On $b$-colorings in regular graphs

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A B S T R A C T

A $b$-coloring is a coloring of the vertices of a graph such that each color class contains a vertex that has a neighbor in all other color classes. El-Sahili and Kouider have conjectured that every $d$-regular graph with girth at least 5 has a $b$-coloring with $d + 1$ colors. We show that the Petersen graph infirms this conjecture, and we propose a new formulation of this question and give a positive answer for small degree.

1. Introduction

A proper coloring of a graph $G = (V, E)$ is a mapping $c$ from $V$ to the set of positive integers (colors) such that any two adjacent vertices are mapped to different colors. Each set of vertices colored with one color is a stable set of vertices of $G$, so a coloring is a partition of $V$ into stable sets. The smallest number $k$ for which $G$ admits a coloring with $k$ colors is the chromatic number $\chi(G)$ of $G$.

Many graph invariants related to colorings have been defined. Most of them try to minimize the number of colors used to color the vertices under some constraints. For some other invariants, it is meaningful to try to maximize this number. The $b$-chromatic number is such an example. When we try to color the vertices of a graph, a simple trick consists in starting from a coloring and trying to decrease the number of colors by reducing them in some way, for example by merging two color classes. This motivated the introduction of the achromatic number by Harary and Hedetniemi [6]: the achromatic number of a graph $G$ is the largest integer $k$ such that $G$ admits a coloring with $k$ colors for which there is an edge between any two color classes. Clearly, the process of merging suggested above is impossible if we have such a coloring. So the achromatic number is a measure of how hard it is to obtain a coloring with few colors. This inspired Irving and Manlove [8,13] to consider another procedure, which consists in trying to reduce the number of colors by transferring all vertices from one color class to other classes. A $b$-coloring is a proper coloring such that every color class $i$ contains at least one vertex that has a neighbor in all the other classes. Any such vertex will be called a $b$-dominating vertex of color $i$. The $b$-chromatic number $b(G)$ is the largest integer $k$ such that $G$ admits a $b$-coloring with $k$ colors.

For a graph $G$, and for any vertex $v$ of $G$, the neighborhood of $v$ is the set $N(v) = \{u \in V(G) \mid uv \in E(G)\}$ and the degree of $v$ is $\deg(v) = |N(v)|$. Let $\Delta(G)$ be the maximum degree in $G$, and let $m(G)$ be the largest integer $k$ such that $G$ has $k$ vertices of degree at least $k - 1$. It is easy to see that every graph $G$ satisfies $b(G) \leq m(G) \leq \Delta(G) + 1$.

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(the first inequality follows from the fact that if $G$ has any $b$-coloring with $k$ colors then it has $k$ vertices of degree at least $k - 1$; the second inequality follows from the definition of $m(G)$), Irving and Manlove [8,13] proved that every tree $T$ has $b$-chromatic number $b(T)$ equal to either $m(T)$ or $m(T) - 1$, and their proof is a polynomial-time algorithm that computes the value of $b(T)$. On the other hand, Kratochvíl, Tuza and Voigt [12] proved that it is NP-complete to decide if $b(G) = m(G)$, even when restricted to the class of connected bipartite graphs such that $m(G) = \Delta(G) + 1$. These NP-completeness results have incited researchers to establish bounds on the $b$-chromatic number in general or to find exact or approximate values for subclasses of graphs [1–4,7,9–11].

Here we focus on a recent problem concerning regular graphs. A graph $G$ is $d$-regular if every vertex of $G$ has degree equal to $d$. Note that every $d$-regular graph $G$ satisfies $m(G) = d + 1$. The girth of $G$ is the length of a shortest cycle in $G$. In [3], El-Sahli and Kouider pose the following question:

**Question** ([3]). Is it true that every $d$-regular graph $G$ with girth $g(G) \geq 5$ satisfies $b(G) = d + 1$?

We observe that the Petersen graph offers a negative answer to this question. However, we also propose some positive results in the case where $d \leq 6$ which suggest that the Petersen graph might be the only counterexample to the question.

2. Preliminary results

For integer $k \geq 3$, we let $C_k$ denote the cycle with $k$ vertices.

**Theorem 1.** The Petersen graph has $b$-chromatic number 3.

**Proof.** Let $G$ be the Petersen graph, with vertices $v_1, \ldots, v_5, w_1, \ldots, w_5$ such that $v_1, v_2, v_3, v_4, v_5$ induce a 5-cycle in this order, $w_1, w_2, w_3, w_4, w_5$ induce a 5-cycle in this order, and $v_i w_1$ is an edge for each $i = 1, \ldots, 5$. Since $\chi(G) = 3$, we have $b(G) \geq 3$. Suppose that $G$ admits a $b$-coloring with 4 colors. For $j = 1, \ldots, 4$, let $d_j$ be a $b$-dominating vertex of color $j$, and let $D = \{d_1, \ldots, d_4\}$. Note that each vertex $d_j$ must have exactly one neighbor of each of the three colors different from $j$.

