Hadamard Upper Bound (HUB) on Optimum Joint Decoding Capacity of Wyner Gaussian Cellular MAC

M. Zeeshan Shakir # †, Tariq S. Durrani †, Mohamed-Slim Alouini #

# Div. of Physical Sciences and Engineering, King Abdullah University of Science and Technology, Makkah Province, Thuwal 23599-6900, Kingdom of Saudi Arabia
{muhammad.shakir, slim.alouini}@kaust.edu.sa

† University of Strathclyde, Dept. of Electronic and Electrical Engineering
204 George Street, Glasgow, G1 1XW, United Kingdom
{mshakir, tsd}@eee.strath.ac.uk

Abstract

This paper presents an original analytical expression for an upper bound on the optimum joint decoding capacity of Wyner Circular Gaussian Cellular Multiple Access Channel (C-GCMAC) for uniformly distributed Mobile Terminals (MTs). This upper bound is referred to as Hadamard Upper Bound (HUB) and is a novel application of the Hadamard inequality established by exploiting the Hadamard operation between the channel fading $H$ and channel slow gain $\Omega$ matrices. This paper demonstrates that the theoretical upper bound converges to the actual capacity under constraints like low range of signal to noise ratios and limiting channel slow gain among the MTs and the Base Station (BS) of interest. The behaviour of the theoretical upper bound is critically observed when the inter-cell and the intra-cell time sharing schemes are employed. In this context, we employ an approximation approach to evaluate the effect of the MT distribution on optimal joint decoding capacity for a variable user-density in C-GCMAC. This paper demonstrates that the analytical HUB based on the proposed approximation approach converges to the theoretical upper bound results for medium to high range of signal to noise ratios and shows a comparable tighter bound on optimum sum-rate capacity.

I. INTRODUCTION

The ever growing demand for wireless communication services has necessitated the development of systems with high bandwidth and power efficiency [1] and [2]. In the last decade, The authors would like to acknowledge the financial support of Picsel Technologies Ltd, Glasgow, UK and KAUST, KSA.
recent milestones in the information theory of wireless communication systems with multiple antenna and multiple users have offered additional newfound hope to meet this demand [3]–[11]. Multiple Input Multiple Output (MIMO) technology provides substantial gains over single antenna communications systems, however the cost of deploying multiple antennas at the mobile terminals (MTs) in a network can be prohibitive, at least in the immediate future [3], [8]. In this context, distributed MIMO approach is a means of realizing the gains of MIMO with single antenna terminals in a network allowing a gradual migration to a true MIMO network. This approach requires some level of cooperation among the network terminals which can be accomplished through suitably designed protocols [4]–[6], [12]–[16]. Towards this end, in the last few decades, numerous papers have been written to analyze various cellular models using information theoretic argument to gain insight into the implications on performance of the system parameters. For an extensive survey on this literature, the reader is referred to [5], [6], [17]–[19] and references there in.

The analytical framework of this paper is inspired by analytically tractable model for multi-cell processing (MCP) as proposed in [7] where Wyner incorporated the fundamental aspects of cellular networks into the framework of the well known Gaussian multiple access channel (MAC) to form Gaussian cellular MAC (GCMAC). The majority of the multi-cell decoding cellular models preserves a fundamental assumption which has initially appeared in Wyner’s model, namely (i) only interference from two adjacent cell is considered; (ii) random user locations and therefore path loss variations are ignored; (iii) the interference intensity form each neighboring base station (BS) is characterized by a single fixed parameter $0 \leq \Omega \leq 1$ i.e. the collocation of MTs. Although this model produces more tractable mathematical models, but still it is unrealistic with respect to current practical cellular systems.

A. Background and Related Work

Wyner model was first used in [7] to derive the capacity of uplink cellular networks with MCP, where it is shown that intra-cell time division multiple access (TDMA) is optimal and achieves the capacity. It was generalized in [20] to account for flat-fading. It is proved that wideband
transmission is advantageous over intra-cell TDMA and that fading increases capacity when the number of users is sufficiently large. In [5] and [6], the Wyner model is used to analyze the throughput of cellular networks under single-cell processing (SCP) and two-cell-site processing (TCSP). Later on, scaling results for the sum capacity were derived under the Wyner model with MIMO links in [21]. Recently, the Wyner model is extended to incorporate shadowing in [22].

Despite the fairly large amount of literature based on the Wyner model, to our knowledge no effort has ever been made to validate this simple model for realistic cellular environment. Namely, the information theoretic bounds on Wyner model by exploiting the variable-user density across the cells for finite number of cooperative BSs. A similar kind of attempt has been made in [23] where random matrix theory (RMT) has been used to derive sum-rate capacity where as the main contribution of our paper is to offer non-asymptotic based approach to derive information theoretic bounds on Wyner C-GCMAC model.

B. Contributions

In this paper, we consider a circular version of Wyner GCMAC (by wrap around the linear Wyner model to form a circle) which we will refer to as Circular GCMAC (C-GCMAC). We consider an architecture where the Base Stations (BSs) can cooperate to jointly decode all user’s data (macro-diversity). Thus, we dispense with cellular structure altogether and consider the entire network of the BSs and the users as a network-MIMO system. Each user has a link to each BS and BSs cooperate to jointly decode all user’s data. In first part of this paper, we study the derivation of Hadamard inequality and its application to derive Hadamard upper bound (HUB) on optimum joint decoding capacity which we referred to as theoretical Hadamard upper bound throughout this paper. The theoretical results of this paper are exploited further to study the effect of variable path gains offered by each user in adjacent cells to the BS of interest (due to variable-user density) and derive analytical form of upper bound. The performance analysis of first part of this paper includes the presentation of capacity expressions over multi-user and single user decoding strategies with and without intra-cell and inter-cell TDMA schemes to determine the existence of such upper bound. In the second part of this paper, we approximate the probability
density function (PDF) of Hadamard product of channel fading matrix $G$ and channel slow gain matrix $\Omega$ and derive the analytical form of HUB. The closed form representation of HUB is represented in form of Meijer’s G-Function. The performance and comparison analysis of analytical work includes the presentation of information theoretic bounds over the range of signal to noise ratios (SNRs) and calculation of the mean area spectral efficiency (ASE) over the range of cell radii for the system under considerations.

The summary of main contributions of this paper are 1) to derive an theoretical and analytical upper bounds on the optimum joint decoding capacity of Wyner C-GCMAC by exploiting Hadamard inequality for finite cellular network-MIMO setup and 2) to alleviate the Wyner’s original assumption by assuming variable-user density across the cells i.e. the MTs are uniformly distributed across the cells in C-GCMAC model.

