Using Sum of Squares Decomposition for Stability of Hybrid Systems

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SUMMARY This paper deals with stability analysis of hybrid systems. Such systems are characterized by a combination of continuous dynamics and logic based switching between discrete modes. Lyapunov theory is a well-known methodology for the stability analysis of linear and nonlinear systems in control system literature. Construction of Lyapunov functions for hybrid systems is generally a difficult task, but once these functions are defined, stabilization of the system is straightforward. The sum of squares (SOS) decomposition and semidefinite programming has also provided an efficient methodology for analysis of nonlinear systems. The computational method used in this paper relies on the SOS decomposition of multivariate polynomials. By using SOS, we construct a (some) Lyapunov function(s) for the hybrid system. The reduction techniques provide numerical solution of large-scale instances; otherwise they will be practically unsolvable. The introduced method can be used for hybrid systems with linear or nonlinear vector fields. Some examples are given to demonstrate the capabilities of the proposed approach.

key words: stability, hybrid systems, Lyapunov function, sum of squares

1. Introduction

In hybrid systems, the behavior of interest is governed by interacting continuous and discrete dynamic processes. There are several reasons for using hybrid models to represent dynamic behaviors of interest. Reducing complexity was and still is an important attraction for dealing with hybrid systems. Hybrid control systems typically arise from the interaction of discrete planning algorithms and continuous processes, which arise frequently in computer-aided control of continuous processes in manufacturing, communication networks and industrial process control systems. The study of hybrid control systems is essential in designing sequential supervisory controllers for continuous systems, and it is central in designing intelligent control systems with a high degree of autonomy [1].

Stability of the closed-loop system is probably the most fundamental objective for control system design. It is well known that efficient control design methods can only be developed when the stability properties of the system are fully recognized. In linear/nonlinear systems, Lyapunov functions are used to investigate the stability of equilibrium points of the system or the invariance of certain sets. There are many notions of stability, especially in nonlinear systems ranging, from local stability to uniform global asymptotic stability. Problems of proving the different notions of stability for the nonlinear case are, in general, of extreme difficult and even sometimes unsolvable [2].

The main approach used in the literature for stability of hybrid and switched systems is Lyapunov theory. The stability of such systems depends on, in general, the switching strategy. Switching between stable system structures does not necessarily imply a stable closed loop behavior [3], [4]. In contrast, unstable system structures can be stabilized by designing a proper switch strategy [5]. In [6] a method has been defined for proving stability and controller design for switched nonlinear systems based on linearization. In [7], the fact that Lyapunov function will decrease along with any hybrid system trajectory is used for proving hybrid system stability. The common Lyapunov function technique is also discussed in brief in [8], [9]. Controller synthesis methodology that guarantees the existence of a single Lyapunov function for the closed loop hybrid system is proposed in [10]. An approach similar to common Lyapunov function is used in [11] for a hybrid control systems.

An algorithmic methodology was proposed recently, allows systems analysis by algorithmically constructing a Lyapunov function as an evidence for stability of the zero equilibrium using the sum of squares (SOS) decomposition and SOSTOOLS [12].

In this paper, some theorems concerning the stability of hybrid systems using Lyapunov functions with SOS decomposition are proposed. Lyapunov functions can guarantee the stability of the hybrid systems that satisfy conditions of each theorem. Some features of the proposed methods are:

• They provide less conservative test for proving stability. It is not necessary that the Lyapunov functions be continuous during the switching surface.

• They can be used for systems with linear or nonlinear vector fields.

• A first attempt is made to verify the conditions for a single Lyapunov function for all discrete states of a hybrid system. Then, the hybrid system will be stable regardless of switching. However, this attempt fails if, for instance, some of the subsystems are unstable and defined in the region containing the origin as an interior point, or possibly the system becomes unstable for some switching of the vector fields. Another attempt can then be made by using one Lyapunov function for each discrete state. This attempt is also fruitless if some of the subsystems are unstable. When these two
attempts fail, some other partitioning using gradient vector method may be used to verify the stability conditions of the system successfully.

2. Hybrid Systems

A hybrid control system is a control system where the plant or the controller contains discrete modes that together with some continuous equations govern the behavior of the system. Typical examples of such systems with varying degrees of complexity include computer disk drives, stepper motors, constrained robotic systems, intelligent vehicle in highways, sampled-data systems, discrete event systems and many other types of systems [13].