Suppose that $D$ induces a stable set. So, up to symmetry, we can assume that $D = \{v_1, v_3, w_4, w_5\}$, with $d_1 = v_1$, $d_2 = v_3$, $d_3 = w_4$, $d_4 = w_5$. Without loss of generality, $v_2$ has color 2, and so $w_2$ does not have color 2. Since $v_1$ is $b$-dominating, it must have a neighbor of color 4, which can only be $w_1$, and a neighbor of color 3, which can only be $v_5$. Since $v_3$ is $b$-dominating, it must have a neighbor of color 4, which can only be $v_4$, and a neighbor of color 1, which can only be $w_3$. But then $w_3$ cannot have a neighbor of color 2, a contradiction. So $D$ does not induce a stable set.

We may assume that $d_1, d_2$ are adjacent, and so, up to symmetry, that $d_1 = v_1$ and $d_2 = v_2$ (since all edges of the Petersen graph play the same role). Now, it is easy to see that, wherever $d_3$ may be, there is a $C_5$ of $G$ that contains $d_1, d_2, d_3$. So (since all $C_5$’s of the Petersen graph play the same role), we can assume that $d_3$ is one of $v_3, v_4, v_5$. Up to symmetry, this leads to two cases.

**Case 1:** $d_3 = v_3$. Since $v_3$ is $b$-dominating, it has a neighbor of color 4, which can only be $w_2$. One of $v_4, v_5$, say $v_5$, does not have color 4. Since $v_1$ is $b$-dominating, it has a neighbor of color 4, which can only be $w_1$, and a neighbor of color 3, which can only be $v_5$. Since $v_3$ is $b$-dominating, it has a neighbor of color 4, which can only be $v_4$, and a neighbor of color 1, which can only be $w_3$. Note that $w_5$ can only have color 2. But now, the vertices of color 4 are $w_1, w_2, v_4$ and no other vertex, and each of these three vertices has two neighbors of the same color, so none of them can be $b$-dominating, a contradiction.

**Case 2:** $d_3 = v_4$. Since $v_1$ is $b$-dominating, it has a neighbor of color 3, which can only be $w_1$, and a neighbor of color 4, which can only be $v_5$. Likewise $v_3$ has a neighbor of color 3, which can only be $w_2$, and a neighbor of color 4, which can only be $v_3$. But then $v_4$ has two neighbors of color 4, so it cannot be $b$-dominating, a contradiction. 

We will need the following result, for which we give a new proof. Our proof is rather similar to the proof of Proposition 2 in [3]; we include it here for the sake of completeness and because it prepares the more complicated proof of Theorem 5.

**Theorem 2 ([9]).** Every $d$-regular graph $G$ with girth $g(G) \geq 6$ has a $b$-coloring with $d + 1$ colors.

**Proof.** Let $x$ be a vertex of $G$, and let $x_1, \ldots, x_d$ be its neighbors. For each $i = 1, \ldots, d$, let $N_i = N(x_i) \setminus \{x\}$. Then each $N_i$ is a stable set, for otherwise $G$ would contain a cycle of length three. Then any two $N_i$’s are disjoint, for otherwise $G$ would contain a cycle of length four; and there is no edge between them, for otherwise $G$ would contain a cycle of length five. We construct a coloring with $d + 1$ colors $0, 1, \ldots, d$ as follows. Assign color 0 to $x$, color $i$ to $x_i$ ($i = 1, \ldots, d$), and assign to the vertices of $N_i$ the colors from $[1, \ldots, d] \setminus \{i\}$, in a one-to-one fashion. Finally, color the remaining vertices in arbitrary order, assigning to each $v$ a color from $[0, 1, \ldots, d]$ different from the colors already assigned to its neighbors. Clearly, we obtain a $b$-coloring with $d + 1$ colors in which the vertices $x, x_1, \ldots, x_d$ are $b$-dominating. 

3. A new proof of El-Sahli and Kouider’s theorem

El-Sahli and Kouider [3] proved the following theorem.

**Theorem 3 ([3]).** If $G$ is a $d$-regular graph with girth $g(G) \geq 5$ and $G$ contains no $C_6$, then $b(G) = d + 1$. 

We propose here a new proof of \textbf{Theorem 3}, using a classical theorem of \textit{Vizing on list coloring}. Let $L$ be a mapping that assigns to each vertex $v$ of a graph $G$ a set $L(v)$ of admissible colors. An \textit{L-coloring} is a coloring $c$ of the vertices of $G$ such that $c(v) \in L(v)$ for every vertex $v$ of $G$. If $G$ admits an L-coloring it is said to be \textit{L-colorable}. Given an integer $k$, $G$ is $k$-\textit{list-colorable} if it admits an L-coloring for every $L$ such that $|L(v)| \geq k$ for every $v \in V$. The list-chromatic number $\chi_l(G)$ is the smallest integer $k$ such that $G$ is $k$-list-colorable.

