifted wavelet $\psi$ corresponding to the lifted filter $g$ to have at least $p + 1$ vanishing moments is that the Fourier transform $\hat{s}(\omega)$ of $s[m]$ satisfies

$$\frac{d^k \hat{s}(\omega)}{d\omega^k} \bigg|_{\omega=0} = 0, \quad 0 \leq k < p \quad (4.3)$$

and

$$\frac{d^p \hat{s}(\omega)}{d\omega^p} \bigg|_{\omega=0} = \frac{1}{\hat{h}^o(0)} \frac{d^p \hat{g}^o(\omega)}{d\omega^p} \bigg|_{\omega=0}. \quad (4.4)$$

**Proof.** The proof of assertion (1) is obvious. We prove (2). Since $\psi^o$ has $p$ vanishing moments, so

$$\frac{d^k \hat{g}^o(\omega)}{d\omega^k} \bigg|_{\omega=0} = 0, \quad 0 \leq k < p, \quad (4.5)$$

and

$$\frac{d^p \hat{g}^o(\omega)}{d\omega^p} \bigg|_{\omega=0} \neq 0. \quad (4.6)$$
Assume that the Fourier transform \( \hat{s}(\omega) \) of \( s[m] \) satisfies (4.3) and (4.4). To prove the assertion (2) it is enough to show that \( \psi \) has at least \( p + 1 \) vanishing moments.

The Fourier transform of the third line of (3.1) yields

\[
\hat{g}(\omega) = \hat{g}^o(\omega) - \hat{s}(\omega)\hat{h}(\omega). \tag{4.7}
\]

Differentiating (4.7), we get

\[
\frac{d^k \hat{g}(\omega)}{d\omega^k} = \frac{d^k \hat{g}^o(\omega)}{d\omega^k} - \sum_{m=0}^{k} \binom{k}{m} \frac{d^m \hat{s}(\omega)}{d\omega^m} \frac{d^{k-m} \hat{h}(\omega)}{d\omega^{k-m}}. \tag{4.8}
\]

In the case 0 ≤ \( k < p \), it follows from (4.5) and (4.8)

\[
\frac{d^k \hat{g}(\omega)}{d\omega^k} \bigg|_{\omega=0} = 0, \quad 0 \leq k < p.
\]

In case of \( k = p \), it follows from (4.3) and (4.8) that

\[
\frac{d^p \hat{g}(\omega)}{d\omega^p} \bigg|_{\omega=0} = 0,
\]

which means that \( \psi \) has at least \( p + 1 \) vanishing moments.

Applying Theorem 5 repeatedly, one can construct dyadic highpass analysis filters with \( q(>p) \) vanishing moments.

Theorem 5 deals with dyadic primal wavelets. The following theorem is concerned with dual wavelets.

Theorem 6.

(1) If \( \tilde{\hat{g}}^o(\omega) \) in Theorem 2 satisfies \( \tilde{\hat{g}}^o(0) \neq 0 \), then a sufficient condition for \( \tilde{g}(0) = 0 \) is

\[
\tilde{r}(0) = \frac{\tilde{\hat{g}}^o(0)}{\hat{h}^o(0)}.
\]

(2) If the dual wavelet \( \tilde{\psi}^o \) corresponding to the wavelet filter \( \tilde{\hat{g}}^o \) has \( p \) vanishing moments, then a sufficient condition for the lifted dual wavelet \( \tilde{\psi} \) corresponding to
\hat{g} to have at least \( p + 1 \) vanishing moments is that the Fourier transform \( \hat{r}^{(m)}(\omega) \) of \( r[m] \) satisfies
\[
\frac{d^k \hat{r}(\omega)}{d\omega^k} \bigg|_{\omega=0} = 0, \quad 0 \leq k < p
\]
and
\[
\frac{d^p \hat{r}(\omega)}{d\omega^p} \bigg|_{\omega=0} = \frac{1}{\hat{h}^o(0)} \frac{d^p \hat{g}^o(\omega)}{d\omega^p} \bigg|_{\omega=0}.
\]

**Proof.** The proof is similar to that of Theorem 5.

In Section 5, spline dyadic wavelet filters with higher numbers of vanishing moments are constructed by the successive use of Theorem 1 and 2.