The hybrid system model considered for stability in this work is the same as the one used by Pettersson and is described by:

\[
\dot{x} = f(x(t), m(t)) \\
m(t) = \phi(x(t), m(t))
\]

(1)

where \( x \in \mathbb{R}^n, m \in M = \{m_1, \ldots, m_l\} \). Here, \( x \) is the continuous state and \( m \) is the discrete state. The notation \( \phi \) indicates that \( m(t) \) is piecewise constant from the right side. The changes on values of \( m \) usually result in abrupt changes in the vector field \( f \). For a system described by (1), it is possible to distinguish between a switched system and a hybrid system. If for each \( x \in \mathbb{R}^n \) only one \( m_i \in M \) is possible, then a switched system is obtained. For a switched system, thus, the discrete state \( m \) is redundant. However, if there are some \( x \in \mathbb{R}^n \) for which several discrete states are possible, then the system is called a hybrid system.

Instead of describing the change of \( m \) by \( \phi \) according to (1), it is common to describe the change by switch sets \( S_{ij} \), that are related to \( \phi \) by:

\[
S_{ij} = \{x \in \mathbb{R}^n | m_j = \phi(x, m_i)\}
\]

(2)

Typically, the sets \( S_{ij} \) are given by hypersurfaces \( s_{ij}(x) = \eta x + \lambda = 0 \), where \( \eta \) is the constant normal vector and \( \lambda \) is a constant vector. The hybrid system in (1) evolves from the initial conditions \( (x_0, m_0) \in I_0 \), where \( I_0 \) represents the set of all possible initial conditions. The evolution can be described as follows: Starting at \( x(t_0), m(t_0) \) at time \( t_0 \), the continuous trajectory evolves according to \( \dot{x} = f(x, m) \). If \( x \) reaches some \( x_j \in S_{ij} \) at time \( t_l \), then the state becomes \( (x_j, m_j) \) from which the process continues according to \( \dot{x} = f(x, m) \).

The evolution of the discrete states from an initial state \( (x_0, m_0) \in I_0 \) can be described by a switching sequence:

\[
\Delta_{(x_0, m_0)} = (\mu_0, t_0), (\mu_1, t_1), \ldots, (\mu_k, t_k) \in M, k \in N
\]

(3)

where \( t_k < t_{k+1}, \mu_k \neq \mu_{k+1} \forall k \in N, \) and \( \mu_0 = m_0 \). The notation \( \mu_k, t_k \) means that \( \dot{x}(t) = f(x(t), \mu_k) \) for \( t_k \leq t \leq t_{k+1} \).

Without loss of generality, it is assumed that the equilibrium point is located at the origin of the continuous state space. This means that if \( x(t_c) = 0 \) for some \( t_c \), then \( x(t) = 0 \) \( \forall t \geq t_c \).

3. Stability Analysis

Stability in the sense of Lyapunov is guaranteed if the system dynamics are such that the “energy” of the system is non-increasing with time. This fact will be used for formulating stability conditions for the hybrid systems.

**Definition 1 (Lyapunov Stability of Equilibrium Point):** An equilibrium point \( \mathbb{T} \) is Lyapunov stable if for any \( R > 0 \), there exists an \( r, 0 < r < R \), such that if \( x_0 \) is inside \( B(\mathbb{T}, r) \), then \( z(t) \) is inside \( B(\mathbb{T}, R) \) for all \( t > 0 \).

**Definition 2 (Lyapunov Function):** A candidate Lyapunov function for the system \( \dot{x} = f(x) \) at the equilibrium point \( \mathbb{T} \) is a real valued function \( V \), defined over a region \( \Omega \) of the state space that contains \( \mathbb{T} \) and satisfies the two requirements:

- Continuity: \( V \) is continuous and, in the case of a continuous-time system, \( V \) has continuous derivative.
- Positive Definiteness: \( V(x) \) has a unique minimum at \( \mathbb{T} \) with respect to all other points in \( \Omega \). Without loss of generality, henceforth assume \( V(\mathbb{T}) = 0 \).

**Theorem 1** (Lyapunov Theorem): If there exists a Lyapunov function \( V(x) \) in the region \( B(\mathbb{T}, R), R > 0 \), then the equilibrium point \( \mathbb{T} \) is Lyapunov stable [14].

3.1 Stability Analysis of Hybrid Systems

An important observation is that it is possible for the hybrid system to be unstable even when all the subsystems are stable. On the other hand, it is possible to stabilize hybrid system even when all the subsystems are unstable [15].