\textbf{Theorem 4} ([5,15]). Let $G$ be a connected graph different from a complete graph and from an odd cycle. Then $G$ is $\Delta$-list-colorable, where \(\Delta\) is the maximum degree in $G$.

\textbf{Proof of Theorem 3.} The theorem holds trivially when $d = 0$ or 1. If $d = 2$, then $G$ is a disjoint union of cycles, all of length $5$ or more. Then $b(G) = 3$. Indeed, in order to obtain a $b$-coloring with 3 colors, it suffices to give colors 1, 2, 3, 1 to five consecutive vertices in one cycle, and to color the rest of $G$ with colors 1, 2, 3. Now we assume that $d \geq 3$.

Pick a vertex $v$ of $G$ and let its neighborhood be $N(v) = \{v_1, v_2, \ldots, v_\ell\}$. For each $i = 1, \ldots, \ell$, let $N_i = N(v_i) \setminus \{v\}$, and let $G_i$ be the subgraph of $G$ induced by $N_i \cup \cdots \cup N_\ell$. We make a few observations about $G_i$. Consider any vertex $x$ in $G_i$; so $x \in N_i$ for some $i \in \{1, \ldots, \ell\}$. Vertex $x$ has no neighbor in $N_i$, for otherwise $G_i$ would have a cycle of length three. Vertex $x$ cannot have two neighbors $y, z \in N_i (j \neq i)$, for otherwise $x, y, v_i, z$ would induce a cycle in $G$; and $x$ cannot have two neighbors $y \in N_i$, $z \in N_k$ (with $i, j, k$ pairwise different), for otherwise $x, y, v_i, v_k, z$ would induce a cycle in $G$. Thus, in $G_i$, every vertex has degree at most 1.

For $i = 1, \ldots, \ell$, set $E_i = \{xy \mid x, y \in N(v_i) \setminus \{v\}, x \neq y\}$. Then let $H_i$ be the graph obtained from $G_i$, by adding to its edge set all the elements of $E_1 \cup \cdots \cup E_\ell$ (so each $N_i$ is a clique in $H_i$). It follows from the preceding observations that, in $H_i$, every vertex has degree at most $d - 1$. Moreover, if $d \geq 4$, then the largest cliques in $H_i$ are induced by the sets $N_i (i = 1, \ldots, \ell)$, which have size $d - 1$; and if $d = 3$ then the largest cliques in $H_i$ have size 2. We claim that:

$$\chi_l(H_i) \leq d - 1. \quad (1)$$

Consider the contrary case. Since the maximum degree in $H_i$ is at most $d - 1$, by \textbf{Theorem 4} we must have either (a) some component of $H_i$ is a complete graph $K_d$, or (b) $d - 1 = 2$ and some component of $H_i$ is an odd cycle. However, (a) is impossible because the largest cliques in $H_i$ have size $d - 1$. So suppose that (b) holds. Then $d = 3$, so the graph $H_i$ has six vertices, and so the only possible odd cycle in $H_i$ is $C_5$, but then some vertex of this $C_5$ has two neighbors in $G_i$, a contradiction. Thus (1) holds.

We define a list assignment $L$ on $H_i$ as follows. If $x$ is any vertex of $H_i$, we have $x \in N_i$ for some $i \in \{1, \ldots, \ell\}$, and we assign the list $L(x) = \{1, \ldots, d\} \setminus \{i\}$ to $x$. Thus each vertex $x$ of $H_i$ satisfies $|L(x)| = d - 1$. By (1), there is an L-coloring of $H_i$. Extend $c$ to a coloring of $G$ as follows. Give color 0 to $v$ and color 1 to $v_i (i = 1, \ldots, \ell)$; then color the remaining vertices in arbitrary order, giving to each vertex a color from $\{0, 1, \ldots, d\}$ different from the colors already assigned to its neighbors. Then we obtain a coloring of $G$ with $d + 1$ colors, and clearly the vertices $v, v_1, \ldots, v_d$ are $b$-dominating. Thus $b(G) = d + 1$.  

\section{When the degree is small}

\textbf{Theorem 5.} Let $G$ be a $d$-regular graph with girth $g(G) \geq 5$, different from the Petersen graph, and with $d \leq 6$. Then $b(G) = d + 1$.

\textbf{Proof.} We distinguish one case for each value of the degree.

\textbf{Case 1:} $d = 1$. Then $G$ is a matching and clearly $b(G) = 2 = d + 1$.