5. **Design of spline dyadic wavelet filters with vanishing moments**

Now employing the constructive results of the previous section, we design filters for spline dyadic wavelets having higher numbers of vanishing moments, which are used for denoising digital images in the following section. We opted for spline dyadic wavelets because these are smooth and symmetric. We choose
\[
\hat{h}(\omega) = \sqrt{2} e^{-i\varepsilon \omega / 2} (\cos \frac{\omega}{2})^{m+1}, \tag{5.1}
\]
where \( m \) is a nonnegative integer, and
\[
\varepsilon = \begin{cases} 
1, & \text{if } m \text{ is even}, \\
0, & \text{otherwise}.
\end{cases}
\]

We choose the dual scaling filter \( \hat{\tilde{h}}[k] \) such that
\[
\hat{\tilde{h}}(\omega) = \hat{h}(\omega). \tag{5.2}
\]

The product \( \hat{g}^*(\omega)\hat{\tilde{g}}(\omega) \) is constrained by DPR condition (2.9). If we choose
\[
\hat{\tilde{g}}(\omega) = (-i)^r \sqrt{2} e^{-i\frac{\tau + 1}{2} \omega} (\sin \frac{\omega}{2})^r, \tag{5.3}
\]
where \( r = 1, 2 \) and
\[
\tau = \begin{cases} 
1, & r = 1, \\
0, & r = 2,
\end{cases}
\]
we have
\[
\hat{\tilde{g}}(\omega) = (-i)^r \sqrt{2} e^{-i\frac{\tau + 1}{2} \omega} \times (\sin \frac{\omega}{2})^{2r} \sum_{l=0}^{m} (\cos \frac{\omega}{2})^{2l}. \tag{5.4}
\]

The spline dyadic wavelet filters described in (5.1) through (5.4) are Finite Impulse Response (FIR) filters. Table 1 lists the values of the filters in case of \( r = 2 \) and \( m = 1 \), where suffix 'o' on each filter means 'old'.
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Table 1. Spline dyadic wavelet filters in case of $r = 2, m = 1.$

<table>
<thead>
<tr>
<th>$k$</th>
<th>$h_0^0[k] / \sqrt{2}$</th>
<th>$g_0^0[k] / \sqrt{2}$</th>
<th>$h_0^1[k] / \sqrt{2}$</th>
<th>$g_0^1[k] / \sqrt{2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1</td>
<td>0.25</td>
<td>0.25</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>0.50</td>
<td>-0.25</td>
<td>0.50</td>
<td>0.25</td>
</tr>
<tr>
<td>1</td>
<td>0.25</td>
<td>0.50</td>
<td>0.25</td>
<td>1.50</td>
</tr>
<tr>
<td>2</td>
<td></td>
<td>-0.25</td>
<td></td>
<td>0.25</td>
</tr>
</tbody>
</table>

Since $\hat{g}_0^0(\omega)$ satisfies
\[
\left. \frac{d^k \hat{g}_0^0(\omega)}{d\omega^k} \right|_{\omega=0} = 0, \quad 0 \leq k \leq 1,
\]
the filter $g_0^0[k]$ shown in Table 1 has 2 vanishing moments. From this filter, we can design a highpass analysis filter with 3 or more vanishing moments using Theorem 5. The condition that $\hat{s}(\omega)$ must satisfy is
\[
\left. \frac{d^k \hat{s}(\omega)}{d\omega^k} \right|_{\omega=0} = 0, \quad 0 \leq k \leq 1,
\]
\[
\left. \frac{d^2 \hat{s}(\omega)}{d\omega^2} \right|_{\omega=0} = \frac{1}{\sqrt{2}} \left. \frac{d^2 \hat{g}_0^0(\omega)}{d\omega^2} \right|_{\omega=0} = \frac{1}{2}.
\]
Considering this condition and the symmetry of $\hat{s}(\omega)$, we choose
\[
\hat{s}(\omega) = e^{-i\omega} \sin^2 \omega/4.
\]
\[
\hat{g}^l(\omega) = \frac{1}{\sqrt{2}} \hat{s}(\omega).
\]

Table 2. Highpass analysis filters with 4 vanishing moments.

