Wave Hierarchies in Viscoelasticity

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Abstract—An evolution operator \( L_n \) with \( n \) arbitrary, typical of several models, is analyzed. When \( n = 1 \), the operator characterizes the standard linear solid of viscoelasticity, whose properties are already established in previous papers. The fundamental solution \( E_n \) of \( L_n \) is explicitly obtained and it's estimated in terms of the fundamental solution \( E_1 \) of \( L_1 \). So, whatever \( n \) may be, asymptotic properties and maximum theorems are achieved. These results are applied to the Rouse model and reptation model, which describe different aspects of polymer chains. © 2004 Elsevier Ltd. All rights reserved.

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1. INTRODUCTION

Let \( L_n \) be the \((2 + n)\) order operator,

\[
L_n = \partial_t^{(n)} (\partial_t - \frac{c_n^2}{n} \partial_{xx}) + a_{n-1} \partial_t^{(n-1)} (\partial_t - \frac{c_{n-1}^2}{n-1} \partial_{xx}) + a_0 (\partial_t - \frac{c_0^2}{0} \partial_{xx}),
\]

where \( a_k (k = 0, \ldots, n - 1) \) are positive constants.

According to the value of \( n \), (1.1) describes several physical phenomena. As an example, when \( n = 1 \), \( L_n \) can be found in dynamic of relaxing gases, in magnetohydrodynamics, in hereditary electromagnetism (see [1] and references therein) and in isotropic viscoelasticity where, (1.1) models the evolution of the standard linear solid (S.L.S.) (see, i.e., [2,3]).

In all these models, \( c_k \) represents the characterized speeds depending on the materials properties of the medium and in many physical problems, it results in \( c_0^2 < c_1^2 < \cdots < c_{n-1}^2 < c_n^2 \), as is typical of wave hierarchies [4].

When \( n = 1 \), the operator (1.1) is strictly-hyperbolic and it has been widely analyzed in [1]. Its fundamental solution, \( E_1 \), has been explicitly determined and singular perturbation problems, together with asymptotic properties, have been estimated.

The aim of this paper is to draw generalizations of the wide analysis related to S.L.S. to the case of (1.1) with \( n \) arbitrary. For this, a conditioned equivalence between (1.1) and an integrodifferential operator \( \mathcal{M} \) related to an appropriate memory function \( g_n(t) \), is considered. Owing to the hypotheses of fading memory, every function \( g_n(t) \) can be approximated by Dirichlet polynomials [5,6] with appropriate restrictions on the coefficients of this expansion. These limitations need that the differential operator \( L_n \) is typical of wave hierarchies [7,8].
By this equivalence, whatever \( n \) may be, the fundamental solution \( \mathcal{E}_n \) of (1.1) is explicitly achieved and it is estimated in terms of \( \mathcal{E}_1 \). So, the maximum properties and asymptotic estimates established for the standard linear solid can be applied to operator \( \mathcal{L}_n \) defined in (1.1). Moreover, boundary layer problems, typical of dissipative media, can be rigorously estimated [9,10].

These results are applied to the Rouse model and the reptation model, which describe different aspects of polymer chains and have met with reasonable success [11–13].

2. DIFFERENTIAL CONSTITUTIVE EQUATION

Let \( B \) a linear, isotropic, homogeneous system and let \( \mathbf{u}(x,t) \) be the displacement field from a nondeformed reference configuration \( B_0 \). If \( \mathbf{u} = u(x,t)\mathbf{i} \) and \( \rho_0 \) denotes the mass density in \( B_0 \), the equations of one-dimensional motions of \( B \) are

\[
\rho_0 u_{tt} = \sigma + f, \quad \varepsilon = u_x,
\]

where \( f = f_i \) is the known body force, while \( \sigma \) and \( \varepsilon \) are the only nonvanishing components of the stress and the strain tensors.

When the viscoelastic behavior of \( B \) is of rate-type, the well-known stress-strain constitutive relation is

\[
\sum_{k=0}^{n} a_k \partial_t^k \sigma = \sum_{k=0}^{n} \alpha_k \partial_t^k \varepsilon,
\]

with \( a_k, \alpha_k \) constant (\( a_n, \alpha_n \neq 0 \)).

