Conditions for a CAPM equilibrium with positive prices

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Abstract

This paper examines the conditions required to guarantee positive prices in the CAPM. Positive prices imply an upper bound on the equity premium. This upper bound depends on the degree of diversity of firms’ fundamentals, and it is independent of investors’ preferences. In economies with realistically diverse assets the only positive-price CAPM equilibrium theoretically possible is a degenerate one, with zero equity premium. Furthermore, when specific standard investors’ preferences are assumed, the CAPM equilibrium with positive prices may be altogether impossible. A possible solution to these fundamental problems may be offered by the segmented-market version of the model.

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0. Introduction

The Sharpe [34] – Lintner [18] capital asset pricing model (CAPM) is no-doubt one of the cornerstones of financial economics. As Roll [30] and Roll and Ross [32] have shown, empirical testing of the CAPM is not straightforward, and there is an ongoing debate among researchers about the empirical validity of the model.\textsuperscript{1} Nonetheless, The CAPM is the foundation of the core ideas of optimal diversification and non-diversifiable risk. In this paper we address the

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\textsuperscript{1}See, for example, Roll [29], Black et al. [6], Miller and Scholes [23], Levy [15], Amihud et al. [2], Fama and French [9], Jagannathan and Wang [14], and Fama and French [10]. See Markowitz [21] for a recent discussion.

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theoretical, rather than the empirical, consistency of the CAPM. We characterize conditions on firms’ fundamentals and investors’ preferences under which a CAPM equilibrium with positive prices is theoretically possible.

One of the key results of the CAPM is the identity of the optimal unlevered mean–variance tangency portfolio with the market portfolio. Thus, although the CAPM does not explicitly restrict short positions, in order for the model to be self-consistent the tangency portfolio must be a positive portfolio, i.e. the portfolio weights of all assets in the tangency portfolio must be positive. Several researchers have investigated the conditions on expected returns and the return covariance matrix which ensure a positive tangency portfolio. In practice, the tangency portfolio based on empirical estimates of the expected returns and return covariances is typically found to involve many short positions. This is a well-known and very robust result. However, proponents of the CAPM suggest that this apparent contradiction to the model may be due to empirical measurement problems. They stress that the ex ante expected returns and covariances are not observable parameters, but rather they are determined in equilibrium by the end-of-period firm value distributions and by the pricing of the assets. It is argued that given a set of end-of-period value distributions, positive prices can be determined in such a way that the CAPM holds.

The main purpose of this paper is to examine this assertion. Namely, we ask the following question. Consider an economy with given firm fundamentals (end-of-period value distributions), and assume that all of the CAPM assumptions hold. Is there a set of positive prices which guarantees the CAPM equilibrium?

Lintner [18] shows that given a set of firms’ fundamentals (i.e. their end-of-period value distributions) prices can be determined such that the CAPM risk-return relationship holds. This result is very encouraging, and it represents the way most financial economists think about the CAPM equilibrium. However, it leads to two other important questions: (1) Does the set of prices which ensures the CAPM risk-return relationship also ensure that prices are positive? (2) Given investors’ preferences, does a CAPM equilibrium always exist? This paper addresses these questions.

We offer two main results: (1) Under the CAPM assumptions with no restriction on preferences, for realistically diverse firms’ fundamentals the only positive-price CAPM equilibrium possible is one with practically zero equity premium. This is, of course, in contradiction to the basic CAPM notion of risk and return, and to the empirically observed equity premium. (2) When the analysis is confined to specific standard preferences, we show that the CAPM equilibrium with positive prices may be altogether impossible.

In a series of illuminating papers, Nielsen [24–27] examines the conditions for the existence and the uniqueness of the CAPM equilibrium for an exogenously given set of firm fundamentals. In particular, Nielsen [27] derives conditions on preferences that ensure positive CAPM prices. He points out that the model has the peculiar property that preferences for assets may be non-monotone—getting more of a positive mean asset for free may actually reduce utility if the increased variance overpowers the increased mean. This is the reason that equilibrium prices may generally be negative. Given the set of firm fundamentals, Nielsen derives bounds on investors’ risk-aversion which ensure positive prices. These bounds on risk-aversion, in turn, can be translated to limits on the equity premium. Thus, from the fundamentals themselves an upper bound can be set on the equity premium, and this upper bound must hold regardless of preference specification. This is the basis of our main finding, result (1) above.

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2 See Roll [30], Roll and Ross [31], Rudd [33], Green [13], and Best and Grauer [4,5].