<table>
<thead>
<tr>
<th>$k$</th>
<th>$h_0^0[k] / \sqrt{2}$</th>
<th>$g_0^0[k] / \sqrt{2}$</th>
<th>$h_0^1[k] / \sqrt{2}$</th>
<th>$g_0^1[k] / \sqrt{2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-3</td>
<td></td>
<td>-0.015625</td>
<td></td>
<td></td>
</tr>
<tr>
<td>-2</td>
<td>0.015625</td>
<td>-0.093750</td>
<td></td>
<td></td>
</tr>
<tr>
<td>-1</td>
<td>0.25</td>
<td>0.031250</td>
<td>0.265625</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>0.50</td>
<td>-0.265625</td>
<td>0.687500</td>
<td>0.25</td>
</tr>
<tr>
<td>1</td>
<td>0.25</td>
<td>0.437500</td>
<td>0.265625</td>
<td>1.50</td>
</tr>
<tr>
<td>2</td>
<td>-0.265625</td>
<td>-0.093750</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>0.031250</td>
<td>-0.015625</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>0.015625</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
It means that the filter $g_l[k]$ listed in Table 2 has 4 vanishing moments. Applying Theorem 5 again, we can design a highpass analysis filter $g_{ll}[k]$ with 5 or more vanishing moments. Indeed, $\hat{s}(\omega)$ must satisfy
\[
\frac{d^k \hat{s}(\omega)}{d\omega^k} \bigg|_{\omega=0} = 0, \quad 0 \leq k \leq 3,
\]
\[
\frac{d^4 \hat{s}(\omega)}{d\omega^4} \bigg|_{\omega=0} = 3\sqrt{2}.
\]
From this condition and symmetry of $\hat{s}(\omega)$, we can choose $\hat{s}(\omega) = e^{-i\omega \sin^4 \omega/8}$. This yields highpass analysis filters with 6 vanishing moments, which are listed in Table 3.

<table>
<thead>
<tr>
<th>$k$</th>
<th>$h_{\sqrt{2}}$</th>
<th>$g_{\sqrt{2}}$</th>
<th>$h'_{\sqrt{2}}$</th>
<th>$g'_{\sqrt{2}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-5</td>
<td>0.001953125</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>-4</td>
<td>-0.001953125</td>
<td>0.011718750</td>
<td></td>
<td></td>
</tr>
<tr>
<td>-3</td>
<td>-0.003906250</td>
<td>-0.021484375</td>
<td></td>
<td></td>
</tr>
<tr>
<td>-2</td>
<td>0.021484375</td>
<td>-0.140625000</td>
<td></td>
<td></td>
</tr>
<tr>
<td>-1</td>
<td>0.25</td>
<td>0.046875000</td>
<td>0.269531250</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>0.50</td>
<td>-0.269531250</td>
<td>0.757812500</td>
<td>0.25</td>
</tr>
<tr>
<td>1</td>
<td>0.25</td>
<td>0.414062500</td>
<td>0.269531250</td>
<td>1.50</td>
</tr>
<tr>
<td>2</td>
<td>-0.269531250</td>
<td>-0.140625000</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>0.046875000</td>
<td>-0.021484375</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>0.021484375</td>
<td>0.011718750</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>-0.003906250</td>
<td>0.001953125</td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>-0.001953125</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Of course, we can construct highpass synthesis filters with even higher numbers of vanishing moments using Theorem 5. However, these filters do not give good denoising results in simulation. So they are omitted.