This paper is organized as follows. In section II, system model for Wyner C-GCMAC is recast in Hadamard matrix framework. Next in Section III, the Hadamard inequality is derived as Theorem 3.3 based on Theorem 3.1 and Corollary 3.2. While in Section IV, a novel application of Hadamard inequality is employed to derive the theoretical upper bound on optimum joint decoding capacity. This followed by the several simulation results for single user and multi-user scenarios that validate the analysis and illustrate the effect of various time sharing schemes on the performance of the optimum joint decoding capacity for the system under consideration. While in section V, the theoretical results on Hadamard upper bound are further exploited to derive a novel analytical expression for upper bound on optimum joint decoding capacity by using Hadamard inequality. This is followed by numerical examples and discussions in section VI that validate the simulation and analytical results, and illustrate the accuracy of the proposed approximation based approach for realistic cellular network-MIMO systems. Conclusions are presented in section VII.

**Notation:** Throughout the paper, $\mathbb{R}^{N\times 1}$ and $\mathbb{C}^{N\times 1}$ denote $N$ dimensional real and complex vector spaces respectively. Furthermore, $\mathbb{P}^{N\times 1}$ denotes $N$ dimensional permutation vector which has 1 at some specific position in each column. Moreover, the matrices are represented by upper boldface letters, as an example, the $N \times M$ matrix $A$ with $N$ rows and $M$ columns are represented...
as $A^{N \times M}$. Similarly, vectors are represented by lowercase boldface italic version of the original matrix, as an example, a $N \times 1$ column vector $a$ is represented as $a^{N \times 1}$. An element of matrix or a vector is represented by the non-boldface letter representing the respective vector structure with subscripted row and column indices, as an example $a_{n,m}$ refers to the element referenced by row $n$ and column $m$ of a matrix $A^{N \times M}$. Similarly, $a_k$ refers to element $k$ of the vector $a^{N \times 1}$. Scalar variables are always represented by non-boldface italic characters. The following standard matrix function are defined as follows: $(\cdot)^{T}$ denotes the non-Hermitian transpose; $(\cdot)^{H}$ denotes the Hermitian transpose; $\text{tr}(\cdot)$ denotes the trace of a square matrix; $\det(\cdot)$ denotes the determinant of a square matrix and $\mathbb{E}[]$ denotes the expectation operator and $(\circ)$ denotes the Hadamard operation between the two matrices.

II. WYNER CIRCULAR GAUSSIAN CELLULAR MAC MODEL

A. System Model

Consider a Circular Gaussian Cellular MAC (C-GCMAC) where $N = 6$ cell are arranged in a circle as shown in Fig. [24], [25]. Assuming each cell contains $K$ users such that there are $M = NK$ users in the network-MIMO system. Wyner's model of cellular network used a single parameter to represent the signal strength of inter-cell interference where the path gain is to the closed BS is 1, and the path gain to the adjacent BSs is $\Omega$ and it is zero elsewhere [7]. Wyner considered optimal joint precessing of all BSs by exploiting BS cooperation. Later, Shamai and Wyner considered a similar model with frequency flat fading scenario and more conventional decoding schemes [5] and [6]. Thus, assuming a perfect symbol, frame synchronization, and $K$ users in each cell, at a given time instant the received signal at each of the BS is

$$y_j = \sum_{i=1}^{K} h_{B_j T_j}^{i} x_{j+i}^{1} + \sum_{i=\pm 1}^{K} h_{B_j T_{j+i}}^{i} x_{j+i}^{1} + z_j$$

where $\{B_j\}_{j=1}^{N}$ are the BSs in each cell and $\{T_j\}_{j=1}^{N}$ are the $T_j$ Mobile Terminals (MTs) located in $j^{th}$ cell for $j = 1, 2, \ldots, N$ MTs in each cell and $x_j^{1}$ is the transmitted complex symbol form $^1T_{j+1} \triangleq T_{j+1 \mod N}$. 

October 19, 2010 DRAFT
$K^{th}$ transmitter in $j^{th}$ cell and each $z_j \sim CN(0, \sigma_z^2)$. Each transmitted symbol is subject to the average power constraint $\mathbb{E}[\|x_j^l\|^2] \leq P$ for all $(j, i) = (1, \ldots, N) \times (1, \ldots, K)$. Also, $h_{BjT_j}$ is the intra-cell channel gain between the $l^{th}$ MT and $T_j$ and BS $B_j$ in $j^{th}$ cell and $h_{BjT_{j+i}}$ is the inter-cell channel gain between the $l^{th}$ MT $T_{j+i}$ in $(j+i)^{th}$ cell for $i = \pm 1$ and BS $B_j$. In general, we model the intra-cell and inter-cell channel gains as a Hadamard product of two terms $h_{BjT_{j+i}} = (g_{BjT_{j+i}} \circ \Omega_{BjT_{j+i}})$ for $i = (0, \pm 1)$ where $\Omega_{BjT_{j+i}} \in U(0, 1)$ denotes frequency flat-path gain that depends on distance between the BSs and the MTs which are calculated according to the normalized path loss model

$$\Omega_{BjT_{j+i}} = \left( \frac{d_{BjT_j}}{d_{BjT_{j+i}}} \right)^{\eta/2}$$

(2)

where $d_{BjT_j}$ and $d_{BjT_{j+i}}$ are the distances along the line of sight of the transmission path between the intra-cell and inter-cell MTs to the respective BS of the interest respectively and $\eta$ is the path loss exponent and we assumed it is 4 for urban cellular environment [2]. The gain $g_{BjT_{j+i}} \in CN(0, 1)$ is the small scale fading coefficient that depends on the local scattering environment around the MTs such that the fading coefficients are assumed to have unit power. It is to note that these tw components of the resultant composite channel are mutually independent as they are due to different propagation effects. Therefore, the C-GCMAC model in (1) can be extended to account for fading as follows,

$$y_j = \sum_{i=1}^{K} \left( g_{BjT_j}^l \circ \Omega_{BjT_j}^l \right) x_j^l + \sum_{i=\pm 1} \sum_{l=1}^{K} \left( g_{BjT_{j+i}}^l \circ \Omega_{BjT_{j+i}}^l \right) x_{j+i}^l + z_j$$

(3)

For notation convenience the entire signal model over C-GCMAC can be more compactly expressed as a vector memoryless channel of the form

$$y = Hx + z$$

(4)

where $y \in \mathbb{C}^{N \times 1}, x \in \mathbb{C}^{NK \times 1}, z \in \mathbb{C}^{N \times 1}$ and $H \in \mathbb{C}^{N \times NK}$. The composite channel matrix $H$ is throughout this paper, $H_{N,K}, G_{N,K}$ and $\Omega_{N,K}$ refers to the channel matrices corresponding to $N$ number of cells and $K$ users per cell in a C-GCMAC. For brevity, the channel matrices will be expressed as $H$, $G$ and $\Omega$ respectively unless it is necessary to emphasis the number of cells and the number of users.
is defined as the Hadamard product matrix of the channel fading and channel slow gain matrices given by

\[ H_{N,K} \triangleq (G_{N,K} \circ \Omega_{N,K}) \]  

(5)

where \( G_{N,K} \in \mathbb{C}^{N \times NK} \) and \( \Omega_{N,K} \in \mathbb{R}^{N \times NK} \). The modeling of channel slow gain matrix \( \Omega_{N,K} \) for single and multi-user environments can be well understood form following Lemma.