Then, by (2.1),(2.2), the displacement field \( \mathbf{u}(x,t) \) is solution of the higher-order equation like,

\[
\mathcal{L}_n \mathbf{v} \equiv \sum_{k=0}^{n} a_k \partial_t^k (v_{tt} - c_k v_{xx}) = F,
\]

where

\[
c_k = \frac{\alpha_k}{\rho_0 a_k}, \quad F = \left( \frac{1}{\rho_0} \right) \sum_{k=0}^{n} a_k \partial_t^k f.
\]

The constitutive relation (2.2) includes various classical mechanical models, as Maxwell and Kelvin-Voigt models [14]. Moreover, when \( n = 1 \) and

\[
0 < c_0 < c_1, \quad \eta = \frac{a_1}{a_0} > 0,
\]

one has the case of the standard linear solid (S.L.S.), which is modelled by the strictly-hyperbolic third-order equation,

\[
\mathcal{L}_1 \mathbf{v} \equiv \eta \partial_t (v_{tt} - c_1 v_{xx}) + v_{tt} - c_0 v_{xx} = \left( \frac{1}{a_0} \right) F.
\]

The fundamental solution \( \mathcal{E}_1 \) of this operator has been obtained also in [1], when \( x \in \mathbb{R}^2 \) or \( x \in \mathbb{R}^3 \). Further, numerous basic properties of \( \mathcal{E}_1 \) have been rigorously estimated and the wave behavior of S.L.S. is now acquired.

When \( n > 1 \) and all the \( c_k \) are positive, then, waves of different orders appear and their roles must be clarified in order to see how each set is modified by the presence of the other. Obviously, wave or dispersive behavior depend on the requirements of the coefficients \( a_k \) and \( c_k \) due to physical properties of the system \( B \).

For this, we will analyze the restrictions imposed on the constants \( a_k \) and \( c_k \) by usual hypotheses of fading memory for \( B \).
3. INTEGRAL CONSTITUTIVE EQUATION

When the strain amplitudes are not too large, the behavior of most viscoelastic media is fairly well modelled by linear hereditary equations like,

$$\varepsilon(t) = J(0) \sigma(t) + \int_{-\infty}^{t} J(t-\tau) \sigma(\tau) \, d\tau,$$

(3.1)

where $J(t)$ denotes the creep-compliance and the integral term needs the knowledge of the past history of the stress.

Usually, according to fading memory hypotheses, $J(t)$ is a positive fast-decreasing function. For instance, several real materials as polymers, rubbers, bitumines, have satisfactory representations by means of chains of S.L.S. elements in series or parallel \[3,11\]. In the series case, the creep function is

$$J_n(t) = J_n(0) \left[ 1 + \sum_{k=1}^{n} \frac{B_k}{J_n(0)} (1-e^{-\beta_k t}) \right],$$

(3.2)

where $n$ is the number of elements in the chain, $\tau_k = \beta_k^{-1}$ are the characteristic times and $J_n(0)$ denotes the elastic compliances.

Then, if one puts

$$c^2 = \left[ \rho_0 J_n(0) \right]^{-1}, \quad F_* = c^2 \left[ J_n(0) f + \int_{-\infty}^{0} J_n(t-\tau) \sigma_x(\tau) \, d\tau \int_{0}^{t} J(t-\tau) f(\tau) \, d\tau \right],$$

(3.3)

by (2.1),(3.1),(3.2) one deduces,

$$\mathcal{M}u = c^2 u_{xx} - u_{tt} - \int_{0}^{t} g(t-\tau) u_{rr} \, d\tau = -F_*(x,t),$$

(3.4)

with

$$g = g_n(t) = \sum_{k=1}^{n} B_k e^{-\beta_k t} = \frac{j_n(t)}{J_n(0)}.$$  

(3.5)

In this memory function, $n$ is quite arbitrary and constants $B_k$ and frequencies $\beta_k$ are, such that

$$0 < \beta_1 < \beta_2 \cdots < \beta_n; \quad B_k > 0, \quad \forall k = 1,2 \ldots n. \quad (3.6)$$

These hypotheses assure that

$$g(t) > 0, \quad \dot{g} < 0, \quad \ddot{g} > 0, \quad \forall t \geq 0$$

(3.7)

according to the convexity assumption considered by Dafermos \[15\].