\textbf{Case 2:} $d = 2$. Then $G$ is a disjoint union of cycles, all of length at least 5. We can obtain a $b$-coloring with 3 colors by assigning colors 1, 2, 3, 1, 2 to five consecutive vertices in one cycle of $G$, and to color the rest of $G$ with colors 1, 2, 3 greedily. So $b(G) = 3$.

Now let $d \geq 3$. We may assume that $G$ contains a cycle of length 5, for otherwise the result follows from \textbf{Theorem 2}. Therefore in Cases 3, 4, and 5 below, we let $x_1, \ldots, x_5$ be five vertices of $G$ that induce a cycle $C$ in this order.

\textbf{Case 3:} $d = 3$. For each $i = 1, \ldots, 5$, let $u_i$ be the neighbor of $x_i$ that is not in the cycle $C$. First suppose that the edge $u_1u_2$ exists for every $i = 1, \ldots, 5$. Then the vertices $x_1, \ldots, x_5, u_1, \ldots, u_5$ induce the Petersen graph and form one connected component of $G$. Since $G$ itself is not the Petersen graph, it must have another component $Z$. In that case, give color 1 to $x_1, x_3, x_2, x_5, x_3$, and color 4 to $x_1, u_1, u_2, u_3$ and to some vertex $z$ of $Z$, and give colors 1, 2, 3 to the neighbors of $z$. So $x_1, x_2, x_3, z$ are $b$-dominating vertices of colors 1, 2, 3, 4 respectively, and this coloring can be extended to a coloring of $G$ with four colors in any greedy way.

Now we can assume, up to symmetry, that $u_1u_3$ is not an edge of $G$. We construct a $b$-coloring with 4 colors such that $x_1, x_2, x_3, u_2$ are $b$-dominating vertices of colors 1, 2, 3, 4 respectively. To do this, we first give color 1 to $x_1, x_3$, color 2 to $x_2, x_3$, and color 4 to $u_1, u_2, u_3$. Note that $x_1, x_2, x_3$ are $b$-dominating vertices of colors 1, 2, 3 respectively. Now consider $u_2$. Let $a, b$ be the two neighbors of $u_2$ different from $x_2$. (Possibly $[a, b] \cap \{u_2, \} \neq \emptyset$.) Note that $a$ and $b$ are not adjacent to $x_1, x_2, x_3$, for otherwise $G$ would contain a cycle of length 3 or 4. Moreover, and for the same reason, each of $a, b$ is adjacent to at most one of $x_4, x_5$; and if each of them is adjacent to one of $x_4, x_5$ then it is not to the same vertex; in other words the edge set between $[a, b]$ and $[x_4, x_5]$ is a matching of size at most two. So it is possible to give color 1 to one of $a, b$ and color 3 to the other without having two adjacent vertices of the same color. Now $u_2$ is a $b$-dominating vertex of color 4. Finally this coloring can be extended to a coloring of $G$ with four colors in any greedy way.
Case 4: $d = 4$. For each $i = 1, \ldots, 5$, let $A_i$ be the set of the two neighbors of $x_i$ that are not in $C$. Here all subscripts on the $A_i$'s are understood modulo 5 and from the set $\{1, \ldots, 5\}$. Since $G$ contains no cycle of length 3 or 4, it is easy to see that:

- $A_i$ is a stable set; 
- $A_i \cap A_j = \emptyset$ if $i \neq j$;  
- There is no edge between $A_i$ and $A_{i+1}$; 

We construct a $b$-coloring of $G$ with five colors such that $x_1, \ldots, x_5$ are $b$-dominating vertices of colors $1, \ldots, 5$ respectively, as follows. For each $i = 1, \ldots, 5$, assign color $i$ to $x_i$ and colors $i+2$ and $i+3$ (modulo 5) to the two vertices of $A_i$. Fact (4), and the fact that the two colors assigned to the vertices of $A_i$ are different from the two colors assigned to $A_{i+2}$ resp. $A_{i+3}$, ensures that no two adjacent vertices in $A_1 \cup \cdots \cup A_5$ receive the same color. Thus all vertices of $A_1, \ldots, A_5$ have received a color, and each of $x_1, \ldots, x_5$ has neighbors of all colors other than its own. Finally, since the uncoulored vertices have degree 4, we can color them successively with one of the five colors, in any greedy way. Thus we obtain a $b$-coloring of $G$ with five colors.

Case 5: $d = 5$. For each $i = 1, \ldots, 5$, let $A_i$ be the set of the three neighbors of $x_i$ that are not in $C$. All subscripts on $A_1, \ldots, A_5$ are understood modulo 5 and from the set $\{1, \ldots, 5\}$. Since $G$ contains no cycle of length 3 or 4, it is easy to see that:

- $A_i$ is a stable set; 
- $A_i \cap A_j = \emptyset$ if $i \neq j$; 
- There is no edge between $A_i$ and $A_{i+1}$; 
- Every vertex different from $x_i$ has at most one neighbor in $A_i$.