**Lemma 2.1: (Modeling of Channel Slow Gain Matrix)**

Let \( S \) be the circular permutation operator, viewed as \( N \times N \) matrix relative to the standard basis for \( \mathbb{R}^N \). For given circular cellular setup where initially we assumed \( K = 1 \) and \( N = 6 \) such that there are \( M = NK = 6 \) users in system. Let \( \{e_1, e_2, \ldots, e_6\} \) be the standard row basis vector for \( \mathbb{R}^6 \) such that \( e_i = Se_{i+1} \) for \( i = 1, 2, \ldots, N \). Therefore, the circular shift operator matrix \( S \) relative to the defined row basis vectors, can be expressed as [26], [27].

\[
S = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\]  

(6)

The matrix \( S \) is real and orthogonal, hence \( S^{-1} = S^T \) and also the basis vectors are orthogonal for \( \mathbb{R}^6 \). In symmetrical Wyner model, the variable slow gain between the MTs \( T_{j+i} \) for \( i = 0, \pm 1 \) and the respective BSs \( B_j \) can be viewed as a row vector of the resultant \( N \times M \) circular channel slow gain matrix \( \Omega \) and can be expressed as \( \Omega (1,:) = (\Omega_{B_j T_j}, \Omega_{B_j T_{j+i}} 0, 0, 0, \Omega_{B_j T_{j-i}}) \) where \( \Omega_{B_j T_j} \) is the slow gain between the intra-cell MT and the respective BS and \( \Omega_{B_j T_{j+i}} \) and \( \Omega_{B_j T_{j+i}} \) for \( i = \pm 1 \) are the channel slow gain between the MTs \( T_{j+i} \) for \( i = \pm 1 \) in the adjacent cells on the right side and left side of the BS of the interest respectively. In this setup, it is known that the circular matrix \( \Omega \) can be expressed as a linear combination of powers of the shift operator \( S \) [26], [27]. Therefore, the resultant circular symmetrical channel slow gain matrix in this scenario...
can be expressed as

$$\Omega_{N,1} = I_N + \Omega_{B_jT_j+1}S + \Omega_{B_jT_j-1}S^T$$  \hspace{1cm} (7)$$

Similarly, the channel slow gain model can be extended for unsymmetrical scenario as follows

$$\Omega_{N,1} = I_N + \hat{\Omega}_{N,1} \circ S + \hat{\Omega}_{N,1} \circ S^T$$  \hspace{1cm} (8)$$

where $I_N$ is $N \times N$ identity matrix; $S$ is the shift operator; $\Omega_{B_jT_j \pm 1} \in \mathcal{U}(0, 1)$ and $\hat{\Omega}_{N,1} \in \mathcal{U}(0, 1)$

Furthermore, for multi-user scenario the symmetric model may be formulated as

$$\Omega_{N,K} = 1_K \otimes I_N + \{\Omega_{B_jT_j+1}^l\}^K_{l=1} \otimes S + \{\Omega_{B_jT_j-1}^l\}^K_{l=1} \otimes S^T$$  \hspace{1cm} (9)$$

Similarly, the channel slow gain model can be extended for unsymmetrical scenario as follows

$$\Omega_{N,K} = 1_K \otimes I_N + \hat{\Omega}_{N,K} \circ \{S \otimes 1_K\} + \hat{\Omega}_{N,K} \circ \{S^T \otimes 1_K\}$$  \hspace{1cm} (10)$$

where $1_K$ denotes $1 \times K$ all ones vector and $(\otimes)$ denotes the Kronecker product.

**B. Terminology**

In this paper, we consider different system settings, which are explained as follows:

i. **Intra-cell TDMA**: once user per cell is allowed to transmit at any time instant while the users in different cells can transmit simultaneously.

ii. **Inter-cell TDMA**: one cell is active at any time instant and all the local users inside that cell are allowed to transmit simultaneously while the users in different cells are inactive at that time instant.

iii. **Channel Slow Gain ($\Omega$)**: normalized path loss offered by MTs in adjacent cells to the BS of interest.

iv. **Multi-cell Processing (MCP)**: for the uplink, a joint receiver has access to all the received signals and an optimal decoder decodes all the transmit signals jointly; while for the downlink, the transmit signal from each BS contains information for all users.
v. **Single-cell Processing (SCP):** for the uplink, BSs only process transmit signals from their own cells and treat inter-cell interference as Gaussian noise; while for the downlink, BSs transmit signals with information only intended for their local users.

### III. INFORMATION THEORY AND HADAMARD INEQUALITY

In this section, a novel expression for the upper bound on the sum-rate capacity based on Hadamard inequality is derived [12]. The upper bound is referred to as Hadamard upper bound (HUB) throughout the paper in discussions and analysis. Let us assume firstly that the receiver has perfect channel state information (CSI) while the transmitter knows neither the statistics nor the instantaneous CSI. In this case, a sensible choice for the transmitter is to split the total amount of power equally among all data streams and, consequently, an equal-power transmission scheme takes place [4]–[6]. The justification for adopting this scheme, though not optimal, originates from the so-called maxmin property which demonstrates the robustness of the above mentioned technique for maximizing the capacity of the worst fading matrix [3]–[6]. Under these circumstances, the most commonly used figure of merit in the analysis of MIMO systems is the normalized total sum-rate constraint, which in this paper is referred to as the optimum joint decoding capacity. Following the argument in [8], one can easily show that sum-rate capacity of the system of interest is

\[
C_{\text{opt}} (p(H), \gamma) = \frac{1}{N} \mathcal{I}(x; y|H),
\]

\[
= \frac{1}{N} \mathbb{E} \left[ \log_2 \det \left( I_N + \gamma H H^H \right) \right].
\]

(11)

(12)

where \( p(H) \) signifies that the channel matrix is ergodic with density \( p(H) \); \( I_N \) is a \( N \times N \) identity matrix and \( \gamma \) is the SNR. Here, the BSs are assumed to be able to jointly decode the received signals in order to detect the transmitted vector \( x \). Applying the Hadamard decomposition [5], the Hadamard form of (12) may be expressed as

\[
C_{\text{opt}} (p(H), \gamma) = \frac{1}{N} \mathbb{E} \left[ \log_2 \det \left( I_N + \gamma (G \circ \Omega) (G \circ \Omega)^H \right) \right].
\]

(13)
**Theorem 3.1: (Hadamard Product)**

Let $G$ and $\Omega$ be an arbitrary $N \times M$ matrices. Then \[ \text{and} \]

$$G \circ \Omega = \mathcal{P}_N^T (G \otimes \Omega) \mathcal{P}_M. \quad (14)$$

where $\mathcal{P}_N$ and $\mathcal{P}_M$ are $N^2 \times N$ and $M^2 \times M$ partial permutation matrices respectively. The $j^{th}$ column of $\mathcal{P}_N$ and $\mathcal{P}_M$ has 1 in its $(j-1)N+j$ position and zero elsewhere. The following corollary lists several useful properties of the partial permutation matrices $\mathcal{P}_N$ and $\mathcal{P}_M$.