We observe that the representation (3.5) of the memory function $g$ is not restrictive because well-known Muntz’s and Schwartz’s theorems \[5,6\] imply that whatever $C^0(\mathbb{R}^+)$ function can be uniformly represented by means of Dirichlet polynomials. Moreover, as $n$ is arbitrary, the constants $B_k, \beta_k$ can be determined in order to fit the experimental curves for $g(t)$ to any prefixed degree of approximation \[11\].

By (3.2),(3.6) one has

$$J_n(\infty) = J_n(0) \left[ 1 + \sum_{k=1}^{n} \frac{B_k}{J_n(0)} \right] > J_n(0).$$

(3.8)
4. FADING MEMORY AND WAVE HIERARCHIES

Let the initial data related to (2.3) and (3.4) be null and let

\[ P(s) = \sum_{k=0}^{n} \mu_k s^k, \quad Q(s) = \sum_{k=0}^{n} \lambda_k s^k, \quad (4.1) \]

with

\[ \mu_k = \frac{a_k}{a_n}, \quad \lambda_k = \frac{a_k c_k}{a_n c_n}, \quad (k = 0, \ldots, n), \quad (4.2) \]

so that \( \mu_n = \lambda_n = 1 \).

Further, let

\[ P(s) = \sum_{k=0}^{n} \frac{B_k}{s + \beta_k}, \quad (4.3) \]

be the Laplace transform of the memory function (3.5).

Then, if one applies the Laplace transformation to (2.3) and (3.4), it results as

\[ \frac{\hat{u}_{xx}}{c^2} - \frac{s^2}{c_n} \frac{P(s)}{Q(s)} \hat{u} = -\frac{1}{a_n c_n} \frac{\hat{F}}{Q(s)}, \quad (4.4) \]

\[ \frac{\hat{u}_{xx}}{c^2} [1 + G(s)] \hat{u} = -\frac{\hat{F}_x}{c^2}, \quad (4.5) \]

where \((\hat{\cdot})\) denotes the \(L\)-transform of \((\cdot)\).

By comparing (4.4),(4.5) one deduces

\[ \frac{P(s)}{Q(s)} = \frac{c_n}{c^2} [1 + G(s)], \quad (4.6) \]

and the polynomial identity implies \(c_n = c^2\) and

\[ \lambda_0 = \beta_1 \beta_2 \cdots \beta_n, \]

\[ \vdots \]

\[ \lambda_{n-2} = \beta_1 \beta_2 + \beta_1 \beta_3 + \cdots + \beta_{n-1} \beta_n, \]

\[ \lambda_{n-1} = \beta_1 + \cdots + \beta_n. \quad (4.7) \]

So, owing to (3.6), all the \(\lambda_k\) are positive. Further, as for \(\mu_k\), one has

\[ \mu_0 = \lambda_0 + B_1 (\beta_2 \cdots \beta_n) + \cdots + B_n (\beta_1 \cdots \beta_{n-1}), \]

\[ \vdots \]

\[ \mu_{n-2} = \lambda_{n-2} + B_1 (\beta_2 + \cdots + \beta_n) + \cdots + B_n (\beta_1 + \cdots + \beta_{n-1}), \]

\[ \mu_{n-1} = \lambda_{n-1} + B_1 + \cdots + B_n. \quad (4.8) \]

and (3.6),(4.8) imply, too, \(0 < \lambda_k < \mu_k\) \((k = 0, \ldots, n - 1)\). As a consequence,

\[ 0 < c_k < c_n = c^2 \quad (k = 0, \ldots, n - 1). \quad (4.9) \]

At last, by (4.7),(4.8), it follows,

\[ \frac{\lambda_0}{\mu_0} < \frac{\lambda_1}{\mu_1} < \frac{\lambda_{n-1}}{\mu_{n-1}}, \]

and so,

\[ 0 < c_0 < c_1 \cdots < c_n. \quad (4.10) \]
So, the following property holds.