3 See, for example, Levy [16], and Frost and Savarino [11,12].
The structure of this paper is as follows. In Section 1 we review and extend Nielsen’s [25] results demonstrating that the CAPM equilibrium is not unique. It is shown that given a single set of firms’ fundamentals there are infinitely many price vectors that ensure the CAPM risk-return relationship. We build on this result and characterize the set of all equilibria that satisfy the CAPM risk-return relationship. We show that the market portfolios that correspond to these equilibria lie on a straight line in the mean–standard-deviation plane. In Section 2 we show that only a bottom segment of this straight line corresponds to equilibria with positive prices. Market portfolios beyond this bottom segment represent mathematical solutions which satisfy the CAPM risk-return relationship, but involve negative prices for some of the assets. The size of the segment of positive-price CAPM equilibria is determined by the firms’ fundamentals, as described in Theorem 1. We show that for realistic sets of firms’ fundamentals the subset of positive-price equilibria shrinks to a single degenerate equilibrium with zero equity premium. These results are general, and are not based on any assumptions regarding investors’ endowments or preferences (other than risk-aversion). In Section 3 we show that when specific preferences are assumed, the set of possible equilibria is further restricted. In fact, for many sets of firms’ fundamentals and standard investors’ preferences a CAPM equilibrium with positive prices is simply impossible. Section 4 concludes the paper, and suggests that the segmented-market extension of the CAPM (see [15,22,20,35]) may offer a solution to the problems discussed in the paper. Thus, while the CAPM suffers from theoretical internal-consistency problems, its extension, the segmented-market CAPM, can offer a solution to these inconsistencies.

1. Multiple CAPM equilibria

Consider an economy composed of \( n \) firms with stochastic end-of-period liquidation values. The end-of-period value of firm \( i \) is denoted by \( \tilde{V}_{i1} \) and it is normally distributed \(^4\) with mean \( \bar{V}_{i1} \) and standard deviation \( \sigma_{i} \) (the bar on top of \( \sigma \) distinguishes the standard deviation of the end-of-period value from the more conventional standard deviation of rates of return, which is denoted by \( \sigma_t \)). Lintner [18] shows that in such an economy firms can be priced such that the CAPM holds. Lintner derives the CAPM firm market values as

\[
V_{i0} = \frac{\bar{V}_{i1} - \gamma \sum_{j=1}^{n} \sigma_{ij}}{1 + r_f},
\]

(1)

where \( V_{i0} \) is the market value of firm \( i \) at time zero, \( r_f \) is the risk-free interest rate, \( \sigma_{ij} \) is the covariance between the end-of-period values of firms \( i \) and \( j \) (\( \sigma_{ij} \equiv \text{Cov}(\tilde{V}_{i1}, \tilde{V}_{j1}) \)), and \( \gamma \) is defined by Lintner as “the market price of risk” (see Lintner Eq. (17), p. 600). \(^5\) This constitutes

\(^4\) The CAPM is typically based on the assumption of normal return distributions or alternatively on the assumption of quadratic preference (see [8,28,3] for generalizations). Each of these alternatives has its pros and cons. As the normal distribution has infinite support, this case suffers from the problem of allowing negative total returns (or negative end-of-period values). Quadratic utility is characterized by increasing absolute risk-aversion, which is inconsistent with empirical and experimental observation. However, Levy and Markowitz [17] show that mean–variance analysis provides an excellent approximation for expected utility maximization even if preferences are not quadratic and the return distributions are not normal. Some studies simply assume mean–variance preferences, rather than viewing them as a result of expected utility maximization (see, for example, [26,1,19]). The analysis below does not depend on the framework used as the basis for mean-variance analysis.

\(^5\) We follow Lintner’s notation: tilda stands for a random variable and bar stands for the expected value, with the distinction that a bar over \( \sigma \) stands for the standard deviation and covariance of the end-of-period total value (rather than the standard deviation and covariance of rates of return).
an equilibrium in the usual sense, i.e., investors maximize expected utility and markets clear. Note, however, that the firm values, \( V_{i0} \), are not necessarily positive.

The market values \( V_{i0} \) in (1) simultaneously determine the rates of return, the standard deviation and covariances of returns, the proportion of each firm in the market portfolio, and consequently its beta. Namely, for a given market value \( V_{i0} \), the rate of return on firm \( i \) is given by

\[
\tilde{r}_i = \frac{\tilde{V}_i - V_{i0}}{V_{i0}},
\]

which implies the following expected return, standard deviation, and covariance of returns:

\[
\bar{r}_i = \frac{\bar{V}_i - V_{i0}}{V_{i0}}, \quad \sigma_i = \frac{\bar{\sigma}_i}{V_{i0}} \quad \text{and} \quad \sigma_{ij} = \frac{\bar{\sigma}_{ij}}{V_{i0}V_{j0}}.
\]

Firm \( i \)'s proportion in the market portfolio, denoted by \( x_i \), is given by

\[
x_i = \frac{V_{i0}}{\sum_{j=1}^{n} V_{j0}} = \frac{V_{i0}}{T_0},
\]

where \( T_0 \) is the total market value of all firms at period 0: \( T_0 \equiv \sum_{j=1}^{n} V_{j0} \).