For each $i = 1, \ldots, 5$, we can find a neighbor $s_i$ of $x_i$ such that the set $S = \{s_1, \ldots, s_5\}$ is a stable set, as follows. Pick any $s_1 \in A_1$, then $s_2 \in A_1 \setminus N(s_1), s_3 \in A_5 \setminus N(s_2), s_4 \in A_2 \setminus N(s_3)$, and $s_5 \in A_4 \setminus (N(s_1) \cup N(s_2))$. Such vertices exist because of (7) and (8). It follows from this construction that $S = \{s_1, \ldots, s_5\}$ is indeed a stable set. We rename vertex $s_1$ as $x_0$. For $i = 1, \ldots, 5$, let $B_i = A_i \setminus \{s_i\}$; so $|B_i| = 2$. Let $B_6 = N(x_0) \setminus \{x_1\}$; so $|B_6| = 4$. Since $G$ contains no cycle of length 3 or 4, it is easy to see that:

- $B_6$ is a stable set; 
- $B_6 \cap (B_1 \cup B_2 \cup B_3) = \emptyset$; 
- $|B_6 \cap B_i| \leq 1$ for $i \in \{3, 4\}$; 
- There is no edge between $B_6$ and $B_1$; 
- Every vertex different from $x_0$ has at most one neighbor in $B_6$.

Note that condition (7) implies that:

- There is no edge between $B_i$ and $B_{i+1}$ ($i \in \{1, \ldots, 5\}$, modulo 5);

and conditions (8) and (13) imply that:

- The edges between $B_i$ and $B_{i+1}$ form a matching ($i, j \in \{1, \ldots, 6\}, i \neq j$).

We construct a $b$-coloring of $G$ with six colors such that $x_1, \ldots, x_6$ will be $b$-dominating vertices of colors $1, \ldots, 6$ respectively. We start by assigning color $i$ to $x_i$ for each $i = 1, \ldots, 5$ and color 6 to the vertices of $S = \{s_1, \ldots, s_5\}$. Now we must find a way to assign colors $i+2$ and $i+3$ (modulo 5) to the two vertices of $B_i$, for each $i = 1, \ldots, 5$, and colors $2, 3, 4, 5$ to the four vertices of $B_6$. We view this as a list-coloring problem, where each vertex of $B_i$ ($i = 1, \ldots, 5$) has a list of allowed colors $L_i = \{i+2, i+3\}$ and each vertex of $B_6$ has a list of allowed colors $L_6 = \{2, 3, 4, 5\}$. See Fig. 1. In that figure, each box represents a set $B_j$ with its list $L_j$; a line between two boxes means that there may be edges between the corresponding sets, subject to condition (15); and no line between two boxes illustrates conditions (12) and (14). During our coloring procedure, we will say that a vertex $x$ loses a color $j$ if this color must be removed from the list of allowed colors for $x$ (because it has been assigned to a neighbor of $x$).
Recall from (11) that $B_0$ may have one common vertex with any of $B_1$, $B_4$. Let the vertices of $B_0$ be called $a$, $b$, $c$, $d$ such that: if $B_0 \cap B_3 \neq \emptyset$, then $a$ is the (unique) vertex in that intersection; and if $B_0 \cap B_4 \neq \emptyset$, then $b$ is the (unique) vertex in that intersection. Assign colors 5, 2, 3, 4 to $a$, $b$, $c$, $d$ respectively.

By (14), and since the sets of colors we want to assign to $B_i$ and $B_{i+2}$ (modulo 5) are disjoint, we can ignore the edges between any two such sets. Therefore we may color the sets $B_1$, \ldots, $B_5$ independently from each other and no conflict will arise between any two sets.

By (12), we can color the two vertices of $B_1$ with colors 3 and 4. Because of the assignment in $B_0$, and by (15), for each $j \in \{4, 5\}$ at most one vertex of $B_2$ loses color $j$, and that is a different vertex for each $j$. So it is possible to color the two vertices of $B_2$ with colors 4 and 5. The same holds for $B_3$ with colors 2 and 3. Now consider $B_4$. If $a \in B_3$, then $a$ is a vertex of color 5 in $B_3$ and the remaining vertex of $B_3$ can be colored 1. If $a \notin B_3$ then one vertex of $B_3$ may lose color 5, but it is still possible to color the two vertices of $B_3$ with colors 1 and 5. The same holds for $B_4$ with colors 1, 2. Thus all vertices of $B_1$, \ldots, $B_6$ have received a color, and each of $x_1, \ldots, x_6$ has neighbors of all colors other than its own. Finally, since the uncolored vertices have degree 5, we can color them successively with one of the six colors, in any greedy way. Thus we obtain a $b$-coloring of $G$ with six colors.