**Property 4.1.** Hypotheses of fading memory (3.5), (3.6) imply that the differential operator (2.3) is typical of wave hierarchies.

Conversely, the inverse transformation of (4.7), (4.8) requires care.

When the differential equation (2.3) is prefixed, in order to obtain the dual hereditary equation (3.4) with a memory function $g(t)$, satisfying (3.5), (3.6), appropriate restrictions on the constants $a_k, c_k$ must be imposed.

At first, (4.3), (4.6) imply

$$
P(s) = B_0 + \sum_{k=1}^{n} \frac{B_k}{s + \gamma_k},
$$

where all the roots $s = -\gamma_k$ of $Q(s)$ are real and simple, with $\gamma_k > 0$. Moreover, the conditions $B_k > 0$, which are sufficient to verify (3.7), involve further limitations.

**Example 4.1.** When $n = 1$, one has $c^2 = c_1$, $B_0 = 1$, and

$$
\beta_1 = \frac{a_0 c_0}{a_1 c_1} > 0, \quad B_1 = \frac{a_0}{a_1} \left(1 - \frac{c_0}{c_1}\right) > 0,
$$

which represent the known restrictions typical of S.L.S.

**Example 4.2.** When $n = 2$, one has $c^2 = c_2$, $B_0 = 1$, and $\beta_1, \beta_2$ are real, iff

$$
\omega^2 = (a_1 c_1)^2 - 4 (a_0 c_0) (a_2 c_2) > 0.
$$

Then, it results

$$
\beta_1 = \frac{1}{2a_2 c_2} (a_1 c_1 - \omega), \quad \beta_2 = \frac{1}{2a_2 c_2} (a_1 c_1 + \omega),
$$

so that, $0 < \beta_1 < \beta_2$. Further,

$$
B_i = \frac{(-1)^{i-1}}{\omega} \left[ a_0 (c_2 - c_0) - a_1 \beta_i (c_2 - c_1) \right] \quad (i = 1, 2).
$$

Thus, it is $B_1 > 0$, $B_2 > 0$, iff

$$
\beta_1 < \frac{a_0}{a_1} \frac{c_2 - c_0}{c_2 - c_1} < \beta_2.
$$

Therefore, the fourth-order operator,

$$
a_2 (u_{tt} - c_2 u_{xx})_{tt} + a_1 (u_{tt} - c_1 u_{xx})_t + a_0 (u_{tt} - c_0 u_{xx}),
$$

can be analyzed by (3.4)–(3.6) when the constants, $a_k, c_k$, satisfy (4.13) and (4.16).

5. **Estimates for the Hereditary Model**

Let $B_n$ be the viscoelastic model characterized by the memory function $g_n$ in (3.5); the case $n = 1$ corresponds to the L.S.L. $B_1$.

In [16, 17], the fundamental solution $E_n$ of operator $M$ in (3.4) has been explicitly determined, whatever $n$ may be. If $\eta$ is the step-function and $I_0$ is the modified Bessel function of first kind, it results as

$$
E_n = E_n (\beta_1 \ldots \beta_n, B_1 \ldots B_n) = \frac{1}{2c} \eta (t - r) (A_1 + A_2),
$$

with

$$
A_1 = e^{-g_0 t/2} I_0 \left( \frac{g_0}{2} \sqrt{t^2 - r^2} \right),
$$

$$
A_2 = \frac{1}{\pi} \int_0^\pi d\theta \int_r^t e^{-g_0 z} H (z, t - w) \, dw,
$$
and \( g_0 = g(0), r = |x|/c, 2z = w - \cos \theta (w^2 - r^2)^{1/2} \). Further, if

\[
\phi_k (z,t) = e^{-\beta_k t} \sqrt{\frac{B_k \beta_k z}{t}} I_1 \left( 2 \sqrt{B_k \beta_k z t} \right),
\]

(5.4)

one has

\[
H (z,t) = \sum_k \phi_k + \sum_{k_1, k_2} \phi_{k_1} * \phi_{k_2} + \cdots,
\]

(5.5)

where sums are computed according to the simple combination of the indices \( k_1, k_2, \ldots, k_n \), and \( * \) denotes the convolution with respect to \( t \).