The beta of firm \( i \) is given by

\[
\beta_i = \frac{\text{Cov}(\tilde{r}_i, \bar{R}_m)}{\sigma_m^2} = \frac{\text{Cov}(\tilde{r}_i, \sum_{j=1}^{n} x_j \tilde{r}_j)}{\sum_{i=1}^{n} x_i \sigma_{ij}} = \frac{1}{T_0} \sum_{j=1}^{n} V_{j0} \sigma_{ij} = \frac{\bar{V}_i}{V_{i0}} \sum_{j=1}^{n} \bar{\sigma}_{ij} = \frac{T_0 \sum_{j=1}^{n} \bar{\sigma}_{ij}}{\sum_{i=1}^{n} \bar{\sigma}_{ij}},
\]

and the market portfolio expected rate of return and variance are

\[
\bar{R}_m = \sum_{i=1}^{n} x_i \bar{r}_i = \sum_{i=1}^{n} \frac{V_{i0}}{T_0} \left( \frac{\bar{V}_i - V_{i0}}{V_{i0}} \right) = \sum_{i=1}^{n} \bar{V}_i - 1,
\]

and

\[
\sigma_m^2 = \sum_{i=1}^{n} x_i x_j \sigma_{ij} = \frac{1}{T_0^2} \sum_{j=1}^{n} \bar{\sigma}_{ij}.
\]

The market price of risk, \( \gamma \), is given by

\[
\gamma = \frac{\bar{R}_m - r_f}{T_0 \sigma_m^2}.
\]

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6 Eq. (1) can be written as \( \gamma \sum_{j=1}^{n} \bar{\sigma}_{ij} = \bar{V}_i - (1 + r_f)V_{i0} \). Summing both sides over all assets \( i \) we obtain

\[
\gamma \sum_{i=1}^{n} x_i \bar{r}_i = \sum_{i=1}^{n} \bar{V}_i - (1 + r_f) \sum_{i=1}^{n} V_{i0},
\]

which can be written as \( \gamma = \frac{\sum_{i=1}^{n} V_{i0} - (1 + r_f)}{T_0} \sum_{i=1}^{n} \bar{\sigma}_{ij} = \frac{\bar{R}_m - r_f}{T_0 \sigma_m^2} \).
Lintner [18] shows that the pricing formula (1) generates expected returns and betas which satisfy the well-known Sharpe–Lintner CAPM security market line (SML) relationship:

$$\bar{r}_i = r_f + \beta_i (\bar{R}_m - r_f).$$

(8)

Note, however, that the vector of firm values which satisfies the CAPM risk-return relationship is not unique (as shown by Nielsen [25], and Bottazzi et al. [7]). Different values of $\gamma$, when implemented in Eq. (1) generate different sets of firm values, each of which guarantees a different CAPM solution, with different betas and expected returns, and a different market portfolio. Thus, given a set of fundamentals, there is an infinite number of possible CAPM solutions. So far, we have treated $\gamma$, the market price of risk, as a parameter. Obviously, $\gamma$ may depend on the investors’ preferences and on their initial endowments. It is interesting to note that Nielsen [25] has shown that even for a given set of preferences and endowments, multiple equilibria are possible (i.e. $\gamma$ may have different values). While multiple CAPM equilibria are possible, we show below that the market portfolios corresponding to these equilibria are not necessarily positive.

In order to show that under reasonable sets of fundamentals the only positive-price CAPM equilibrium possible is a degenerate one, let us first analyze the infinite set of firm value vectors which satisfy the CAPM risk-return relationship. As a first step we show below that the geometric location of the market portfolios corresponding to all of the CAPM equilibria is on a straight line in the mean–standard deviation plane, originating at the point (mean $= 0$, standard deviation $= -1$), see Fig. 1. To see this, note that from the expression for the market portfolio variance in Eq. (6b) we can write $T_0$ as

$$T_0 = \sqrt{\frac{\sum_{i=1}^{n} \sigma_{ij}}{\sigma_m}}.$$

Plugging this expression for $T_0$ into the formula for the market portfolio expected return in (6a) yields:

$$\bar{R}_m = \left[ \frac{\sum_{i=1}^{n} V_{i1} \sigma_{ij}}{\sqrt{\sum_{i=1}^{n} \sigma_{ij}^2}} \right] \sigma_m - 1. \tag{9}$$

Thus, all equilibrium market portfolios, characterized by $(\sigma_m, \bar{R}_m)$, lie on a straight line in the mean–standard deviation plane, a line which originates at point (0, −1) and has a slope of $\sum_{i=1}^{n} V_{i1} / \sqrt{\sum_{i=1}^{n} \sigma_{ij}^2}$. Under the assumption of risk-aversion, only equilibria with $\bar{R}_m \geq r_f$ are possible, i.e. only the solid segment of the straight line in Fig. 1 represents the set of CAPM equilibria. Let us denote the linear segment representing all the CAPM equilibria by Market Portfolio Line, or MPL.