Now we consider the spline dyadic wavelet in case of $r = 1$ and $m = 2$, whose filters are given in Table 4.
Since $\hat{g}^o(\omega)$ satisfies
\[ \hat{g}^o(0) = 0, \quad \frac{d\hat{g}^o(\omega)}{d\omega} \bigg|_{\omega=0} = \frac{-\sqrt{2}}{2}i, \]
the filter $g^o[k]$ given in Table 4 has just 1 vanishing moment. Theorem 5 enables the construction of a highpass analysis filter $g^l[k]$ with 2 or more vanishing moments.

From the condition
\[ s(0) = 0, \quad \frac{d\hat{s}(\omega)}{d\omega} \bigg|_{\omega=0} = \frac{1}{\sqrt{2}} \frac{d\hat{g}^o(\omega)}{d\omega} \bigg|_{\omega=0} = \frac{i}{2}, \]
we can select $s(\omega) = -i \sin \omega/2$.

Table 5. Highpass analysis filters with 3 vanishing moments.

<table>
<thead>
<tr>
<th>$k$</th>
<th>$h^l[k]$ $\sqrt{2}$</th>
<th>$g^l[k]$ $\sqrt{2}$</th>
<th>$h^o[k]$ $\sqrt{2}$</th>
<th>$g^o[k]$ $\sqrt{2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-3</td>
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<td>0.0546875</td>
<td>-0.03125</td>
<td>-0.0078125</td>
</tr>
<tr>
<td>-2</td>
<td>-0.03125</td>
<td>0.0546875</td>
<td>-0.03125</td>
<td>-0.0078125</td>
</tr>
<tr>
<td>-1</td>
<td>0.125</td>
<td>-0.09375</td>
<td>0.2890625</td>
<td>-0.21875</td>
</tr>
<tr>
<td>0</td>
<td>0.375</td>
<td>-0.526250</td>
<td>0.1484375</td>
<td>-0.68750</td>
</tr>
<tr>
<td>1</td>
<td>0.375</td>
<td>0.526250</td>
<td>0.1484375</td>
<td>0.68750</td>
</tr>
<tr>
<td>2</td>
<td>0.125</td>
<td>0.09375</td>
<td>0.2890625</td>
<td>0.21875</td>
</tr>
<tr>
<td>3</td>
<td>0.03125</td>
<td>0.0546875</td>
<td>0.03125</td>
<td>0.0078125</td>
</tr>
<tr>
<td>4</td>
<td>0</td>
<td>0.0546875</td>
<td>0</td>
<td>0.0078125</td>
</tr>
</tbody>
</table>

With this $s$, $\hat{g}^l(\omega)$ satisfies
\[ \frac{d^k\hat{g}^l(\omega)}{d\omega^k} \bigg|_{\omega=0} = 0, \quad 0 \leq k \leq 2, \]
\[ \frac{d^3\hat{g}^l(\omega)}{d\omega^3} \bigg|_{\omega=0} = -\frac{3\sqrt{2}}{2}i. \]
It follows that the filter $g_l[k]$ shown in Table 5 has 3 vanishing moments. A further application of Theorem 5 to Table 5 constructs a highpass analysis filter $g_{ll}[k]$ with 4 or more vanishing moments. The Fourier transform $\hat{s}(\omega)$ must satisfy

$$\frac{d^k \hat{s}(\omega)}{d\omega^k} \bigg|_{\omega=0} = 0, \quad 0 \leq k \leq 2,$$

$$\frac{d^3 \hat{s}(\omega)}{d\omega^3} \bigg|_{\omega=0} = \frac{1}{\sqrt{2}} \frac{d^3 \hat{g}(\omega)}{d\omega^3} \bigg|_{\omega=0} = -\frac{3}{2}.$$

This condition and symmetry of $\hat{s}(\omega)$ yield $\hat{s}(\omega) = -i \sin^3 \omega/4$. This result gives highpass analysis filters with 5 vanishing moments as shown in Table 6.