Hybrid system stability analysis relies on the classical Lyapunov stability theory. When the hybrid system (1) evolves from an initial state \( (x_0, m_0) \in I_0 \), a sequence of candidate Lyapunov functions will be used according to

\[
\Lambda_{(x_0, m_0)} = (V_0, \tau_0), (V_1, \tau_1), \ldots, V_k \in V, k \in N
\]

(4)

where \( (V_k, \tau_k) \) means that the system energy is measured by the candidate Lyapunov function \( V_k \) for \( \tau_k \leq t \leq \tau_{k+1} \).

**Theorem 2:** Let a hybrid system be described by (1) with \( f(0, m) = 0 \) \( \forall m \in M \). If for all switching sequences \( \Lambda_{(x_0, m_0)} \) in (4) occurring from \( (x_0, m_0) \in I_0 \):

1. \( V_k(x(t)) \leq 0 \)
   \[
   \forall t \in [\tau_k, \tau_{k+1}), \tau_k, \tau_{k+1} \in \pi(\Lambda_{(x_0, m_0)})
   \]

2. \( V_{k+1}(x(\tau_k)) \leq V_k(x(\tau_k)) \)
   \[
   \forall \tau_k \in \pi(\Lambda_{(x_0, m_0)})
   \]

(5)

then the origin is stable in the sense of Lyapunov [5].

Construction of Lyapunov functions is difficult in general, with the exception of special classes of systems. In this paper, we present an algorithm that searches for Lyapunov functions for hybrid systems using the Positivstellensatz and sum of squares decomposition.
4. Sum of Squares Decomposition

One of the main problems in different fields of mathematics is the global nonnegativity of a multivariable function. Concretely, the problem is to give equivalent conditions or a procedure for checking the validity of the proposition:

\[ f(x_1, ..., x_n) \geq 0, \quad \forall x_1, ..., x_n \in \mathbb{R} \quad (6) \]

If a polynomial \( f \) satisfies (6), then an obvious necessary condition is that the degree of the polynomial must be even. A simple sufficient condition for a real valued function \( F(x) \) to be nonnegative everywhere is given by the existence of the sum of squares decomposition:

\[ F(x) = \sum_i f_i^2(x) \quad (7) \]

It is clear that if a given function \( F(x) \) can be written as (7), for some \( f_i \), then it is nonnegative for all values of \( x \). The main idea of the method is to show that the given form \( f(x) \) is a quadratic for some new variable \( z \). These new variables are all the monomials of degree \( \frac{1}{2}m \) given by the different products of the \( x \) variables. Therefore, \( f(x) \) can be represented as

\[ f(x) = z^T Q z \quad (8) \]

where \( Q \) is a constant matrix. \( f(x) \) can be presented as a sum of squares when \( Q \) is a positive semidefinite matrix. Therefore, the problem of checking whether polynomial can be decomposed as a sum of squares is equivalent to verifying whether a certain affine matrix subspace intersects the cone of some positive definite matrices. A sum of squares program is a convex optimization problem of the following form:

Minimize \[ \sum_{j=1}^J w_j c_j \]

Subject to: \[ a_i,0(x) + \sum_{j=1}^J a_{ij}(x)c_j \text{ is SOS} \]

for \( i = 1, ..., I \) \quad (9)

where \( c_j \)'s are scalar real decision variables, \( w_j \)'s are some given real numbers, and \( a_{ij}(x) \)'s are some given multivariate polynomials. The conversion from SOS programs to semidefinite programs (SDPs) can be manually performed for small size problems. It is therefore desirable to have a computational aid that automatically performs this conversion for general SOS programs. This is exactly what SOSTOOLS is useful for. It automates the conversion from SOS program to SDP, calls the SDP solver, and converts the SDP solution back to the solution of the original SOS program [16]. SOSTOOLS is a MATLAB toolbox for formulating and solving sums of squares optimization programs. The SOS programs are solved by using SeDuMi or SDPT3, both well-known semidefinite programming solvers.

5. Proposed Method

As mentioned in Sect. (3), stability analysis can be reduced, using Lyapunov theory, to the existence of a positive definite function, such that its time derivative along the trajectories of the system is negative. Then these two conditions become polynomial nonnegativity conditions. To work around this problem we can use the following proposition.

**Theorem 3:** Given a polynomial \( V(x) \) of degree \( 2d \), Let \( \varphi(x) = \sum_{i=1}^n \sum_{j=1}^d \varepsilon_{ij} x_i^2 \) such that:

\[ \sum_{j=1}^d \varepsilon_{ij} > \gamma \quad \forall i = 1, ..., n \]

with \( \gamma > 0 \), and \( \varepsilon_{ij} \geq 0 \) for all \( i \) and \( j \). Then the condition that

\[ V(x) - \varphi(x) \text{ is SOS,} \quad (10) \]

guarantees the positive definiteness of \( V(x) \) [17].