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7 Multiplying Eq. (1) by $1 + r_f$ yields $V_{01}(1 + r_f) = V_{i1} - \gamma \sum_{j=1}^{n} \sigma_{ij}$. Dividing by $V_{01}$ and rearranging we obtain $r_i = \frac{V_{i1}}{V_{01}} - 1 = r_f + \gamma \sum_{j=1}^{n} \sigma_{ij}$, $r_f = \frac{\bar{R}_m - r_f}{\sigma_m^2}$, and $\sigma_m^2 = \frac{1}{\bar{R}_m} \sum_{i=1}^{n} \sigma_{ij}$. Substituting in (9), yields

$$\bar{R}_m = r_f + \frac{T_0}{\sum_{i=1}^{n} \sigma_{ij}} = r_f + \beta_i (\bar{R}_m - r_f),$$

where the last equality follows from Eq. (5).

8 The multiplicity of equilibria stems from the fact that the pricing equations (1) are dependent, because the market price of risk, $\gamma$, is a function of the asset prices, $V_{0i}$, (see Eq. (6) and (7)). To see this dependency, sum Eq. (1) over all assets and employ relations (6) and (7) to obtain the trivial identity $T_0 = T_0$. Thus, if there are $n$ assets, we have $n$ unknowns ($V_{0i}$), but only $n - 1$ independent equations, i.e. we have an infinite number of possible solutions (see also [25,7]).

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Fig. 1. The set of all potential CAPM equilibria. Each price vector that mathematically satisfies the CAPM risk-return relationship generates a certain market portfolio. The set of all these market portfolios corresponding to potential CAPM equilibria is a straight line in the mean-standard deviation plane, which is called the market portfolio line (or MPL). While all the points on the MPL are mathematical solutions to the CAPM, only those in the segment a–b are equilibria with positive prices. The market portfolios below point a imply risk-loving, and the market portfolios above point b involve negative prices for some of the assets.

The parameter \( \gamma \) uniquely determines the location of the equilibrium point on the MPL. To see this, note that \( \gamma \) is generally a function of \( \overline{R}_m, \sigma_m, \) and \( T_0 \) (see Eq. (7)). However, for portfolios on the MPL \( \sigma_m \) and \( T_0 \) can be written as a function of \( \overline{R}_m \), thus, we can isolate \( \gamma \) as a function of \( \overline{R}_m \):

\[
\gamma = \frac{\overline{R}_m - r_f}{T_0 \sigma_m^2} = \frac{\sum_{i=1}^{n} V_{i1} \sigma_{ij}}{\sum_{i=1,j=1}^{n} \sigma_{ij} \left( \frac{\overline{R}_m + r_f}{R_m + 1} \right)^2},
\]

where the second equality is obtained by employing Eq. (9) and the expression for \( \sigma_m \) in Eq. (6a). (Note that \( \sum_{i=1}^{n} V_{i1} \) and \( \sum_{i=1,j=1}^{n} \sigma_{ij} \) are given by the firms’ fundamentals, and are therefore constant). It is straightforward to verify that \( \frac{\partial \gamma}{\partial \overline{R}_m} > 0 \). Thus, \( \gamma \) is monotonically increasing along the MPL, and each point on the MPL represents a different potential CAPM equilibrium with some \( \gamma \) in the range \( 0 < \gamma < \gamma_{\text{max}} \). Thus, which of these potential equilibria are characterized by positive market values for all firms? To this question we turn next.

\[9 \quad \frac{\partial \gamma}{\partial \overline{R}_m} = \frac{\sum_{i=1}^{n} V_{i1} \sigma_{ij}}{\sum_{i=1,j=1}^{n} \sigma_{ij} \left( \frac{\overline{R}_m + r_f}{R_m + 1} \right)^2} \]

\[10 \text{ Where } \gamma_{\text{max}} \text{ is defined as } \gamma_{\text{max}} = \sum_{i=1}^{n} V_{i1} / \sum_{i=1,j=1}^{n} \sigma_{ij}. \text{ Eq. (10) implies that for } \overline{R}_m = r_f \text{ we have } \gamma = 0, \text{ and for } \overline{R}_m \to \infty \text{ we have } \gamma \to \gamma_{\text{max}}. \text{ Thus, for all equilibria on the MPL } 0 < \gamma < \gamma_{\text{max}}. \]
2. Conditions for an equilibrium with positive prices