**Corollary 3.2: (Hadamard Product)**

For brevity, the partial permutation matrices $\mathcal{P}_N$ and $\mathcal{P}_M$ will be denoted by $\mathcal{P}$ unless it is necessary to emphasize theory order. In the same way, the partial permutation matrices $\mathcal{Q}_N$ and $\mathcal{Q}_M$, defined below, are denoted by $\mathcal{Q}$

i. $\mathcal{P}_N$ and $\mathcal{P}_M$ are the only matrices of zeros and ones that satisfy \[(14)\] for all $G$ and $\Omega$.

ii. $\mathcal{P}^T \mathcal{P} = \mathbf{I}$ and $\mathcal{P} \mathcal{P}^T$ is a diagonal matrix of zeros and ones, so $0 \leq \mathcal{P} \mathcal{P}^T \leq 1$.

iii. There exists a $N^2 \times (N^2 - N^2)$ matrix $\mathcal{Q}_N$ and $M^2 \times (M^2 - M^2)$ matrix $\mathcal{Q}_M$ of zeros and ones such that $\pi \triangleq (\mathcal{P} \mathcal{Q})$ is the permutation matrix. The matrix $\mathcal{Q}$ is not unique but for any choice of $\mathcal{Q}$, following holds;

$$\mathcal{P}^T \mathcal{Q} = 0; \quad \mathcal{Q}^T \mathcal{Q} = \mathbf{I} \quad \mathcal{Q} \mathcal{Q}^T = \mathbf{I} - \mathcal{P} \mathcal{P}^T.$$

iv. Using the properties of a permutation matrix together with the definition of $\pi$ in (iii); we have

$$\pi \pi^T = \left( \begin{array}{c} \mathcal{P} \\ \mathcal{Q} \end{array} \right) \left( \begin{array}{c} \mathcal{P}^T \\ \mathcal{Q}^T \end{array} \right) = \mathcal{P} \mathcal{P}^T + \mathcal{Q} \mathcal{Q}^T = \mathbf{I}.$$

\[3\] As an example, for $N = 6$ and $K = 1$, the partial permutation matrices are $\mathcal{P} \in \mathbb{F}^{36 \times 6}$ and $\mathcal{Q} \in \mathbb{F}^{36 \times 30}$
Theorem 3.3: (Hadamard Inequality)

Let \( G \) and \( \Omega \) be an arbitrary \( N \times M \) matrices. Then

\[
GG^H \circ \Omega \Omega^H = (G \circ \Omega) (G \circ \Omega)^H + \Gamma_{(P,Q)}.
\] (15)

where \((P,Q) = P_N^T (G \otimes \Omega) Q_M Q_M^T (G \otimes \Omega) P_N\) and we called it the Gamma equality function.

From \( 15 \) we can obviously deduce \( 28 \)

\[
GG^H \circ \Omega \Omega^H \geq (G \circ \Omega) (G \circ \Omega)^H.
\] (16)

This inequality is referred to as the Hadamard inequality which will be employed to find the theoretical Hadamard upper bound on the capacity \( 13 \).

Proof: Using the well known property of the Kronecker product,

\[
AC^H \otimes BD^H = (A \otimes B) (C \otimes D)^H \] (31), and Corollary \( 3 \) \( (P_M P_M^T + Q_M Q_M^T) = I \), subsequently, we have

\[
GG^H \otimes \Omega \Omega^H = (G \otimes \Omega) (P_M P_M^T + Q_M Q_M^T) (G \otimes \Omega)^H,
\]

\[
= (G \otimes \Omega) P_M P_M^T (G \otimes \Omega)^H + (G \otimes \Omega) Q_M Q_M^T (G \otimes \Omega)^H,
\]

Multiply each term by partial permutation matrix \( P \) of appropriate order to ensure Theorem \( 3.1 \)

\[
P_N^T GG^H \otimes \Omega \Omega^H P_M = P_M^T (G \otimes \Omega) P_M P_M^T (G \otimes \Omega)^H P_N + P_N^T (G \otimes \Omega) Q_M Q_M^T (G \otimes \Omega)^H P_M,
\]

Recall from Theorem \( 3.1 \) i.e. \( G \circ \Omega = P^T (G \otimes \Omega) P \), we have

\[
GG^H \circ \Omega \Omega^H = (G \circ \Omega) (G \circ \Omega)^H + \Gamma_{(P,Q)}.
\]
bound when various time sharing schemes are employed. The simple upper bound on optimum joint decoding capacity using the Hadamard inequality (Theorem 3.3) is derived as

\[
C_{\text{opt}} (p (\mathbf{H}), \gamma) \leq \bar{C}_{\text{opt}} (p (\mathbf{H}), \gamma),
\]

\[
= \frac{1}{N} \mathbb{E} \left[ \log_2 \det \left( \mathbf{I}_N + \gamma \left( \mathbf{G} \mathbf{G}^H \circ (\Omega \Omega^H) \right) \right) \right].
\]  

(17)  

(18)

Now, in the following sub-sections we analyze the validity of the HUB on optimum decoding capacity w.r.t single and multi-user environments under limiting constraints.