<table>
<thead>
<tr>
<th>$k$</th>
<th>$\frac{\hat{g}[k]}{\sqrt{2}}$</th>
<th>$\frac{\hat{g}[k]}{\sqrt{2}}$</th>
<th>$\frac{\hat{g}[k]}{\sqrt{2}}$</th>
<th>$\frac{\hat{g}[k]}{\sqrt{2}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-5</td>
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<td>0.0009765625</td>
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<td>0.006359375</td>
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<td>-3</td>
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<tr>
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</tr>
<tr>
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</tr>
<tr>
<td>6</td>
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<td>0.0009765625</td>
<td>0.0009765625</td>
<td>0.0009765625</td>
</tr>
</tbody>
</table>

6. Application to noise reduction

In general, a discrete noisy signal can be represented as

$$y(n) = f(n) + \sigma e(n), \quad 0 \leq n \leq N - 1,$$

where $y$ represents noisy signal, $f$ is an unknown deterministic signal, $e$ is standard Gaussian white noise, and $\sigma$ is a noise level. The problem of de-noising a signal is to find a nearly optimal estimator $\hat{f}$ of $f$ that is as smooth as $f^8$. Wavelet transform has been successfully employed to solve this problem $^8,^{13,4}$. In this section, to validate the usefulness of our dyadic lifting schemes, we employ the wavelet filters designed in the previous section for de-noising digital graylevel images. For de-noising images, two different wavelet based approaches exist in the literature $^8,^{13,9,12}$: hard logic masking and softer logic masking. In hard logic masking method, the wavelet
coefficients that are below a certain threshold value are set to zero. This method is not suitable when the noise level is high because it causes considerable edge distortions. In softer logic masking method, the wavelet coefficients that are below a certain threshold value are reduced gradually to zero rather than setting them abruptly to zero. This approach depicts more reasonable de-noising results, and so to demonstrate the usefulness of our lifted filters, we go for this approach. In the following, we describe a modified version of the softer logic masking method of Shi et al.\textsuperscript{13}, which has been partly presented in\textsuperscript{4}.

Let $C_0[n, l]$ be an image corrupted by Gaussian noise. Applying the decomposition formulae (2.10) and (2.11) in horizontal and vertical directions to $C_0[n, l]$ successively, we compute the lowpass components $C_j[n, l]$ and three kinds of highpass components $D^m_j[n, l], 1 \leq m \leq 3, j = 1, 2, 3, \ldots$ The filters involved in (2.10) and (2.11) are the lifted filters $h[k]$ and $g[k]$ with different numbers of vanishing moments constructed in Section 5. The highpass components $D^m_j[n, l], 1 \leq m \leq 3$ are converted into 1-D signals $D^m_j[k], 1 \leq m \leq 3$ to apply the noise reduction method. The threshold masking procedure for $D^m_j[k]$ involves the following steps:

(1) Compute the discrete Fourier transform $\hat{D}^m_j(\omega)$ of $D^m_j[k], j = 1, 2, 3, \ldots$. The amplitudes of $\hat{D}^m_j(\omega)$ are different and to adjust them, we need to rescale $D^m_j[k]$ with respect to some common standard. For this purpose we compute the following normalizing factor.

$$\lambda^m_j = \frac{1}{\max_{\omega \in [0, 2\pi]} |\hat{D}^m_j(\omega)|}.$$

(2) Rescale the 1-D signals $D^m_j[k]$ i.e. $\tilde{D}^m_j[k] = \lambda^m_j D^m_j[k]$.