Each point on the MPL represents a CAPM equilibrium, however, the mathematical CAPM equations do not explicitly require that all prices are positive. In this section, we characterize the subset of equilibria on the MPL with positive prices. We show that this subset is always the lower part of the MPL. We then demonstrate that for diverse economies the only positive-price CAPM equilibrium which is possible is a degenerate one with practically no equity premium, i.e. \( \bar{R}_m = r_f \). In order to prove this claim we need to introduce a new variable which measures the fundamental risk of each asset. Let us elaborate.

Define the “fundamental risk” of firm \( i \) as \( z_i \equiv \sum_{j=1}^{n} \frac{\sigma_{ij}}{V_i} \), i.e. \( z_i \) is the sum of firm \( i \)'s covariances with all firms (in terms of the end-of-period values) divided by its expected end-of-period value. The higher \( z_i \), the riskier the firm in terms of its fundamentals. Firm \( i \)'s fundamental risk is closely related to its beta. However, unlike \( \beta_i \), which is formulated in terms of returns, and which therefore depends on pricing and the specific equilibrium point on the MPL, \( z_i \) depends only on the exogenous firms' fundamentals. Denote the riskiness of the riskiest firm, i.e. the firm with highest fundamental risk, by \( z_{\text{max}} \equiv \max_i \{z_i\} \). Let us also denote the ratio between \( z_{\text{max}} \) and the “average riskiness” of all assets in the market by \( \delta \): \( \delta \equiv z_{\text{max}} / \left( \sum_{i=1}^{n} \frac{\sigma_{ij}}{V_i} \right) \).\(^{11}\)

Theorem 1 shows that \( \delta \) determines the subset of CAPM equilibria with positive prices.

**Theorem 1.** The set of CAPM equilibria with positive prices is a subset of the MPL: it is the lower segment of the MPL corresponding to \( \gamma \) in the range \( 0 < \gamma < 1/z_{\text{max}} \). In terms of the market portfolio expected return, this subset corresponds to the segment of the MPL with \( \bar{R}_m < \frac{r_f + \frac{1}{2}}{1 - \frac{\delta}{\delta}} \). (see segment a–b in Fig. 1). Thus, positive-prices imply an upper bound on \( \bar{R}_m \) and on the equity premium.

**Proof.** Positive prices imply \( V_{i0} > 0 \) for all assets \( i \). From Eq. (1) it is evident that \( V_{i0} > 0 \) implies that \( \bar{V}_{i1} > \gamma \sum_{j=1}^{n} \bar{\sigma}_{ij} \), or \( \gamma < \frac{\bar{V}_{i1}}{\sum_{j=1}^{n} \bar{\sigma}_{ij}} = \frac{1}{z_i} \).\(^{12,13}\) The requirement that \( V_{i0} > 0 \) for all assets implies that \( \gamma < 1/z_i \) for all \( i \), or \( \gamma < 1/z_{\text{max}} \).

Thus, all the market portfolios with positive prices are portfolios corresponding to \( \gamma < 1/z_{\text{max}} \). The expected return of any market portfolio can be generally written as

\[
1 + \bar{R}_m = \frac{\sum_{i=1}^{n} \bar{V}_{i1}}{\sum_{i=1}^{n} V_{i0}} = \frac{\sum_{i=1}^{n} \bar{V}_{i1} / \sum_{i=1}^{n} V_{i0}}{1 - \frac{\gamma}{1 + r_f} \frac{\sum_{i=1}^{n} \bar{\sigma}_{ij}}{\sum_{i=1}^{n} \bar{V}_{i1}}} = (1 + r_f) \left[ \frac{1}{1 - \frac{\gamma}{1 + r_f} \frac{\sum_{i=1}^{n} \bar{\sigma}_{ij}}{\sum_{i=1}^{n} \bar{V}_{i1}}} \right]. \tag{11}
\]

\(^{11}\) Note that the expression for the “average riskiness” in the market is the value weighted average of \( z \) over all assets: \( \frac{\sum_{i=1}^{n} \bar{V}_{i1} z_i}{\sum_{i=1}^{n} \bar{V}_{i1}} = \frac{\sum_{i=1}^{n} \bar{\sigma}_{ij} z_i}{\sum_{i=1}^{n} \bar{V}_{i1}} \).