A. Single User Environment

i. Low Inter-Cell Interference

For a single user case, as the inter-cell interference levels among the MTs and the BSs is negligible i.e. \( \Omega \to 0 \), the Hadamard upper bound on the optimum joint decoding capacity approaches the actual capacity since \( \mathbf{G} \) and \( \Omega \) becomes diagonal matrices and (16) holds equality results such that

\[
\mathbf{G} \mathbf{G}^H \circ \Omega \Omega^H = (\mathbf{G} \circ \Omega) (\mathbf{G} \circ \Omega)^H.
\]  

(19)

*Proof:* To arrive at (19), we first notice from (15) that \( \mathcal{P}_N^T (\mathbf{G} \otimes \Omega) \mathcal{Q}_M \mathcal{Q}_M^T = 0 \) when \( \mathbf{G} \) and \( \Omega \) are diagonal matrices. Using corollary 3.2, \( \mathcal{Q} \mathcal{Q}^T = \mathbf{I} - \mathcal{P} \mathcal{P}^T \), subsequently, we have \( \mathcal{P}_N^T (\mathbf{G} \otimes \Omega) (\mathbf{I} - \mathcal{P} \mathcal{P}^T) = 0 \) such that

\[
\mathcal{P}_N^T (\mathbf{G} \otimes \Omega) = \mathcal{P}_N^T (\mathbf{G} \otimes \Omega) \mathcal{P} \mathcal{P}^T,
\]

Multiply both sides by \( (\mathbf{G} \otimes \Omega^H) \mathcal{P}_N \), we have

\[
\mathcal{P}_N^T (\mathbf{G} \otimes \Omega) (\mathbf{G} \otimes \Omega)^H \mathcal{P}_N = \mathcal{P}_N^T (\mathbf{G} \otimes \Omega) \mathcal{P} \mathcal{P}^T (\mathbf{G} \otimes \Omega)^H \mathcal{P}_N,
\]
Using the well property of Kronecker product $A C^H \otimes B D^H = (A \otimes B) (C \otimes D)^H$, we have

$$P_N^T (G G^H \otimes \Omega \Omega^H) P_N = P_N^T (G \otimes \Omega) P P^T (G \otimes \Omega)^H P_N,$$

Using Theorem 3.1 finally we arrived at

$$G G^H \circ \Omega \Omega^H = (G \circ \Omega) (G \circ \Omega)^H.$$

Therefore, by using (19) we have

$$\bar{C}_{opt} (p(H), \gamma) = \lim_{\Omega \to \infty} \frac{1}{N} \mathbb{E} \left[ \log_2 \det \left( \mathbf{I}_N + \gamma (G G^H) \circ (\Omega \Omega^H) \right) \right],$$

$$= C_{opt} (p(H), \gamma).$$

The summary of the theoretical HUB on optimum joint decoding capacity over flat faded C-GCMAC for $K = 1$ is shown in Fig. 2. The curves are obtained over 10,000 Monte Carlo simulation trials of channel matrix $H$. It can be seen that the theoretical upper bound is relatively lose at medium to high SNRs as compared bound at low SNRs for $\Omega U (0, 1)$ (compare the solid curve using (13) with the red dashed curve using (18)). This upper bound using (16) is the consequence of the fact that the determinant is increasing in the space of semi-definite positive matrices $G$ and $\Omega$. It can be seen that in the limiting environment such as when $\Omega \to 0$, the theoretical upper bound approaches to the actual optimum joint decoding capacity (compare the curve with square markers and the dashed-dotted curve). It is to note that the channel slow gain $\Omega$ among the MTs in the adjacent cells and BS of interest may be negligible when users are located at the edge of the adjacent cells.

ii. **Tightness of HUB - Low SNR Regime**

In this sub-section, we will show that the theoretical HUB on optimum joint decoding
capacity converges to the actual sum-rate capacity at lower range of signal to noise ratios whereas at higher range of signal to noise ratios, the offset from the actual sum-rate capacity is almost constant. In general, if $\Delta$ is the absolute gain inserted by the theoretical upper bound on $C_{\text{opt}}$ which may be expressed as

$$\Delta = \bar{C}_{\text{opt}}(p(H), \gamma K) - C_{\text{opt}}(p(H), \gamma),$$  \hspace{1cm} (22)

and asymptotically tends to zero as $\gamma \to 0$, given as

$$\Delta = \lim_{\gamma \to 0} \gamma \frac{1}{N} \mathbb{E} [\text{tr} (\Gamma_{PQ})].$$  \hspace{1cm} (23)

**Proof:** Using (22), we have

$$\Delta = \frac{1}{N} \mathbb{E} \left[ \log_2 \left( \frac{\det (I_N + \gamma G G^H \circ \Omega \Omega^H)}{\det (I_N + \gamma G \circ \Omega G \circ \Omega^H)} \right) \right],$$

$$= \frac{1}{N} \mathbb{E} \left[ \log_2 \left( \frac{1 + \gamma \text{tr} (G G^H \circ \Omega \Omega^H) + O_1(\gamma^2)}{1 + \gamma \text{tr} (G \circ \Omega G \circ \Omega^H) + O_1(\gamma^2)} \right) \right],$$

where we have made use of property $\det (I + \gamma A) = 1 + \gamma \text{tr} A + O(\gamma^2)$, hence using (15), the tightness on the bound becomes

$$= \frac{1}{N} \mathbb{E} \left[ \log_2 \left( 1 + \frac{\gamma \text{tr} (\Gamma_{PQ})}{1 + \gamma \text{tr} (G \circ \Omega G \circ \Omega^H)} \right) \right],$$

$$= \frac{1}{N} \mathbb{E} \left[ \log_2 (1 + \gamma \text{tr} (\Gamma_{PQ})) \right],$$

In limiting case, using Taylor series expansion and ignoring higher order terms of $\gamma$, we have

$$\Delta = \lim_{\gamma \to 0} \gamma \frac{1}{N} \mathbb{E} [\text{tr} (\Gamma_{PQ})].$$
It is demonstrated in Fig. 2 that as $\gamma \rightarrow 0$, the gain inserted by upper bound $\Delta \approx 0$. It can be seen from the figure that the theoretical HUB on optimum capacity is loose at higher range of signal to noise ratios and comparably tighter at lower range of signal to noise ratios and hence $C_{\text{opt}}(p(H), \gamma K) \approx C_{\text{opt}}(p(H), \gamma)$.

iii. Inter-Cell TDMA Scheme

Note that (19) holds if and only if $\Gamma_{PQ} = 0$, which is mathematically equivalent to $\mathcal{P}_N^T(G \otimes \Omega) Q_M Q_M^T = 0$. It is found that $\Gamma_{PQ} = 0$ only when $G_{N,1}$ and $\Omega_{N,1}$ are diagonal matrices for single user case i.e. $K = 1$, when intra-cell TDMA is employed i.e. $\Omega = 0$. This is considered as a special case in Circular-GCMAC decoding when each BS only decodes own local users (intra-cell users) and there is no inter-cell interference from the adjacent cells. Hence, the resultant channel matrix is a diagonal matrix such that for the given $G_{N,1}$ and $\Omega_{N,1}$ (19) holds such that

$$C_{\text{opt}}^{\text{TDMA}}(p(H), \gamma K) = \tilde{C}_{\text{opt}}(p(H), \gamma K) = C_{\text{opt}}(p(H), \gamma).$$ (24)

The same has been shown in Fig. 2 (compare the curve with square markers, the curve with plus markers and the dashed-dotted curve).

