Note

t-Designs with few intersection numbers

Alexander Pott
Mathematisches Institut der Justus-Liebig-Universität Gießen, Arndtstr 7, W-6300 Gießen, Germany

Mohan Shrikhande
Mathematics Department, Central Michigan University, Mount Pleasant, MI 48859, USA

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Abstract


We give a method to obtain new i-designs from t-designs with j distinct intersection numbers if i + j - 1 does not exceed t.

1. Introduction

We assume familiarity with t-designs, see for instance [2]. Let \( \Sigma_r(t, k, v) \) be a t-design with v points, block size k and where any t points are contained in exactly \( \lambda \) blocks. We will allow possibly repeated blocks. It is well known that every t-design is also an s-design with \( s < t \) in which any s points are contained in \( \lambda_s \) blocks, where

\[
\lambda_s = \frac{(v-s)}{(k-s)} \lambda_{s+1}.
\]

For distinct blocks \( X, Y \) of the t-design we call the cardinality of \( X \cap Y \) an intersection number. Let \( x_1, x_2, \ldots, x_j \) be all the distinct intersection numbers of the t-design. Ray-Chaudhuri and Wilson [5] showed that in any nontrivial 2i-design there are at least i distinct intersection numbers with equality if and only if the number of blocks equals \( \binom{v}{t} \). Cameron [4] has shown that any nontrivial \((2i+1)\)-design has at least \( i+1 \) distinct intersection numbers.

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Let $B$ be a fixed block of $S(t, k, v) = D$ with possible intersection numbers $x_1, x_2, \ldots, x_j$. We define an incidence structure $D_i(B)$ whose points are the elements outside $B$ and whose blocks are the blocks of $D$ which intersect $B$ in exactly $x_i$ points.

The main results of this note are the following theorem and corollary.

**Theorem A.** Let $S(t, k, v) = D$ be a $t$-design with distinct intersection numbers $x_1, x_2, \ldots, x_j$. Then $D_i(B)$ is an $i$-design (with possibly repeated blocks) if $i + j - 1 \leq t$ and $i \leq v - k$.

**Corollary B.** Let $D$ satisfy the hypothesis of Theorem A. Then $D_i(B)$ has no repeated blocks under any one of the following conditions:

1. $D$ has no repeated blocks and $x_i = 0$.
2. $i = 1$ and $k - x_i \equiv t$.

The proof of Theorem A is given in Section 2. Corollary B is then immediate. We end this note with some examples concerning the Witt designs and Hadamard 3-designs.

2. The proof of Theorem A

Let $D$ satisfy the assumptions of Theorem A. Since $i \leq v - k$, one may choose a set $S$ of $i$ points outside $B$; let $\lambda_i^{(x)}$ be the number of blocks through $S$ meeting $B$ in exactly $x_l$ points ($l = 1, 2, \ldots, j$). We shall show that the numbers $\lambda_i^{(x)}$ are independent of the choice of $S$. Since $D$ is an $i$-design, we clearly have

$$\lambda_i^{(x_1)} + \lambda_i^{(x_2)} + \cdots + \lambda_i^{(x_j)} = \lambda_i.$$

We next consider flags $(p, C)$ where $p \in B \cap C$ and $S \subseteq C$ and obtain by counting in two ways:

$$x_1\lambda_i^{(x_1)} + x_2\lambda_i^{(x_2)} + \cdots + x_j\lambda_i^{(x_j)} = k\lambda_{i+1}.$$

By similar counting of the occurrences of $m$-tuples, $m = 2, 3, \ldots, j - 1$, of points in $B \cap C$, where $C$ is a block with $S \subseteq C$, we obtain the following further equations:

$$\sum_{n=1}^{j} x_n(x_n - 1) \cdots (x_n - m + 1)\lambda_i^{(x_n)} = k(k - 1) \cdots (k - m + 1)\lambda_{i+m}.$$

Since $i + j - 1 \leq t$, we thus have a system of $j$ linear equations in $j$ unknowns $\lambda_i^{(x_1)}, \lambda_i^{(x_2)}, \ldots, \lambda_i^{(x_j)}$. The coefficient matrix of this system can be easily transformed into a Vandermonde matrix which is nonsingular, since $x_1, x_2, \ldots, x_j$ are distinct. Hence $\lambda_i^{(x_1)}, \lambda_i^{(x_2)}, \ldots, \lambda_i^{(x_j)}$ depend only on the design parameters and the intersection numbers. This completes the proof of Theorem A.
As an illustration of our theorem we can construct the table below using the Witt designs and Hadamard 3-designs. We indicate that there are no repeated blocks in \( D_t(B) \) by a *.

<table>
<thead>
<tr>
<th>( S_t(t, k, v) )</th>
<th>Intersection numbers</th>
<th>( D_t(B) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( S_t(5, 8, 24) )</td>
<td>0, 2, 4</td>
<td>( S_s(3, 8, 16)^<em>, s_{16}(3, 6, 16)^</em>, s_5(3, 4, 16) )</td>
</tr>
<tr>
<td>( S_t(4, 7, 23) )</td>
<td>1, 3</td>
<td>( s_5(3, 6, 16)^<em>, s_5(3, 4, 16)^</em> )</td>
</tr>
<tr>
<td>( S_s(3, 6, 22) )</td>
<td>0, 2</td>
<td>( s_2(2, 6, 16)^<em>, s_2(2, 4, 16)^</em> )</td>
</tr>
<tr>
<td>( S_t(5, 6, 12) )</td>
<td>0, 2, 3, 4</td>
<td>( S_s(2, 6, 6)^*, s_{16}(2, 4, 6), s_5(2, 3, 6), s_5(2, 2, 6) )</td>
</tr>
<tr>
<td>( S_{n-1}(3, 2n, 4n) )</td>
<td>0, n</td>
<td>( S_s(2, 2n, 2n)^*, s_{2n-1}(2, n, 2n) )</td>
</tr>
</tbody>
</table>

**Remark.** The 5-designs of Denniston [2] unfortunately yield only 1-designs \( D_t(B) \).

Cameron [3] used the 2-design \( D_3(B) \) in the special case \( t = 3, j = 2 \) and \( x_1 = 0 \) as a tool in classifying extensions of symmetric 2-designs, see also [1]. The authors hope that the designs \( D_t(B) \) may prove helpful in classifying quasi-symmetric 3-designs with positive intersection numbers, see for example [6].

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**References**