(3) Normalize $\tilde{D}^m_j[k]$ as

$$\tilde{D}^m_j[k] = \frac{|\tilde{D}^m_j[k]|}{\max_{j,m} |\tilde{D}^m_j[k]|}.$$

(4) Define a threshold function $a(t)$ by

$$a(t) = \frac{1}{1 + \alpha e^{-\beta t}}$$

where $\alpha$ and $\beta$ are parameters.

(5) Compute

$$D^{m,*}_j[k] = D^m_j[k] a\left(\frac{\tilde{D}^m_j[k] - \zeta}{1 - \zeta}\right)$$

where $\zeta$ is a normalized adaptive threshold and is set equal to $\sigma \sqrt{\frac{2 \log N}{N}}$. Here $N$ is the number of pixels in the image, and the noise variance $\sigma^2$ is computed by

$$\sigma^2 = \frac{Var\{D^m_j[k]\}}{0.6745},$$

\textit{The dyadic lifting schemes and the denoising of digital images}
where the factor 0.6745 is chosen by the calibration with the Gaussian distribution.

After performing the above steps, convert 1-D signals $D^m_*[k]$, $1 \leq m \leq 3$ into 2-D image signals $D_j^m*[n,l]$, $1 \leq m \leq 3$ and reconstruct $\hat{C}_j^[n,l]$ by substituting these components and lowpass components $C_j[n,l]$ into the 2-D version of the inverse formula (3.7). Repeat this procedure until the noise free image $C^*_0[n,l]$ is recovered.

In practice, the performance of de-noising procedures typically deteriorates when the signal is short. In case of dyadic wavelet transform, the size of subbands is the same as that of the original. On the other hand, in case of down-sampling type wavelet transform, the size of the wavelet coefficients at the coarser scales is reduced significantly because at each scale the size is cut into half. This offers a justification for using the dyadic wavelet transform instead of the down-sampling type of wavelet transform for de-noising.

7. Experimental results

We employ the well-known Lena, Barbara, and Peppers images each of size 512×512 as benchmarks. For comparison, we choose two state-of-the-art methods by Shi et al. and Mallat. Gaussian noise is added to each image. Images shown in Figures 1(b)–3(b) are corrupted images. The dyadic wavelet decomposition and reconstruction of each image is carried out up to scale 3 i.e. $j = 1, 2, 3$ exploiting the spline dyadic wavelet filters designed in Section 5 listed in Tables 1 through 6. The threshold masking procedure described in Section 6 is applied only to the highpass components with various values assigned to the sigmoid function parameters $\alpha$ and $\beta$ for different images; the values of $\alpha$ and $\beta$ assigned for different images have been shown Table 7. Finally, the images are reconstructed using the lowpass components and thresholded highpass components employing inverse fast wavelet transform given by (3.7) and synthesis filters listed in Tables 1 through 6. De-noising results have been shown in Figures 1(c, d, e, f, g, h)- 3(c, d, e, f, g, h).

The traditional standard for evaluating the image quality is the mean square error (MSE), which possesses a simple mathematical structure. For a discrete signal $f(n)$ and its approximation $\tilde{f}(n)$, $0 \leq n \leq N - 1$, MSE is defined to be

$$MSE = \frac{1}{N} \sum_{n=0}^{N-1} [\tilde{f}(n) - f(n)]^2.$$  

Employing $MSE$, we use the following evaluation measure to estimate the image quality

$$PSNR = 10 \log_{10}[255 \times 255 / MSE].$$

Higher the value of $PSNR$ the better the de-noising capability.