B. Multi-User Environment

In this sub-section, we demonstrate the behaviour of the theoretical HUB when two implementation forms of time sharing schemes are employed. One is referred to as inter-cell TDMA, intra-cell narrowband scheme (TDMA, NB) and other is inter-cell TDMA, intra-cell wideband scheme. We will refer this scheme as inter-cell time sharing, wideband scheme, (ICTS, WB) throughout the discussions. It is to note that SCP is employed to determine the application of new bound for realistic cellular environment.

i. Inter-Cell TDMA, Intra-Cell Narrow-band scheme (TDMA, NB)

In multi-user case, when there are $K$ active users in each cell, then the channel matrix is

$^4$Terms with higher order of $\gamma$ are ignored $\Leftrightarrow \gamma^x \approx 0 \forall x = 2, 3, \ldots$. [31].
no longer diagonal and hence (19) is not valid and \( \Gamma_{PQ} \neq 0 \). However, the results of single user case is still valid when intra-cell TDMA is employed in combination with inter-cell TDMA (TDMA, NB). If the multi-user channel matrix \( H_{N,K} \) is expressed as (5), then by exploiting the TDMA, NB scheme the rectangular channel matrix \( H_{N,K} \) may be reduced to \( H_{N,1} \) and can be expressed as

\[
H_{N,1} = \left( G_{N,1} \circ \Omega_{N,1} \right) .
\]

(25)

where \( G_{N,1} \) and \( \Omega_{N,1} \) are exactly diagonal matrices as discussed earlier in single user case. The capacity in this case becomes

\[
C_{opt}^{TDMA,NB} \left( p(H), \gamma K \right) = \frac{1}{N} E \left[ \log_2 \det \left( I_N + \gamma H_{N,1} H_{N,1}^H \right) \right] ,
\]

(26)

\[
= C_{opt}^{TDMA,NB} \left( p(H), \gamma K \right) .
\]

(27)

Using the Hadamard inequality, the upper bound on TDMA, NB sum-rate capacity is equal to the actual sum-rate capacity offered by this scheduling scheme. The scenario is simulated and shown in Fig. 3. It is to note that the capacity in this figure is normalized with respect to number of users and number of cells. It can be seen that the actual sum-rate capacity and the upper bound on the optimum capacity are identical for \( K = 5 \) and \( K = 10 \) (compare the curves with circle markers with the black solid curves).

ii. Inter-Cell Time Sharing, Wide-band scheme, (ICTS, WB)

It is well known that the increase in number of users to be decoded jointly increases the channel capacity [5], [6], [13]–[16]. Let us consider a scenario without intra-cell TDMA i.e. there are \( K \) active users in each cell and they are allowed to transmit simultaneously. Mathematically, the local intra-cell users are located along the main diagonal of a rectangular channel matrix \( H_{N,K} \). The capacity in this case when only inter-cell TDMA scheme
(ICTS, WB) is employed becomes

\[
C_{\text{opt}}^{\text{ICTS,WB}} (p(H), \gamma) = \frac{1}{N} \mathbb{E} \left[ \log_2 \det \left( I_N + \gamma \mathbf{H}_{N,K} \mathbf{H}_{N,K}^H \right) \right],
\]

(28)

\[
< C_{\text{opt}}^{\text{ICTS,WB}} (p(H), \gamma K).
\]

(29)

The capacity by employing ICTS, WB scheme for \( K = 5 \) and \( K = 10 \) is shown in Fig. 3(a) and Fig. 3(b) respectively. The theoretical upper bound on the capacity using Hadamard inequality by employing ICTS, WB is also shown in this figure (compare the solid curve with the dashed curve). It is observed that the theoretical upper bound on ICTS, WB capacity increases with the increase in number of intra-cell users to be jointly decoded in the multi-user case. An example, at \( \gamma = 20 \) dB and for \( K = 5 \) the relative increment in sum-rate due to Hadamard upper bound is 6.5 % and similarly for \( K = 10 \) the relative increment is raised to 12 %. Thus, using an inequality (16), multi-user decoding offers \( \log_2 (K) \) times higher non-achievable capacity as compared to actual capacity offered by this scheme. Also, the overall performance of ICTS scheduling is superior to the TDMA scheme due to wideband intra-cell scheme (compare the dashed-dotted curves with the solid curves). The results are summarized in Table I to clearly validate the existence of HUB.

V. ANALYTICAL HADAMARD UPPER BOUND (HUB)

In this section, we approximate the PDF of Hadamard product of channel fading matrix \( \mathbf{H} \) and channel slow gain matrix \( \mathbf{\Omega} \) as the PDF of the trace of the Hadamard product of two matrices. The closed form expression of the new bound HUB is expressed in the form of Meijer’s G-Function representation. Recall from (18) (section IV), the simple upper bound on optimum joint
decoding capacity \( C_{\text{opt}}(p(H), \gamma) \) using the Hadamard inequality (Theorem 3.3) is derived as

\[
C_{\text{opt}}(p(H), \gamma) \leq \bar{C}_{\text{opt}}(p(H), \gamma),
\]

\[
= \frac{1}{N} \mathbb{E} \left[ \log_2 \det \left( I_N + \gamma \left( \mathbf{G} \mathbf{G}^H \circ (\mathbf{\Omega} \mathbf{\Omega}^H) \right) \right) \right],
\]

\[
= \frac{1}{N} \mathbb{E} \left[ \log_2 \left( 1 + \gamma \left( \bar{\mathbf{G}} \circ \bar{\mathbf{\Omega}} \right) \right) \right],
\]

where we have made use of property \( \det (I + \gamma A) = 1 + \gamma \text{tr} A + O(\gamma^2) \). Ignoring the terms with higher order of \( \gamma \); \( \text{tr} \left( \bar{\mathbf{G}} \circ \bar{\mathbf{\Omega}} \right) \) denotes the trace of the Hadamard product of the composite channel matrix \( \left( \bar{\mathbf{G}} \circ \bar{\mathbf{\Omega}} \right) \) and

\[
\frac{1}{N} \nu(\bar{G} \circ \bar{\Omega}) (\gamma) = \frac{1}{N} \mathbb{E} \left[ \log_2 \left( 1 + \gamma \left( \bar{\mathbf{G}} \circ \bar{\mathbf{\Omega}} \right) \right) \right],
\]

\[
= \int_0^\infty \log_2 \left( 1 + \gamma \text{tr} \left( \bar{\mathbf{G}} \circ \bar{\mathbf{\Omega}} \right) \right) dF_{\bar{G} \circ \bar{\Omega}} \left( \text{tr} \left( \bar{\mathbf{G}} \circ \bar{\mathbf{\Omega}} \right) \right),
\]

is the Shannon transform of a random square Hadamard composite matrix \( \left( \bar{\mathbf{G}} \circ \bar{\mathbf{\Omega}} \right) \) and distributed according to the cumulative distribution function (CDF) denoted by \( F_{\bar{G} \circ \bar{\Omega}} \left( \text{tr} \left( \bar{\mathbf{G}} \circ \bar{\mathbf{\Omega}} \right) \right) \).