Moreover, the fundamental solution \( E_n \) related to \( B_n \) and defined in (5.1)-(5.5), can be rigorously estimated in terms of the fundamental solution \( E_1 \) related to an appropriate S.L.S. \( B^*_1 \) defined by

\[
g_1 = be^{-\beta_1 t} \quad \text{with } b = \beta_1 \sum_{k=1}^n \frac{B_k}{\beta_k}.
\]

(5.6)

In fact, if \( \Gamma \) is the open forward characteristic cone \( \{ (t, x) : t > 0, |x| < ct \} \), and

\[
\chi_n = \prod_{k=2}^n \left( \frac{B_k}{\beta_1} \right)^2,
\]

then, the following theorem holds.

**Theorem 5.1.** If the memory function is given by (3.5),(3.6), then, the fundamental solution \( E_n \) of \( \mathcal{M} \) is a never negative \( C^\infty (\Gamma) \) function and it satisfies the estimate,

\[
0 < E_n (\beta_1 \ldots \beta_n, B_1 \ldots B_n) < \chi_n E_1 (\beta_1, b),
\]

(5.7)

everywhere in \( \Gamma \) and whatever \( n \) may be.

**Remark 5.1.** The model \( B^*_1 \) defined by (5.6) is physically meaningful. In fact, memory function \( g_1 \) is related just to the obliviator because \( \tau_1 = \beta_1^{-1} \) is the longest characteristic time.

Furthermore, \( B_n \) and \( B^*_1 \) verify the same hypotheses of fading memory and by (3.5),(5.6), it results as,

\[
\int_0^\infty g_n (t) \, dt = \int_0^\infty g_1 (t) \, dt = \sum_{k=1}^n \frac{B_k}{\beta_k}.
\]

(5.8)

Moreover, the integral (5.8) affects the asymptotic analysis of hereditary equation [7,18].

**Remark 5.2.** By known properties of asymptotic behaviour of convolutions, the constitutive relation (3.1) implies

\[
\lim_{t \to \infty} \varepsilon (x,t) = J_n (0) \left[ 1 + \int_0^\infty g_n (t) \, dt \right] \lim_{t \to \infty} \sigma (x,t),
\]

(5.9)

provided that \( \lim_{t \to \infty} \sigma \) exists. Then, \( B_n \) and the model \( B^*_1 \) exhibit the same asymptotic behaviour (5.9).

Further, by (3.5),(3.8), it results

\[
\frac{J_n (\infty)}{J_n (0)} = 1 + \sum_{k=1}^n \frac{B_k}{\beta_k} = \frac{J_1 (\infty)}{J_1 (0)}
\]

(5.10)

and so, (5.10) implies \( J_n (\infty) \sigma (\infty) = \varepsilon (\infty) \). Consequently, when \( t \) is large, the behaviour of \( B_n \) is typical of an elastic material with modulus \( J_n (\infty) \).

[ ]
6. ESTIMATES RELATED TO WAVE HIERARCHIES

When operator $\mathcal{L}_n$ is reduced to the hereditary operator $\mathcal{M}$ of (3.4), then, estimates of Theorem 5.1 can be applied to wave hierarchies. Obviously, the equivalence is conditioned by inverse transformation of (4.7), (4.8) together with (3.6) (see n.4).

Let $\mathcal{L}_1^*$ be operator (2.6) related to the S.L.S. $\mathcal{B}_1^*$ characterized by

$$\eta = \frac{1}{\beta_1 + b}, \quad a_0 = \frac{\beta_1 + b}{\beta_1}, \quad c_0 = \frac{c^2 \beta_1}{\beta_1 + b}, \quad c_1 = c^2.$$  \hspace{1cm} (6.1)

Now, let $\mathcal{L}_n$ be the differential operator given by (2.3) whatever $n$ may be, and let $\mathcal{P}_n$ be a prefixed boundary value problem related to $\mathcal{L}_n$. The meaningful aspects of qualitative analysis of the solution of $\mathcal{P}_n$ can be obtained by means of Theorem 5.1 and by the known properties of $\mathcal{L}_1^*$ [1].