In the remaining part of this section, Sect. 5.1 describes stability of hybrid system with a single Lyapunov function. In Sect. 5.2, stability with a single Lyapunov function for each discrete state is achieved. Section 5.3 describes how to stabilize a (some) discrete state(s) of hybrid system by several Lyapunov functions.

5.1 Stability with a Single Lyapunov Function

If \( M \) consists of several discrete states, it is sometimes possible to use only a single Lyapunov function to show the stability of the hybrid system. The first step of the procedure is to find a single Lyapunov function. If such function exists, the system is stable (asymptotically stable) no matter how the switching occur. In the following theorem the sufficient conditions for ensuring the existence of a common Lyapunov function for hybrid systems is defined.

**Theorem 4:** Assume that there exists polynomial \( V(x) \), such that \( V(0) = 0 \). If the following conditions are satisfied:

\[ L1. \quad V(x) - \sum_{i=1}^L b_i(x)g_i(x) \text{ is SOS,} \]

\[ L2. \quad \sum_{i=1}^L \left( \frac{\partial V}{\partial x} f_i(x) + c_i(x)g_i(x) \right) \text{ is SOS,} \quad (11) \]

where \( g_i(x), b_i(x) \) and \( c_i(x) \) are SOS functions similar to \( \varphi(x) \) in theorem (3). The existence of the Lyapunov function \( V \) for the system \( \dot{x} = \sum_{i=1}^I \alpha_i f_i(x) \) guarantees that the origin is a stable (asymptotically stable) equilibrium point for arbitrary \( \alpha_i \geq 0 \).

**Proof:** From conditions (11) we have:
Using the SOSTOOLS the following Lyapunov function:

\[
V(x) = -5.13x_1 - 1.07x_2 + 13.52x_1^2 + 2.29x_1x_2 + 5.95x_2^2 + 0.39x_1^2 + 1.78x_2^2
\]

was found for both subsystems. Simulation result of the hybrid system with initial state \((x_0, m_0) = \begin{bmatrix} -3 & -3 \end{bmatrix}^T, m_1\) and the level sets of the obtained Lyapunov function is shown in Fig. 2. The origin is reached in a finite time interval that depends on the distance of the initial state to the origin. For systems satisfying conditions of theorem (4), a single Lyapunov function can guarantee the stability of hybrid systems under any arbitrary switching. However, in many cases such a common function does not exist, especially if the dynamics of the modes differ strongly. Also, there is a chance to find such a common Lyapunov function only if all modes have stable dynamics. As this is often not the case (in fact: stable hybrid systems can have unstable modes), a more refined approach is needed. The next thing to do is trying to find one candidate Lyapunov function for each discrete state. In this way the number of Lyapunov functions is the same as the number of discrete states.
5.2 Stability with a Single Lyapunov Function for Each Discrete State

Another attempt may then be to partition the hybrid state space such that only one region corresponds to each discrete state, i.e. one Lyapunov function for each discrete state has to be found. In this subsection, we give a brief introduction to conditions where we need to use one Lyapunov function for each discrete state. Details can be found in [5].

**Definition 3:** Let $R = \{R_1,...,R_k\}$ be a partitioning of hybrid system. For each pair of regions $R_i$ and $R_j$, define the neighboring regions $T_{ij}$ ($i \neq j$)

$$T_{ij} = \{(x,m) | \exists \varepsilon > 0 : (x(t-\varepsilon),m(t-\varepsilon)) \in R_i \text{ and } (x(t+\varepsilon),m(t+\varepsilon)) \in R_j, \text{ when } \varepsilon \rightarrow 0, \varepsilon > 0\}$$

(16)

**Definition 4:** A set $R^d_i$ is defined by:

$$R^d_i = \left\{ x \in \mathbb{R}^n | \frac{\partial V_i}{\partial x} f_i(x) \leq 0 \right\}$$

(17)

In this case, every point in the switch set $S_{ij}$ such that $(x,m_i) \in R_i$ and $(x,m_j) \in R_j$ is included in $T_{ij}$. Hence by investigating the set of states $x \in S_{ij}$, the hybrid state $(x,m_i)$ is included in $T_{ij}$ if the trajectory enters another region. If $(x,m_i) \in R_i$ and $(x,m_j) \in R_j$, then $(x,m_i) \in T_{ij}$, or $x \in S_{ij} \cap R^d_i \cap R^d_j$. Note that it is necessary that $R^d_i \cap R^d_j \neq \emptyset$ for $T_{ij}$ not to be empty. When the partitioning is made in such a way that only one region corresponds to each discrete state, then every state $(x,m_i)$ where $x \in S_{ij}$ is included in some of the sets $T_{ij}$ and therefore theorem (5) can be used for proving the stability of system.