Also, where \( \bar{\gamma} = \gamma N^2 \) and \( \gamma = P/\sigma^2 \) is the mobile terminal transmit power over receiver noise ratio. Using trace inequality \( (32) \), we have an upper bound on \( (32) \) as

\[
C_{\text{opt}}(p(H), \gamma) \leq \bar{C}_{\text{opt}}(p(H), \gamma),
\]

\[
= \frac{1}{N} \mathbb{E} \left[ \log_2 \left( 1 + \gamma \text{tr} \left( \bar{\mathbf{G}} \right) \text{tr} \left( \bar{\mathbf{\Omega}} \right) \right) \right].
\]

If \( u = x y \); where \( x = \text{tr} \left( \bar{\mathbf{G}} \right) \) and \( y = \text{tr} \left( \bar{\mathbf{\Omega}} \right) \) then \( (34) \) can be expressed as

\[
\bar{C}_{\text{opt}}(p(H), \gamma) = \int_{0}^{\infty} \log_2 \left( 1 + \bar{\gamma} u \right) dF_{\bar{G} \circ \bar{\Omega}} (u),
\]

\[
= \int_{0}^{\infty} \log_2 \left( 1 + \bar{\gamma} u \right) df_{\bar{G} \circ \bar{\Omega}} (u).
\]

where \( f_{\bar{G} \circ \bar{\Omega}} (u) \) is the joint PDF of the \( \text{tr} \left( \bar{\mathbf{G}} \right) \) and \( \text{tr} \left( \bar{\mathbf{\Omega}} \right) \) which is evaluated as follows in the next sub-section.
A. Approximation of PDF of \((\mathcal{G} \circ \mathcal{O})\):

Let \( u = xy \) and \( v = x \), then the Jacobian is given as

\[
J \left( \frac{u, v}{x, y} \right) = \begin{vmatrix} y & x \\ 1 & 0 \end{vmatrix} = -x = -\frac{u}{y}.
\]

(39)

\[
f_{\mathcal{G} \circ \mathcal{O}}(u, v) \, du \, dv = f_{\mathcal{G} \circ \mathcal{O}}(x, y) \, dx \, dy = f_{\mathcal{G} \circ \mathcal{O}}(x, y) \frac{y}{u} \, du \, dv,
\]

(40)

so,

\[
f_{\mathcal{G} \circ \mathcal{O}}(u, v) = \frac{y}{u} f_{\mathcal{G} \circ \mathcal{O}}(x, y).
\]

(41)

where we approximate the PDF of \( f_{\mathcal{G} \circ \mathcal{O}}(x, y) \) of Hadamard product of two random variables \( x \) and \( y \) as a product of Gaussian and Uniform distributions respectively such that

\[
f_{\mathcal{G} \circ \mathcal{O}}(x, y) = \frac{1}{\sqrt{2\pi} u} e^{\left( -\frac{u^2}{2y^2} \right)} U(y).
\]

(42)

Now, the PDF of the Hadamard product of two composite matrices \( \mathcal{G} \) and \( \mathcal{O} \) may be approximated as

\[
f_{\mathcal{G} \circ \mathcal{O}}(x, y) = \frac{1}{\sqrt{2\pi}} \int_0^1 \frac{1}{u} e^{\left( -\frac{u^2}{2y^2} \right)} dy.
\]

(43)

By substituting (43) into (38), the analytical upper bound on optimum joint decoding capacity can be calculated as

\[
\tilde{C}_{opt}(p(H), \gamma) = \frac{1}{\sqrt{2\pi}} \int_0^\infty \int_0^1 \frac{y}{u} \log_2(1 + \gamma u) e^{\left( -\frac{u^2}{2y^2} \right)} dy \, du,
\]

(44)

the solution to integral (44) is shown in closed form as (45); where we have made a use of Meijer’s G-Function( [33]), available in standard scientific software packages such as Mathematica, in order to transform the integral expression to the form in (45) and \( \Theta(\gamma; N) = 1/64\sqrt{2\pi}^2 \gamma^2 \).
\[ \tilde{C}_{\text{opt}}(p(\mathbf{H}), \gamma) = \Theta(\gamma; \mathcal{N}) \left( \gamma^2 \left( G_{4,6}^{5,3} \left( \begin{array}{c|c|c|c|c} \frac{1}{16} \gamma^4 & 0, \frac{1}{4}, \frac{3}{4}, 1 & 0, 0, \frac{1}{4}, \frac{3}{4}, \frac{1}{2} \\ \hline & 0, 0, 0, \frac{1}{4}, \frac{3}{4}, \frac{1}{2} \\ \end{array} \right) + G_{4,6}^{5,3} \left( \begin{array}{c|c|c|c|c} \frac{1}{16} \gamma^4 & 0, \frac{1}{4}, \frac{3}{4}, 1 & 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 0 \\ \hline & 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 0 \\ \end{array} \right) \right) \right) - 4 \sqrt{\pi} G_{4,2}^{3,4} \left( \begin{array}{c|c|c|c|c} \frac{1}{2} \gamma^2 & -1, -\frac{1}{2}, 1 & -1, -1, -\frac{1}{2}, 0 \\ \hline & -1, -1, \frac{1}{2}, 0 \\ \end{array} \right) \right). \] (45)

VI. NUMERICAL EXAMPLES AND CONCLUSION

In this section, we present Monte Carlo simulation results in order to validate the accuracy of the analytical analysis based on approximation approach of upper bound on optimum joint decoding capacity for C-GCMAC with Uniformly distributed MTs. In the context of Monte Carlo finite system simulations, the MTs gains toward the BS of interest are randomly generated according to the considered distribution and the capacity is calculated by the evaluation of capacity formula (13). Using (18) the upper bound on the optimum capacity is calculated using (12). It can be seen from Fig. 4 that the theoretical upper bound converges to the actual capacity under constraints like low SNRs (compare dashed curve with sold curve) (12). In the context of mathematical analysis which is the main contribution of this paper, (45) is utilized to compare the upper bound on optimum joint decoding capacity based on proposed analytical analysis with upper bound based on simulations. It can also be seen from Fig. 4 that the proposed approximation shows comparable results over the entire range of SNR (compare dotted and dashed curves). However, it is to note that the new HUB on optimum joint decoding capacity of multi-cell setup is tighter for higher range of SNRs as compare to low range of SNRs. The proposed approximation based approach is useful to represent the sum-rate capacity for the realistic multi-cell setup i.e. variable user-density and therefore variable channel slow gain towards the BS of interest.