**Case 6:** $d = 6$. The proof here uses similar arguments as in the case $d = 5$, but the situation is more complicated. We can assume that $G$ contains a cycle of length 6, for otherwise the result follows from Theorem 3. Let $x_1, \ldots, x_6$ be six vertices of $G$ that induce a cycle in this order. For each $i = 1, \ldots, 6$, let $A_i = N(x_i) \setminus \{x_{i-1}, x_{i+1}\} \setminus \{x_j\}$; so $|A_i| = 4$. Here all subscripts on $A_1, \ldots, A_6$ are understood modulo 6 and from the set $\{1, \ldots, 6\}$. Since $G$ contains no cycle of length 3 or 4, it is easy to see that:

- $A_i$ is a stable set; \hspace{1cm} (16)
- $A_i \cap A_{i+1} = \emptyset$ and $A_i \cap A_{i+2} = \emptyset$; \hspace{1cm} (17)
- $|A_i \cap A_{i+3}| \leq 1$; \hspace{1cm} (18)
- There is no edge between $A_i$ and $A_{i+1}$; \hspace{1cm} (19)
- Every vertex different from $x_i$ has at most one neighbor in $A_i$. \hspace{1cm} (20)

For each $i = 1, \ldots, 6$, we find a neighbor $s_i$ of $x_i$ such that the set $S = \{s_1, \ldots, s_6\}$ is a stable set, as follows:

- If $A_1 \cap A_4 \neq \emptyset$, let $s_1$ and $s_4$ be equal to the (unique) vertex in $A_1 \cap A_4$. If $A_1 \cap A_4 = \emptyset$, let $s_1$ be any vertex in $A_1$ and $s_4$ be any vertex in $A_4 \setminus N(s_1)$ (such $s_4$ exists by (20)). \hspace{1cm} (21)
- If $A_3 \cap A_6 \neq \emptyset$, let $s_3$ and $s_6$ be equal to the (unique) vertex in $A_3 \cap A_6$. Note that, by (19), this vertex is not adjacent to the vertices $s_1$ and $s_4$ found previously. If $A_3 \cap A_6 = \emptyset$, let $s_3$ be any vertex in $A_3 \setminus (N(s_4) \cup N(s_1))$ (vertices $s_3$ and $s_6$ exist by (20)). \hspace{1cm} (22)
- If $A_5 \cap A_2 \neq \emptyset$, let $s_5$ and $s_2$ be equal to the (unique) vertex in $A_5 \cap A_2$. Note that, by (19), this vertex is not adjacent to any of the vertices $s_1, s_3, s_4, s_6$ found previously. If $A_5 \cap A_2 = \emptyset$, then, by (20), there are at least two vertices in $A_5 \setminus (N(s_1) \cup N(s_3))$ and at least two vertices in $A_2 \setminus (N(s_4) \cup N(s_6))$; and by (20) again, among these four vertices there are non-adjacent vertices $s_5 \in A_5$ and $s_2 \in A_2$. \hspace{1cm} (23)

It follows from this construction that the set $S = \{s_1, \ldots, s_6\}$ is a stable set. We rename vertex $s_1$ as $x_7$. For $i = 1, \ldots, 6$, let $B_i = A_i \setminus \{s_i\}$; so $|B_i| = 3$. Note that $B_1, \ldots, B_6$ are pairwise disjoint by (17), (18) and the definition of $\{s_1, \ldots, s_6\}$. Let $B_7 = N(x_7) \setminus \{x_1\}$; so $|B_7| = 5$. Since $G$ contains no cycle of length 3 or 4, it is easy to see that:

- $B_7$ is a stable set; \hspace{1cm} (24)
- $B_7 \cap (B_1 \cup B_2 \cup B_6) = \emptyset$; \hspace{1cm} (25)
- $|B_7 \cap B_i| \leq 1$ for $i \in \{3, 4, 5\}$; \hspace{1cm} (26)
- There is no edge between $B_7$ and $B_i$; \hspace{1cm} (27)
- Every vertex different from $x_7$ has at most one neighbor in $B_7$. \hspace{1cm} (28)

Note that condition (19) implies that:

There is no edge between $B_i$ and $B_{i+1}$ ($i \in \{1, \ldots, 6\}$, modulo 6); \hspace{1cm} (29)

and conditions (20) and (25) imply that:

The edges between $B_i$ and $B_j$ form a matching ($i, j \in \{1, \ldots, 7\}, i \neq j$). \hspace{1cm} (30)

In particular:

If sets $X \subseteq B_i$ and $Y \subseteq B_j$ are such that $|X| > |Y|$, then some vertex of $X$ has no neighbor in $Y$. \hspace{1cm} (31)