**Theorem 5:** Suppose that for each discrete state of a hybrid system, there exists a polynomial $V_i(x)$ such that $V_i(0) = 0$ and

L1. $V_i(x)$ is SOS,

L2. $-\frac{\partial V_i(x)}{\partial x} f_i(x)$ is SOS,

L3. $V_j(x) \leq V_i(x)$ for all switching sequences $S_{ij}$

(18)

Then the origin of the state space is stable by $V(x(t)) = V_i(x(t))$, if $\dot{x} = f_i(x)$.

**Proof:** Since functions $V_i(x)$ and $-\frac{\partial V_i(x)}{\partial x} f_i(x)$ are SOS then according to definition of sum of squares method, both of them are positive. Then the proof of conditions L1 and L2 are similar to the proof of theorem (1). According to the third condition, the energy decreases at every switching time. Hence $V(x(t)) \leq V(x_0)$ for all $t \geq 0$.

It should be mentioned that according to [17], [18], Lyapunov function must be connective during the switching of vector fields. But in this paper we don’t need connectivity on Lyapunov functions. Considering L3 in theorem (5), these functions guarantee the energy not to increase when there are switching to vector fields. Thus our proposed method is generally less conservative than those proposed in [17].

**Example 3:** (linear vector fields) The following example illustrates the application of theorem (5) for hybrid systems with linear vector fields. Consider the hybrid system $\dot{x} = A_i x$ which is composed of four subsystems [19].

$$A_1 = A_3 = \begin{bmatrix} -0.2 & 1 \\ -10 & -0.2 \end{bmatrix}$$

$$A_2 = A_4 = \begin{bmatrix} -0.2 & 10 \\ -1 & -0.2 \end{bmatrix}$$

The hybrid automaton of system is illustrated in Fig. 3. Using SOSTOOLS, the following Lyapunov functions are found:

$$V_1(x) = V_3(x) = 18.52x_1^2 + 0.2x_1x_2 + 1.89x_2^2$$

$$V_2(x) = V_4(x) = 1.87x_1^2 + 0.2x_1x_2 + 18.32x_2^2$$

Figure 4 shows that the hybrid system is (asymptotically) stable.

**Example 4:** (nonlinear vector fields) To show the flexibility of the introduced stability theory, it is also applied to the
following hybrid system with nonlinear vector fields. Consider the hybrid system \( \dot{x} = f(x) \) which is composed of two nonlinear subsystems:
\[
\begin{align*}
    f_1(x) &= \begin{bmatrix}
        -2x_1 + x_2 \\
        -x_1^2
    \end{bmatrix} \\
    f_2(x) &= \begin{bmatrix}
        -3x_1^3 - x_2^3 \\
        2x_1 - 6x_2
    \end{bmatrix}
\end{align*}
\]

Define the switching functions as follows:
\[

S_{12} = \{ x \in \mathbb{R}^2 | i(t) = 1; \ x_2 = 0.66x_1 \}
\]
\[

S_{21} = \{ x \in \mathbb{R}^2 | i(t) = 2; \ x_2 = 0.66x_1 \}
\]

In Fig. 5, a simulation result of \( f_1(x) \) is shown. This system is unstable and a single Lyapunov function cannot be found for this system. Simulation result of hybrid system is shown in Fig. 6. The origin is reached in a finite time interval that depends on the distance from the initial state to the origin. It can be shown that according to theorem (5), hybrid system is stable. Using SOSTOOLS the following Lyapunov functions are found for \( f_1(x) \) and \( f_2(x) \), respectively,
\[
\begin{align*}
    V_1(x) &= 2.7x_2 + 86x_1^2 + 22x_2^2 - 84x_1x_2 + 5.4x_1^4 \\
            &+ 0.38 \times 10^{-2}x_1^3x_2 + 0.27 \times 10^{-3}x_2^3 + 0.05x_1^3x_2 \\
    V_2(x) &= 4.47x_1^4 + 2.37x_1^2x_2 + 5.21x_1^2x_2^2 + 9.76x_1x_2^3 \\
            &+ 14.53x_2^3
\end{align*}
\]