So, maximum properties, asymptotic behaviour, boundary layer estimates, etc., for the solution of $\mathcal{P}_n$ are deduced by analogous properties related to $\mathcal{L}_1^*$.

Moreover, owing to the equivalence between $\mathcal{L}_n$ and $\mathcal{M}$, it is possible to have explicitly the fundamental solution of $\mathcal{L}_n$, for all $n$. In fact, it suffices to apply the explicit formula (5.1)--(5.3).

As an example, the case of polymeric materials can be considered.

EXAMPLE 6.1. Polymeric materials are very flexible and are easily formed into fibres, thin films, additives for oils, etc. So, their applications to concrete problems are numerous. See [12,19,20]. According to theories of linear viscoelasticity, two models, that describe different aspects of polymer chains, have met a reasonable success: the Rouse model and the reptation model [13].

In both cases, memory function $g(t)$ assumes a form like (3.2), (3.3). In fact, in the reptation model, the stress-relaxation function is

$$g(t) = k \sum_{h=6}^{n} \frac{1}{(2h + 1)^2} e^{-(2h+1)^2(t/\tau_d)},$$  \hspace{1cm} (6.2)

where $k$ is a constant depending on the polymer physics and the value of the “reptation” time $\tau_d$ can be fixed according to elasticity experiments [11].

When the viscoelastic behaviour is represented by the Rouse model, memory function $g(t)$ is given by

$$g(t) = k_1 \sum_{h=1}^{n} e^{2h^2(t/\tau_1)},$$  \hspace{1cm} (6.3)

where the relaxation time $\tau_1$ can be calculated by means of experimental results [13].

So, if one considers the first two steps in the reptation model, it results as $B_1 = k$, $B_2 = B_1/9$, $\beta_1 = 1/\tau_d$, $\beta_2 = 9\beta_1$. Consequently, operator (4.17) is characterized by constants,

$$c_0 = c^2 \frac{81}{81 + 82k_1 \tau_1}, \quad c_1 = c^2 \frac{9}{9 + k_1 \tau_1}, \quad c_2 = c^2,$$

$$a_0 = 1 + \frac{82}{81} k_1 \tau_1, \quad a_1 = \frac{10 \tau_1^2}{9} \left( \frac{1}{\tau_1} + \frac{k_1}{9} \right), \quad a_2 = \frac{\tau_1^2}{9}.$$  \hspace{1cm} (6.4)

Analogously, in the Rouse model, being $B_1 = B_2 = k_1$, $\beta_1 = 2/\tau_1$, $\beta_2 = 4\beta_1$, one has

$$c_0 = c^2 \frac{8}{8 + 5k_1 \tau_1}, \quad c_1 = c^2 \frac{5}{5 + k_1 \tau_1}, \quad c_2 = c^2,$$

$$a_0 = 1 + \frac{5k_1}{8} \tau_1, \quad a_1 = \frac{\tau_1^2}{16} \left( 2k_1 + \frac{10}{\tau_1} \right), \quad a_2 = \frac{\tau_1^2}{16}.$$  \hspace{1cm} (6.5)
The wave hierarchies defined by (6.4) or (6.5) are governed by operator $L_1^*$ of the standard linear solid defined, respectively, by

$$
\begin{align*}
  c_0 &= c^2 \frac{81}{81 + 82k\tau_d}, \\
  c_1 &= c^2, \\
  a_0 &= 1 + \frac{82}{81}k\tau_d, \\
  \eta &= \frac{81\tau_d}{81 + 82k\tau_d}, \\
  c_0 &= c^2 \frac{8}{8 + 5k_1\tau_1}, \\
  c_1 &= c^2, \\
  a_0 &= 1 + \frac{5}{8}k_1\tau_1, \\
  \eta &= \frac{4\tau_1}{8 + 5k_1\tau_1}.
\end{align*}
$$

These results have been confirmed also in [21] for entangled polymers with chain stretch.

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