We construct a $b$-coloring of $G$ with seven colors such that $x_1, \ldots, x_7$ will be $b$-dominating vertices of colors $1, \ldots, 7$ respectively. We start by assigning color $i$ to $x_i$ for each $i = 1, \ldots, 6$ and color 7 to the vertices of $S = \{s_1, \ldots, s_6\}$. Now we must find a way to assign colors $i + 2$, $i + 3$, $i + 4$ (modulo 6) to the three vertices of $B_i$ for each $i = 1, \ldots, 6$, and colors 2, 3, 4, 5, 6 to the five vertices of $B_7$. We view this as a list-coloring problem, where each vertex of $B_i$ ($i = 1, \ldots, 6$) has a list
of allowed colors $L_1 = \{i + 2, i + 3, i + 4\}$ and each vertex of $B_7$ has a list of allowed colors $L_7 = \{2, 3, 4, 5, 6\}$. See Fig. 2. In that figure, each box represents a set $B_i$ with its list $L_i$; a line between two boxes means that there may be edges between the corresponding sets, subject to condition (27); and no line between two boxes illustrates conditions (24) and (26). During our coloring procedure, we will say that a vertex $x$ loses a color $j$ if this color must be removed from the list of allowed colors for $x$.

Recall from (23) that $B_7$ may have one common vertex with any of $B_3, B_4, B_5$. Up to symmetry, we may assume that $|B_7 \cap B_3| \geq |B_7 \cap B_5|$, in other words, if $B_7$ intersects one of $B_3, B_5$ then it intersects $B_3$. Define vertices $a, b, c$ of $B_7$ as follows, where we distinguish two cases:

Case (i): $B_7 \cap B_4 \neq \emptyset$. Let $a$ be the (unique) vertex in $B_7 \cap B_4$. Then, if $B_7 \cap B_3 \neq \emptyset$, let $b$ be the (unique) vertex in $B_7 \cap B_3$ (note that $b$ has no neighbor in $B_4$ by (19)); else, let $b$ be a vertex in $B_7 \setminus \{a\}$ that has no neighbor in $B_4$ (such a vertex exists by (28)). Finally, if $B_7 \cap B_3 = \emptyset$, let $c$ be the vertex in $B_7 \cap B_5$; else, let $c$ be any vertex in $B_7 \setminus \{a, b\}$.

Case (ii): $B_7 \cap B_4 = \emptyset$. If $B_7 \cap B_3 \neq \emptyset$, let $b$ be the (unique) vertex in this intersection (note that $b$ has no neighbor in $B_4$ by (19)); else, let $b$ be a vertex in $B_7$ that has no neighbor in $B_4$ (such a vertex exists by (28)). Then, if $B_7 \cap B_3 \neq \emptyset$, let $c$ be the vertex in this intersection; else, let $c$ be any vertex in $B_7 \setminus \{b\}$. Finally, let $a$ be any vertex in $B_7 \setminus \{b, c\}$.

Note that in all cases vertices $a, b, c$ are well defined and different since $B_3, B_4, B_5$ are pairwise disjoint as mentioned above; and $b$ has no neighbor in $B_4$. Then a vertex $d \in B_7$ is chosen as follows:

If $b \in B_3, c \in B_5, a$ has a neighbor $v_5 \in B_5$ and $v_5$ has a neighbor $v_3 \in B_3$, then choose $d$ in $B_7 \setminus \{a, b, c\}$ and not adjacent to $v_3$ (such a vertex exists by (28)); else let $d$ be any vertex in $B_7 \setminus \{a, b, c\}$.

Finally let $e$ be the remaining vertex of $B_7$. Assign colors $2, 6, 3, 5, 4$ to $a, b, c, d, e$ respectively. Pick a vertex $f_2 \in B_2$ not adjacent to $e$ and a vertex $f_6 \in B_6$ not adjacent to $e$ or $f_2$; such vertices exist by (20). Assign color 4 to $f_2$ and $f_6$.

Let $i \in \{1, \ldots, 6\}$. By (19) there is no edge between $B_i$ and $B_{i-1} \cup B_{i+1}$. Moreover, the sets of colors we want to assign to $B_i$ and to $B_{i+3}$ are disjoint, so we can ignore the edges between these two sets. Therefore we can color $B_1 \cup B_2 \cup B_5$ and $B_2 \cup B_4 \cup B_6$ independently of each other with no conflict between the two sets.