5.3 Stability with Several Lyapunov Functions for the Same Discrete State

If there is no solution to Theorem (5), then several candidate Lyapunov functions may be tested for the same discrete state to show its stability. This is for instance the case when one of the subsystems is unstable and the corresponding discrete state is in the entire continuous state space. When several Lyapunov functions are used for the same discrete state, a further partitioning of the region \( R \), in several subregions is made, and each of these subregions is coupled to different candidate Lyapunov functions. There are infinitely many ways to make this partitioning, but in this paper we introduce a way to get the contour for this partitioning according to the direction of vector fields. To see how the trajectory passes from one region to another at certain states, the vector field directions at these states can be investigated. This follows easily from the definition of the time derivative \( \dot{x} \):
\[
\dot{x} = \lim_{\Delta \to 0} \frac{x(t + \Delta) - x(t)}{\Delta} = f(x, m)
\]
\[
x(t + \Delta) = x(t) + \Delta f(x, m) \quad \Delta \to 0
\]

In general, the investigation of the vector field must be carried out at every neighboring point. However when the neighboring points are separated by differentiable switching function \( S_{ij}(x) = 0 \), the investigation can be reduced to finding the solution to:
\[
\frac{\delta S_{ij}(x)}{\delta x} f(x, m_i) = 0
\]
which partitions the region that \( f(x, m_i) \) is valid into subregions where the vector field has a direction out of or into a specific region. When there is no solution to (20), all points on the region have the same vector field direction.

In fact Eq. (20) is a gradient vector. The main idea for using Eq. (20) is the gradient represents a direction of maximum rate of increase for the function \( f(x) \) at point \( x^* \) and the maximum rate of change of \( f(x) \) at any point \( x^* \) is the magnitude of the gradient vector. This property shows that the gradient vector at any point \( x^* \) represents a direction of maximum increase for function \( f(x) \) and the rate of increase is the magnitude of the vector.

The following theorem is an extension of Lyapunov’s stability theorem, and it can be used, when we need a Lyapunov function (got by SOS decomposition) only in a region of state space.

**Theorem 6**: Consider the system \( \dot{x} = f(x) \) with constraints \( a_k(x) \leq 0, \ k = 1, ..., K \). We assume that \( f(x) \) (apart from the required Lipschitz conditions for the existence of solutions), has no singularity in \( D \), where \( D \) is defined as:
\[
D = \{ x | a_k(x) \leq 0, \ \text{for all} \ k \}
\]
Then there exists \( V(x) \) of degree \( 2d \) and \( W(x) = \sum_{i=1}^{n} \sum_{j=1}^{d} e_{ij} \) such that \( \sum_{j=1}^{d} e_{ij} > \gamma \ \forall i = 1, ..., n \) where \( \gamma \) is a positive number and \( e_{ij} \geq 0 \) for all \( i \) and \( j \).
Then the conditions:

1. \( V(x) - W(x) \) is SOS,
2. \( -\frac{\partial}{\partial x} f(x) + \sum_{k=1}^{K} p_k(x) a_k(x) \) is SOS,
3. \( p_j(x) \) are SOS similar to \( \varphi(x) \) in theorem (3),

will guarantee that the origin of the state space is a stable equilibrium of the system [20].

Suppose that for a hybrid system, for one (some) of discrete state(s) no Lyapunov function can be obtained by using theorem (5). In this case theorem (7) can be used for this (these) state(s). The intention of this theorem is to possibly partition the hybrid state space into a number of subregions and introduce a function in each subregion that measures the hybrid system’s energy.

**Theorem 7**: If there is no Lyapunov function for one (some) of discrete state(s) of hybrid system (1). First, this state is partitioned into \( K \) subregions \( R_{ij} \) be Eq. (20). For each of these subregions there exist \( V_{ij}(x), a_{ij}(x) \leq 0, W_{ij}(x), p_{ij}(x) \) \( j = 1, ..., K \), as mentioned in theorem (6), such that:

1. \( V_{ij}(x) - W_{ij}(x) \) is SOS in subregion \( R_{ij} \),
2. \( -\frac{\partial}{\partial x} f(x) + \sum p_{ij}(x) a_{ij}(x) \) is SOS in subregion \( R_{ij} \),
3. For each pair of subregions \( R_{ij} \) and \( R_{ij(j+1)} \):

\[
V_{ij}(x) \geq V_{ij(j+1)}(x),
\]

then this (these) discrete state(s) of hybrid system is (are) stable by \( V_i(x(t)) = V_j(x(t)), \) if \( m \in R_{ij} \).