A figure of merit utilized in cellular communication which is referred to as mean area spectral efficiency (ASE)

\[ A_c = \frac{C_{\text{opt}}}{\pi R^2} \text{ bits/sec/Hz/Km}^2 \] (46)
averaged over a large number of fading realizations $g_{B_{j}T_{j+i}}$ and channel slow gain $\Omega_{B_{j}T_{j+i}}$ for all $(j, i) = (1 \ldots N) \times (0, \pm 1)$ and $K$ users [34]. Further, we assumed that the range of cell radius $R$ is $0.1 - 1 \ km$ for the system under consideration. Fig. 5(a) and Fig. 5(b) shows that the ASE calculated for $\gamma = -10\ dB$ and $\gamma = 20\ dB$ respectively. It can be seen that the analytical bound HUB on optimum joint decoding capacity based on proposed approximation approach is close to the Monte Carlo simulation results within the entire cell radii for high SNR. On the other side, the HUB is loose up to 500 meters of cell radius and comparably tighter within the higher range of cell radii for low SNR.

VII. Conclusion

The analytical upper bound referred to as Hadamard Upper Bound (HUB) is derived for optimum joint decoding capacity for Wyner C-GCMAC under realistic assumptions: uniformly distributed MTs across the cell; and the BS cooperation among finite number of cells. Using approximation approach to derive HUB, the distribution of trace of composite Hadamard matrix has been derived by employing the Hadamard Inequality. The proposed analysis is validated by using Monte Carlo simulations for variable user-density cellular system. The importance of the methodology presented here lies in the fact that it allows a more realistic representation of the MT’s spatial arrangement. Therefore, this approach can be employed in order to investigate various practical MT distributions and their effect on the sum-rate capacity. New results have been reported for finite number of cooperating BSs and uniformly distributed MTs with variable channel slow gain towards the BS of interest.

APPENDIX A

An Alternative Proof of (16)

Proof: In case that $G$ and $\Omega$ are rank one matrices, we can also derive alternative version of (15) that also proves (16) for such matrices. Let us define $G = u \ v^H$ and $\Omega = w \ z^H$; where $u, v, w, z$ are $N \times 1$ column vectors which corresponds to a vector channel between the user
in any of $N$ cells and $N$ BSs. Then,

$$G \circ \Omega = (u \circ w) (v \circ z)^H$$  \hspace{1cm} (A.47)

also is of rank at most one, and we calculate that

$$((G \circ \Omega) (G \circ \Omega)^H = (u \circ w) (v \circ z)^H (v \circ z) (u \circ w)^H$$  \hspace{1cm} (A.48)

$$= \|v \circ z\|^2 (u \circ w) (u \circ w)^H$$  \hspace{1cm} (A.49)

On the hand, we have

$$GG^H = (u v^H) (u v^H) = \|v\|^2 (u u^H)$$  \hspace{1cm} (A.50)

$$\Omega \Omega^H = (w z^H) (w z^H) = \|z\|^2 (w w^H)$$  \hspace{1cm} (A.51)

This gives the formula

$$\left( G G^H \right) \circ \left( \Omega \Omega^H \right) = \|v\|^2 \|z\|^2 (u u^H) \circ (w w^H)$$  \hspace{1cm} (A.52)

$$= \|v\|^2 \|z\|^2 (u w) \circ (u^H w^H)$$  \hspace{1cm} (A.53)

Comparing the formulas (A.49) and (A.53), we obtain the identity

$$\left( G G^H \right) \circ \left( \Omega \Omega^H \right) = \frac{\|v\|^2 \|z\|^2}{\|v \circ z\|^2} (G \circ \Omega) (G \circ \Omega)^H$$  \hspace{1cm} (A.54)

in this case. In particular, since the norm is sub-multiplicative relative to the Hadamard product:

$$\|v \circ z\| \leq \|v\| \|z\|.$$  \hspace{1cm} (A.55)
REFERENCES


Fig. 1: Uplink of C-GCMAC where BSs are cooperating to decode all user’s data; (the solid line illustrates intra-cell users and the dotted line shows inter-cell users). For simplicity, in this figure there is only $K = 1$ user in each cell.
Fig. 2: Summary of optimum joint decoding capacity and the upper bound on optimum capacity for \( (13) \); the solid curve illustrates capacity using \( (13) \); the dashed curve illustrates capacity using \( (18) \); the curves with square markers and plus markers illustrate capacity using \( (19) \) when and inter-cell TDMA is employed respectively.
Fig. 3: Summary of optimum joint decoding capacity and theoretical upper bound on the optimum capacity for the multi-user case when TDMA, NB and ICTS, WB schemes are employed.
Fig. 4: Summary of Hadamard Upper Bound (HUB) on optimum joint decoding capacity of C-GMAC for variable user-density across the cells; solid curve illustrates actual capacity using (13) obtained by Monte Carlo simulations; dashed and dotted curves illustrate HUB obtained by Monte Carlo simulations and analytical analysis using (18) and (32) respectively. The simulation curves are obtained after averaging 10,000 Monte Carlo trials of the composite channel $\mathbf{H}$. 
Fig. 5: Area mean spectral efficiency ($\text{bits/sec/Hz/Km}^2$) vs. the cell radius.
Table I: Summary of Theoretical Hadamard Upper Bound (HUB)

<table>
<thead>
<tr>
<th>User(s) ($K$)</th>
<th>Constraints for $C_{opt}(p(H);\gamma)=\overline{C}_{opt}(p(H);\gamma)$</th>
<th>Constraints for $C_{opt}(p(H);\gamma)&lt;\overline{C}_{opt}(p(H);\gamma)$</th>
</tr>
</thead>
</table>
| $K = 1$       | i. $\Omega \rightarrow 0$ i.e. low level of inter-cell interference to the BS of interest.  
|               | ii. $\gamma \rightarrow 0$ i.e. the gain inserted by HUB $\Delta \rightarrow 0$ and is given by $\Delta = \lim_{\gamma \rightarrow 0} \gamma E[\text{tr}(\Gamma_{P,Q})]$  
|               | $\Omega \in U(0, 1)$ (variable slow gain among the MTs and the BS of interest due to Uniformly distributed MTs across the cells) |
| $K > 1$       | By employing intra-cell TDMA, inter-cell Narrowband (TDMA, NB) | By employing Inter-cell Time Sharing, Wideband (ICTS, WB) |