Let us consider $B_2, B_4, B_6$. Because of the assignment in $B_7$, and by (20), for each $j \in \{5, 6\}$ at most one vertex of $B_2 \setminus \{f_j\}$ loses color $j$ (a different vertex for each $j$). So it is possible to assign colors 5 and 6 to the two vertices of $B_2 \setminus \{f_2\}$. Likewise, for each $k \in \{2, 3, 4\}$ at most one vertex of $B_4 \setminus \{f_6\}$ loses color $k$ (a different vertex for each $k$), so it is possible to assign colors 2 and 3 to the two vertices of $B_4 \setminus \{f_6\}$. Call $t$ the vertex of $B_4$ that receives color 2; so $t$ is not adjacent to $a$. We are left with coloring the vertices of $B_4$. First suppose that $a \in B_4$ (case (i)). Then $a$ is a vertex of color 2 in $B_4$ (recall that $a, t$ are not adjacent), and the two vertices of $B_4 \setminus \{a\}$ lose color 2. Because of the assignment in $B_2 \cup B_6$, and by (20), at most one vertex of $B_4$ can lose a color (color 6), and by the choice of $b$ no other vertex of $B_4$ can lose color 6. So it is possible to assign colors 1 and 6 to the two vertices of $B_4 \setminus \{a\}$. Now suppose that $a \notin B_4$ (case (ii)). Because of the assignment in $B_2 \cup B_4 \cup B_6$, and by (20), at most two vertices of $B_4$ lose color 2, and by the choice of $b$ at most one loses color 6. So it is possible to assign colors 1, 2, 6 to the three vertices of $B_4$. Thus, in either case all vertices of $B_2 \cup B_4 \cup B_6 \cup B_7$ have received a color, and each of $x_2, x_4, x_5, x_7$ has neighbors of all colors other than its own.

Now we deal with $B_2, B_5$. First suppose that $c \in B_5$, which implies $b \in B_3$ since $|B_7 \cap B_3| \geq |B_7 \cap B_5|$. Then the vertices of $B_3 \setminus \{b\}$ lose color 6 and the vertices of $B_5 \setminus \{c\}$ lose color 3. Because of the assignment in $B_7$, at most one vertex $v_3 \in B_3 \setminus \{b\}$ loses a color (color 5) and at most one vertex $v_5 \in B_5 \setminus \{c\}$ loses a color (color 2). We assign color 1 to $v_3$ and $v_5$. Note that, by the choice of $d$, we may assume that $v_3$ and $v_5$ are not adjacent. Then we assign color 5 to the third vertex of $B_3$ and color 2 to the third vertex of $B_5$. Now suppose that $c \notin B_5$. Because of the assignment in $B_7$, for each $j \in \{2, 3\}$ at most one vertex of $B_5$ loses color $j$ (a different vertex for each $j$), so some vertex $w_5 \in B_5$ loses no color. If $b \in B_3$, then the vertices of $B_3 \setminus \{b\}$ lose color 6 and, because of the assignment in $B_7$, at most one vertex $v_5 \in B_3 \setminus \{b\}$ loses a color (color 5). So we assign color 1 to $v_3$ and color 5 to the remaining vertex of $B_3$. Because of this assignment in $B_3$, at most one vertex of $B_5$ loses color 1. So it is possible to assign colors 1, 2, 3 to the three vertices of $B_3$. If $b \notin B_3$, then for each $j \in \{5, 6\}$ at most one vertex of $B_3$ loses color $j$ (a different vertex for each $j$), so some vertex $w_3 \in B_3$ loses no color. By (20), among the four vertices of $(B_3 \cup B_5) \setminus \{w_3, v_3\}$, there are two non-adjacent vertices, one in $B_3$ and one in $B_5$, to which we assign color 1. Then it is possible to assign color 5 and 6 to the remaining vertices of $B_3$ and colors 2 and 3 to the remaining vertices of $B_5$. 
Now we deal with $B_1$. Recall that (22) and (24) hold. Because of the assignment in $B_3$, at most one vertex of $B_1$ loses a color (color 5) and because of the assignment in $B_2$ at most one vertex of $B_1$ (possibly the same vertex) loses a color (color 3). So it is possible to color the three vertices of $B_1$ with the colors 3, 4, 5. Thus all vertices of $B_1 \cup B_2 \cup B_3$ have received a color, and each of $x_1, x_2, x_3$ has neighbors of all colors other than its own.

Finally, since the uncolored vertices have degree 6, we can color them successively with one of the seven colors, in any greedy way. Thus we obtain a $b$-coloring of $G$ with seven colors. □

The proof above illustrates a technique which can probably not be extended to the general case. Indeed we tried to make a similar proof for graphs with $d = 7$, but the case analysis seems to become inextricable.

**Remark.** In view of the case $d = 7$, we may consider the so-called Hoffman–Singleton graph $HS$, which is the smallest 7-regular graph with girth at least 5. This graph is famous for many interesting properties related to its highly symmetric structure; see [14,16]. It is natural to suspect that $HS$ might be a counterexample to El-Sahili and Kouider’s question. However, the $b$-chromatic number of $HS$ is 8 (it is not hard to construct a $b$-coloring of $HS$ with eight colors such that one vertex and its neighbors are $b$-vertices of colors 1, . . . , 8).

In conclusion we propose the following reformulation of El-Sahili and Kouider’s question:

**Conjecture 1.** Every $d$-regular graph with girth at least 5, different from the Petersen graph, has a $b$-coloring with $d + 1$ colors.

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**References**