**Proof**: The proof is similar to the proof of theorem (5).

**Example 5**: (linear vector fields) Consider the hybrid system \( \dot{x} = A_1 x \) which is composed of two subregions:

\[
A_1 = \begin{bmatrix} 1 & 8 \\ -2 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & -2 \\ 8 & 1 \end{bmatrix}
\]

Both linear sub-systems have eigenvalues of \( 1 \pm 4j \) and thus they are unstable. Define the switching functions as follows:

\[
S_{12} = \{ x \in \mathbb{R}^2 \mid \text{if } x_1 = 0, x_2 = 4.5x_1 \} \\
S_{21} = \{ x \in \mathbb{R}^2 \mid \text{if } x_1 = 0, x_2 = 0.22x_1 \}
\]

A partition of a hybrid state space is shown in Fig. 7 for this example. In regions \( R_1 \) and \( R_2 \) discrete state is \( m_1 \) and for \( R_2 \) and \( R_3 \) discrete state is \( m_1 \) or \( m_2 \) depending on which switching functions were last reached.

For this example it is not possible to show the stability of discrete state \( m_1 \) by only a single Lyapunov function. Since this state is unstable and it is possible in the entire state space except for points in \( S_{12} \), therefore regions are partitioned by Eq. (20).

Deriving switching function \( S_{12} \) leads to the following equation:

\[
\frac{\delta S_{12}}{\delta x} = \begin{bmatrix} -4.5x_1 \\ x_2 \end{bmatrix} = 0
\] (21)

According to Eq. (20), by replacing \( \dot{x} = A_1 x \) in Eq. (21), we have equation \( x_2 = -0.19x_1 \). The same algorithm for \( S_{21} \) and \( \dot{x} = A_2 x \) results in \( x_2 = -5.4x_1 \).

Different subregions for the discrete state \( m_1 \) are shown in Fig. 8. As shown in Fig. 8, subregion \( R_{11} \) is:

\[
R_{11} = \{ (x_1, x_2) \mid x_2 \geq 0 \text{ and } x_2 \geq 4.5x_1 \text{ and } x_2 \geq -5.4x_1 \}
\] (22)

Therefore the functions \( a_{ij}(x) \) for subregion \( R_{1j} \) are defined by Eq. (23).

\[
a_{ij}(x) = \{ -x_2 \leq 0 \text{ and } 4.5x_1 - x_2 \leq 0 \text{ and } -5.4x_1 - x_2 \leq 0 \}
\] (23)

\( a_{ij}(x) \) functions for other subregions can be obtained using same method. Lyapunov functions for the hybrid system can be chosen as (24).

\[
V(x) = \sum_{i=0}^{m} \sum_{j=0}^{n} c_{ij} x_i^j, \quad c_{00} = 0
\] (24)

The \( c_{ij} \) coefficients for each subregion for discrete state \( m_1 \) and \( m_2 \) are given in Table 1.

**Example 6**: (nonlinear vector fields) Consider the hybrid system \( \dot{x} = f_i(x) \) which is composed of two nonlinear sub-systems:
Table 1  Lyapunov coefficients for each subregion of discrete states for example (5).