\(^{12}\) If \( \sum_{j=1}^{n} \bar{\sigma}_{ij} \) happens to be negative, \( V_{i0} > 0 \) implies \( \gamma > \bar{V}_{i1} / \sum_{j=1}^{n} \bar{\sigma}_{ij} \). As \( \bar{R}_m > r_f \) implies \( \gamma > 0 \), \( V_{i0} > 0 \) does not imply an effective constraint on \( \gamma \) in this case.

\(^{13}\) This inequality also appears as a vector inequality in Example 2 of Nielsen [27]. The difference between the two frameworks is that Nielsen assumes an exogenous utility function which is a linear function of the mean and variance (which is, for example, the case for negative exponential utility and normal return distributions). In contrast, in the present framework no assumptions are made regarding preference, and \( \gamma \) is endogenous.
As the market portfolio’s expected return increases monotonically with \( \gamma \) (see the previous section), and as for all positive-price portfolios we have \( \gamma < 1/z_{\text{max}} \), the expected return for all positive-price portfolios is bounded by

\[
1 + \overline{R}_m < (1 + r_f) \left[ \frac{1}{1 - \frac{1}{z_{\text{max}}} \sum_{i,j} \bar{\sigma}_{ij} \sum_i \bar{V}_i} \right] = \frac{1 + r_f}{1 - \frac{1}{\delta}},
\]

or

\[
\overline{R}_m < \frac{r_f + \frac{1}{\delta}}{1 - \frac{1}{\delta}}.
\]

(12)

2.1. Discussion

Note that the subset of CAPM equilibria with positive prices is always non-empty: \( \gamma = 0 \) is a solution which ensures positive prices for all assets. This case, however, is a degenerate solution with no equity premium: \( \overline{R}_m = r_f \) (see Eq. (1) or (11)).

The maximal equity premium is determined by \( \delta \), the ratio of the “riskiness” (as measured by \( z \)) of the riskiest firm and the “average riskiness” \( \left( \sum_{i,j} \bar{\sigma}_{ij} / \sum_i \bar{V}_i \right) \). In a perfectly homogeneous market with identical firms all firms have the same \( z \) and we have \( z_i = z_{\text{max}} \) for all \( i \), and therefore \( \delta = 1 \). In this case Eq. (12) implies \( \frac{r_f + \frac{1}{\delta}}{1 - \frac{1}{\delta}} \rightarrow \infty \), i.e. there is no upper bound on the segment of the MPL representing CAPM equilibria with positive prices. The more heterogeneous the market is in terms of firms’ fundamental risk, the larger we would expect the ratio between the riskiness of the riskiest firm and the average riskiness to be \( \delta \gg 1 \), and therefore the lower the maximal equity premium will be.

The intuition for this result is as follows. The asset with the highest fundamental risk is the “most unattractive” asset: it has a large average covariance relative to its expected end-of-period value. The only way for this asset to be held long in the optimal mean–variance portfolio is for the market price of risk, \( \gamma \), to be very low. This, in turn, implies a very small equity premium for all other assets with “normal” fundamental risk, and as a result the total market equity premium will also be very low.

In the case of realistically large and heterogeneous markets, the subset of positive equilibria is reduced to the degenerate solution with practically no equity premium. Namely, if there is even a single very risky asset in the economy we will have for this asset \( z_{\text{max}} \gg \sum_{i,j} \bar{\sigma}_{ij} / \sum_i \bar{V}_i \), i.e. \( \delta \gg 1 \), and therefore \( \frac{r_f + \frac{1}{\delta}}{1 - \frac{1}{\delta}} \approx r_f \) (see Eq. (12)).

A numerical example may be helpful in illustrating this point. Consider, for example, an economy in which the typical firm has \( \bar{V}_1 = 100 \) and \( \bar{\sigma}_1 = 10 \). Suppose that there are 100 typical firms, and a single risky firm with \( \bar{V}_1 = 1 \) and \( \bar{\sigma}_1 = 10 \). Assume for simplicity that all pair-wise correlations are \( \rho \). For the typical firm \( z_i \) is given by \( z_i = \sum_j \bar{\sigma}_{ij} / \bar{V}_i = \frac{10^2 + 100 \cdot 10 \cdot \rho}{100} = 1 + 100 \rho \). For the risky firm \( z_i \) is \( z_i = \frac{10^2 + 100 \cdot 10 \cdot \rho}{1} = 100(1 + 100 \rho) \). Thus, as expected, \( z \) is larger for the risky firm, and we have \( z_{\text{max}} = 100(1 + 100 \rho) \).
The average riskiness, \( \sum_{i,j} \sigma_{ij} / \sum_{i} \bar{V}_{i} \), is given by
\[
\frac{\sum_{i,j} \sigma_{ij}}{\sum_{i} \bar{V}_{i}} = \frac{\sum_{i,j} \bar{V}_{i1} \sigma_{ij}}{\sum_{i} \bar{V}_{i1}} = \frac{100 \cdot 100 \cdot (1 + 100 \rho) + 1 \cdot 100 (1 + 100 \rho)}{100 \cdot 100 + 1} = 1.01 \cdot (1 + 100 \rho).
\]