Table 7 lists the values of $PSNR$ for the various versions of the three benchmarks recovered from their noisy versions using median filter and the filters listed in Tables
1-6. Maximum PSNR of the Lena images shown in Figures 20 (b, c, d, e) given in \cite{13}, de-noised with down-sampling type wavelets and softer logic masking method is 31.43 db; The maximum PSNR of Lena images shown in Figures 1(c, d, e, f, g, h), de-noised with our method, is 34.96 db. Similarly the maximum PSNR for Barbara image recovered by the method of Shi et al. (See Figure 21 of \cite{13}) is 29.03dB whereas in our case it is 30.8626dB. The maximum PSNR for the Peppers image recovered by the method of Mallat (see Figures 10.6 (e, f), shown in \cite{11}) is 34.3dB whereas in our case it is 36.8682dB. It follows straightaway that in our case maximum value of PSNR is higher than the other methods i.e. our method recovers images with better visual quality. Also take a close look at the PSNR values for different recovered images shown in Figures 1(c, d, e, f, g, h) to 3(c, d, e, f, g, h) corresponding to different wavelet filters. It is clear that as the number of vanishing moments of the highpass analysis filters increases, PSNR also increases i.e. the visual quality of an image enhances and the number of vanishing moments can be increased with dyadic lifting schemes only.

Table 7. PSNR evaluation for Lena, Barbara, and Peppers images of size 512 × 512, in db. Here "Noise" column shows the PSNR of the corresponding image with noise, "Median" column lists against each image its PSNR when it is denoised using median filter. Similarly columns with headings Tables 1 through 6 describe the PSNR of the corresponding images when they are denoised using filters given in Tables 1 through 6 respectively. Columns α and β show the different values of α and β in the thresholding function a(t) for different images.

<table>
<thead>
<tr>
<th>Noised Image</th>
<th>Noise</th>
<th>Median</th>
<th>Table 1</th>
<th>Table 2</th>
<th>Table 3</th>
<th>Table 4</th>
<th>Table 5</th>
<th>Table 6</th>
<th>α</th>
<th>β</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lena</td>
<td>24.4218</td>
<td>31.2787</td>
<td>33.5000</td>
<td>34.6518</td>
<td>34.4631</td>
<td>33.4842</td>
<td>34.9570</td>
<td>34.8272</td>
<td>22</td>
<td>198</td>
</tr>
<tr>
<td>Peppers</td>
<td>28.3282</td>
<td>34.7334</td>
<td>34.8011</td>
<td>36.0876</td>
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<td>35.4734</td>
<td>36.9686</td>
<td>36.8682</td>
<td>25</td>
<td>36</td>
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</table>

8. Conclusion

New Dyadic Lifting Schemes have been proposed, which extend Sweldens Lifting Schemes. To validate the usefulness of the proposed schemes, it has been exploited to propose a design method for custom-design of dyadic wavelets having desirable number of vanishing moments. Using this design method, starting from simple spline dyadic wavelets, filters of spline dyadic wavelets having different numbers of vanishing moments have been designed. These wavelets together with a threshold masking
procedure are applied to the reduction of Gaussian noise from noise corrupted images. Simulation results show better denoising effects of analysis filters with higher numbers of vanishing moments designed using our method. Applications of the filters to other kinds of image processing tasks remain as a future work.

References

Fig. 1. (a) Lena image, (b) noise corrupted image (PSNR=24.42dB), (c) the image restored by the filters listed in Table 1, (d) the image restored by the filters listed in Table 2, (e) the image restored by the filters listed in Table 3, (f) the image restored by the filters listed in Table 4, (g) the image restored by the filters listed in Table 5, (h) the image restored by the filters listed in Table 6.
Fig. 2. (a) Barbara image, (b) noise corrupted image (PSNR=24.52dB), (c) the image restored by the filters listed in Table 1, (d) the image restored by the filters listed in Table 2, (e) the image restored by the filters listed in Table 3, (f) the image restored by the filters listed in Table 4, (g) the image restored by the filters listed in Table 5, (h) the image restored by the filters listed in Table 6.
Fig. 3. (a) Peppers image, (b) noise corrupted image (PSNR=28.23dB), (c) the image restored by the filters listed in Table 1, (d) the image restored by the filters listed in Table 2, (e) the image restored by the filters listed in Table 3, (f) the image restored by the filters listed in Table 4, (g) the image restored by the filters listed in Table 5, (h) the image restored by the filters listed in Table 6.