<table>
<thead>
<tr>
<th>Discrete states (subregion)</th>
<th>$R_{11}$</th>
<th>$R_{12}$</th>
<th>$R_{13}$</th>
<th>$R_{21}$</th>
<th>$R_{22}$</th>
<th>$R_{23}$</th>
<th>$R_{31}$</th>
<th>$R_{32}$</th>
<th>$R_{33}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c_{00}$</td>
<td>0.44</td>
<td>0.13</td>
<td>0.10</td>
<td>0.04</td>
<td>0.13</td>
<td>0.41</td>
<td>0.21</td>
<td>0.31</td>
<td>0.10</td>
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<td>$c_{01}$</td>
<td>-2.15</td>
<td>0.31</td>
<td>0.26</td>
<td>2.15</td>
<td>-0.31</td>
<td>13.5</td>
<td>2.26</td>
<td>0.07</td>
<td>-0.07</td>
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<tr>
<td>$c_{02}$</td>
<td>2.30</td>
<td>0.54</td>
<td>0.54</td>
<td>2.30</td>
<td>0.54</td>
<td>0.54</td>
<td>0.54</td>
<td>0.54</td>
<td>0.54</td>
</tr>
<tr>
<td>$c_{11}$</td>
<td>-3.07</td>
<td>-0.23</td>
<td>-4.74</td>
<td>-17.7</td>
<td>-3.07</td>
<td>-0.23</td>
<td>47.4</td>
<td>-17.7</td>
<td>-0.84</td>
</tr>
<tr>
<td>$c_{03}$</td>
<td>0.55</td>
<td>0.17</td>
<td>0.43</td>
<td>-0.55</td>
<td>0.17</td>
<td>-2.15</td>
<td>-1.43</td>
<td>-0.16</td>
<td>0.16</td>
</tr>
<tr>
<td>$c_{12}$</td>
<td>-0.61</td>
<td>0.22</td>
<td>1.92</td>
<td>0.61</td>
<td>-0.22</td>
<td>139</td>
<td>-1.92</td>
<td>0.29</td>
<td>-0.29</td>
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<tr>
<td>$c_{13}$</td>
<td>1.38</td>
<td>-0.93</td>
<td>-1.17</td>
<td>12.1</td>
<td>-1.38</td>
<td>0.93</td>
<td>117</td>
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<tr>
<td>$c_{04}$</td>
<td>-2.44</td>
<td>0.18</td>
<td>4.72</td>
<td>-11.6</td>
<td>2.44</td>
<td>-0.18</td>
<td>47.2</td>
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<td>-0.07</td>
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<tr>
<td>$c_{40}$</td>
<td>0.10</td>
<td>0.01</td>
<td>0.03</td>
<td>-0.10</td>
<td>0.01</td>
<td>-0.01</td>
<td>0.01</td>
<td>0.01</td>
<td>0.01</td>
</tr>
<tr>
<td>$c_{11}$</td>
<td>0.16</td>
<td>0.19</td>
<td>0.16</td>
<td>0.06</td>
<td>0.19</td>
<td>0.16</td>
<td>0.01</td>
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<tr>
<td>$c_{13}$</td>
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<td>-0.05</td>
<td>-0.31</td>
<td>-0.27</td>
<td>-0.44</td>
<td>-0.05</td>
<td>-0.31</td>
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<tr>
<td>$c_{04}$</td>
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<td>0.02</td>
<td>0.27</td>
<td>0.34</td>
<td>0.29</td>
<td>0.02</td>
<td>0.27</td>
<td>0.34</td>
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</tr>
<tr>
<td>$c_{40}$</td>
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<td>0.00</td>
<td>-0.00</td>
<td>0.00</td>
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</tr>
<tr>
<td>$c_{12}$</td>
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<td>0.00</td>
<td>-0.00</td>
<td>0.00</td>
<td>-0.00</td>
<td>0.00</td>
<td>-0.00</td>
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<tr>
<td>$c_{14}$</td>
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<td>0.00</td>
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<td>-0.00</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>$c_{05}$</td>
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<td>0.00</td>
<td>-0.00</td>
<td>0.00</td>
<td>-0.00</td>
<td>0.00</td>
<td>-0.00</td>
<td>0.00</td>
<td>-0.00</td>
</tr>
</tbody>
</table>

As shown in Fig. 11 the Eq. (26) is satisfied for a region that $f_1(x)$ is not valid. Hence only Eq. (25) for $f_1(x)$ will be used. Different subregions for $f_1(x)$ and $f_2(x)$ are illustrated in Fig. 12. Using SOSTOOLS the following Lyapunov functions are found for different regions of $f_1(x)$ and $f_2(x)$.

Fig. 9  Phase plot and state trajectory of $f_1(x)$ of example (6).

Fig. 10  Phase plot and state trajectory of $f_2(x)$ of example (6).

Fig. 11  Different regions and transition contours of example (6).
polynomials, the Lyapunov conditions are essentially polynomial non-negativity conditions which can be NP hard to test [21], probably one of the reasons for the lack of algorithmic construction of Lyapunov functions. In this paper, new strategies based on semidefinite programming and the sum of squares decomposition are proposed for stability of hybrid systems. The advantage of the proposed procedure is that no special requirements on the system structure are imposed. Using this approach, higher degree Lyapunov functions can be constructed, and the conservatism of searching is reduced for only candidate functions. This method also provides reliable and less conservative results than most existing relaxation methods. We demonstrated the efficiency of our approach by some examples of hybrid systems. The presented stability results are quite general and can be applied for hybrid systems with linear or nonlinear vector fields.

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