Thus, \( \delta = \frac{100 \cdot 100 \rho}{1.01 \cdot (1 + 100 \rho)} \cong 99 \), and the expected return of all positive-price market portfolios is bounded by
\[
\bar{R}_{m} < r_{f} + \frac{1}{99} \cong 1.01 r_{f} + 0.01 \quad \text{(see Eq. (12)).}
\]

Thus, the maximal theoretically possible equity premium is only about 1%. Note that this result is independent of the value of the correlation, \( \rho \).

\( \delta \) determines the maximal equity premium. Note that the value of \( \delta \) and the upper bound on the maximal equity premium possible crucially depend on the single riskiest asset in the market. In realistically large and heterogeneous markets with a variety of many “long-shot” investments, we would expect \( \delta \) to be very large, and therefore the only positive-price CAPM equilibrium theoretically possible is one with practically zero equity premium.

3. Preference considerations

The preceding analysis derives the restrictions on the set of positive-price CAPM equilibria that are imposed by the firms’ fundamentals. These restrictions hold for all risk-averse preferences. If more specific information about the preferences of investors is given, this imposes further restrictions on the set of possible equilibria. In fact, when standard preferences are assumed, a positive-price CAPM equilibrium may be altogether impossible, as shown below.

The three classes of preferences that are probably the most common in the economic literature are the power utility function (with the logarithmic function as a special case), the negative exponential utility function, and the quadratic utility function. The power utility function is not compatible with the CAPM, because it is defined only over positive wealth and it is therefore incompatible with normal return distributions, which have infinite support (and if the return distribution is not normal the mean–variance criterion is not optimal for this preference). The quadratic utility function has the very problematic feature of decreasing utility beyond a certain wealth level. Thus, below we focus on the canonical case which a priori seem consistent with the CAPM: negative exponential utility with normal return distributions. We show that a positive-price CAPM equilibrium may be simply impossible under these preferences. The analysis is very similar to that of Example 2 in Nielsen [27], and it provides a simple graphical interpretation for the no positive-price equilibrium result in the MPL framework.

Consider an economy with a representative investor characterized by a negative exponential utility function with risk-aversion parameter \( \gamma_{0} : U(W) = -e^{-\gamma_{0} W} \). Faced with a risk-free return of \( r_{f} \) and a market portfolio yielding a normal rate of return distribution with mean \( \bar{R}_{m} \) and standard deviation \( \sigma_{m} \), such an investor will optimally invest an amount of
\[
\frac{\bar{R}_{m} - r_{f}}{\gamma_{0} \sigma_{m}^{2}} \text{ dollars in the}
\]

14 For a numerical example which illustrates a case where a positive price CAPM equilibrium is impossible with quadratic preferences, see http://bschool.huji.ac.il/faculty/moshe-l/.
market portfolio. Thus, the total market value, $T_0$, will be equal to this amount:

$$T_0 = \frac{\bar{R}_m - r_f}{\gamma_0 \sigma_m^2}.$$  

(14)

Using this expression for $T_0$ in the formula for the market price of risk, $\gamma$ (see Eq. (7)), we obtain:

$$\gamma = \frac{\bar{R}_m - r_f}{T_0 \sigma_m^2} = \frac{\bar{R}_m - r_f}{\sigma_m^2} = \gamma_0.$$

(15)

Thus, unsurprisingly, in this case the market price of risk is simply the investor’s risk-aversion coefficient $\gamma_0$.

In general, $\gamma$ determines the location of the CAPM equilibrium on the MPL. In the case of negative exponential utility $\gamma = \gamma_0$, and thus the investor’s risk-aversion determines the location on the MPL. If $\gamma_0 < 1/z_{\text{max}}$ then a CAPM equilibrium with positive prices is possible. However, if $\gamma_0 > 1/z_{\text{max}}$ this implies a market portfolio which is above point b on the MPL (see Fig. 1). As Theorem 1 tells us, this market portfolio implies at least one negative price. Indeed, in an economy with $\gamma_0 > 1/z_{\text{max}}$ the positive-price CAPM equilibrium is theoretically impossible. Note that $\gamma_0$ and $z_{\text{max}}$ are related to different basic aspects of the economy: $\gamma_0$ is related to investors’ preferences while $z_{\text{max}}$ is related to the firms’ fundamentals. Yet, the relationship between these two parameters determines the theoretical possibility of the CAPM with positive prices.

4. Conclusions

Most financial economists think of mean–variance optimization directly in terms of the expected returns and the covariances of returns. However, it is clear that these parameters are not exogenously “given”, but rather they are endogenously determined in equilibrium by the firms’ fundamentals (end-of-period value distributions) and by market pricing. While this is implicit in most formulations, Lintner [18] formalizes this analysis explicitly, and shows that under the CAPM assumptions, given a set of firms’ fundamentals, prices can be determined such that the CAPM risk-return relationship holds. However, things are not as simple as they may seem. . .

In this paper we show that given a set of firms’ fundamentals, there are infinitely many potential CAPM equilibria, characterized by market portfolios which lie on a straight line in the mean–standard-deviation plane. We denote this line of potential CAPM equilibria as market portfolio line, or MPL. While all points on the MPL represent equilibria which mathematically satisfy the CAPM risk-return relationship, they do not necessarily guarantee positive prices. The subset of equilibria with positive prices is shown to be only a lower segment of the MPL. We characterize this segment and show that it is determined by $\delta$, the ratio of the “riskiness” of the riskiest asset (in terms of fundamentals) to the “average riskiness” in the market. The more heterogeneous

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15 Denote the rate of return on the market portfolio by $\bar{R}$. The investor’s end-of-period wealth is given by: $\tilde{W} = (W_0 - x)(1 + r_f) + x(w + \bar{R})$, where $W_0$ is the initial wealth and $x$ is the dollar amount invested in the market portfolio. The expected utility is $EU(x) = E[\exp(-\gamma_0 \{ (W_0 - x)(1 + r_f) + x(w + \bar{R}) \})]. As \bar{R} is normally distributed with mean $\bar{R}_m$ and standard deviation $\sigma_m$ we have $EU(x) = -\frac{1}{\sqrt{2\pi \sigma_m^2}} e^{-\gamma_0 (W_0 - x)(1 + r_f)} \int_{-\infty}^{\infty} e^{-\gamma_0 x(1 + \bar{R})} \cdot e^{-\frac{\delta(R - \bar{R}_m)^2}{2\sigma_m^2}} d\bar{R}$. Integrating over $\bar{R}$, and then differentiating with respect to $x$ yields the optimal investment in the market portfolio: $x_{\text{opt}} = \frac{\bar{R}_m - r_f}{\gamma_0 \sigma_m^2}$. 

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firms in the market are, the larger this ratio becomes, and the smaller the set of equilibria with positive prices. For realistically diverse markets with several very risky firms the set of positive-price CAPM equilibria is reduced to a degenerate equilibrium with practically no equity premium ($\bar{R}_m \approx r_f$). The intuition for this result is that the asset with the highest fundamental risk will have a positive price only if the market price of risk is very low, which in turn, implies a very low equity premium.

This analysis is general and makes no assumptions on investors’ preferences other than risk-aversion. Assuming specific commonly used preferences further reduces the set of possible CAPM equilibria. It may be the case that given a set of firms’ fundamentals and standard investors’ preferences no CAPM equilibrium with positive prices is possible at all.

While there is an ongoing debate in the literature regarding the empirical validity of the CAPM, our analysis shows that in many cases the CAPM with positive prices is even theoretically impossible. This is a very troubling result regarding one of the most important models in finance. However, the segmented-market extension of the CAPM (see [15,22,20,35]) may offer a solution. Let us elaborate.

Theorem 1 shows that the restriction on possible CAPM equilibria is a result of the difference in the fundamental risk of different assets. The larger $\delta$, the ratio between the riskiness of the riskiest asset and the “average riskiness”, the tighter this restriction becomes, and the lower the upper bound on the equity premium becomes. In the segmented-market extension of the CAPM investors do not necessarily hold all available assets, and different investor types may focus on different asset classes (see [22]). To see how this may solve the negative price problem and alleviate the restriction on possible equilibria, consider the following example. Suppose for simplicity that there are two kinds of assets: very risky assets, and assets with typical riskiness. Suppose also that there are also two types of investors: those who specialize in the very risky ventures, and those who invest in the typical assets. In this case, the venture investors will focus on the risky assets, and will face a universe of homogeneous (risky) assets. Similarly, the “typical” investors will focus on the typical assets, and will face a universe of homogeneous (typical) assets. In each of these two homogeneous segments, $\delta$ is close to 1, and there is practically no restriction on the equity premium. As a result, the market as a whole may also have a significant equity premium. Hence, market segmentation may restore the possibility of a non-degenerate equilibrium. Thus, the relationship between firms’ fundamentals, investors’ preferences, and the theoretical possibility of the CAPM with positive prices not only implies some important theoretical limitations of the CAPM, but may also point the direction to a possible solution.

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