DENOISING USING PROJECTIONS ONTO THE EPIGRAPH SET OF A CONVEX COST FUNCTION

Mohammad Tofghi, Kivanc Kose*, and A. Enis Cetin

Department of Electrical and Electronic Engineering, Bilkent University, Ankara, Turkey
*Dermatology Department, Memorial Sloan Kettering Cancer Center, New York, USA
tofghi@ee.bilkent.edu.tr, *kosek@mskcc.org, cetin@bilkent.edu.tr

ABSTRACT
A new denoising algorithm based on orthogonal projections onto the epigraph set of a convex cost function is presented. In this algorithm, the dimension of the minimization problem is lifted by one and sets corresponding to the cost function are defined. As the utilized cost function is a convex function in $\mathbb{R}^N$, the corresponding epigraph set is also a convex set in $\mathbb{R}^{N+1}$. The denoising algorithm starts with an arbitrary initial estimate in $\mathbb{R}^{N+1}$. At each step of the iterative denoising, an orthogonal projection is performed onto one of the constraint sets associated with the cost function in a sequential manner. The method provides globally optimal solutions for total-variation, $\ell_1$, $\ell_2$, and entropic cost functions.

Index Terms—Epigraph of a cost function, denoising, Projection onto convex sets, total variation

1. INTRODUCTION
A new denoising algorithm based on orthogonal Projections onto the Epigraph Set of a Convex cost function (PESC) is introduced. In Bregman’s standard POCS approach \cite{1, 2}, the algorithm converges to the intersection of convex constraint sets. In this article, it is shown that it is possible to use a convex cost function in a POCS based framework using the epigraph set and the new framework is used in denoising \cite{3, 5}.

Bregman also developed iterative methods based on the so-called Bregman distance to solve convex optimization problems \cite{6}. In Bregman’s approach, it is necessary to perform a Bregman projection at each step of the algorithm, which may not be easy to compute the Bregman distance in general \cite{5, 7}.

In standard POCS approach, the goal is simply to find a vector, which is in the intersection of convex constraint sets \cite{8, 9, 29}. In each step of the iterative algorithm an orthogonal projection is performed onto one of the convex sets. Bregman showed that successive orthogonal projections converge to a vector, which is in the intersection of all the convex sets. If the sets do not intersect it iterates oscillate between members of the sets \cite{30, 31}. Since, there is no need to compute the Bregman distance in standard POCS, it found applications in many practical problems. In this article, orthogonal projections onto the epigraph set of a convex cost functions are used to solve convex optimization problems instead of the Bregman distance approach.

In PESC approach, the dimension of the signal reconstruction or restoration problem is lifted by one and sets corresponding to a given convex cost function are defined. This approach is graphically illustrated in Fig. 1. If the cost function is a convex function in $\mathbb{R}^N$, the corresponding epigraph set is also a convex set in $\mathbb{R}^{N+1}$.

As a result, the convex minimization problem is reduced to finding the $[w^*, f(w^*)]$ vector of the epigraph set corresponding to the cost function as shown in Fig. 1. As in standard POCS approach, the new iterative optimization method starts with an arbitrary initial estimate in $\mathbb{R}^{N+1}$ and an orthogonal projection is performed onto one of the constraint sets. The resulting vector is then projected onto the epigraph set. This process is continued in a sequential manner at each step of the optimization problem. This method provides globally optimal solutions for convex cost functions such as total-variation \cite{32}, filtered variation \cite{41}, $\ell_1$ \cite{33}, and entropic function \cite{10}. The iteration process is shown in Fig. 1. Regardless of the initial value $w_0$, iterates converge to $[w^*, f(w^*)]$ pair as shown in Fig. 1.

The article is organized as follows. In Section 2 the epigraph of a convex cost function is defined and the convex minimization method based on the PESC approach is introduced. In Section 3 the new denoising method is presented. The new approach does not require a regularization parameter as in other TV based methods \cite{2, 20, 32}. In Section 4 the simulation results and some denoising examples, are presented.

2. EPIGRAPH OF A CONVEX COST FUNCTION
Let us consider a convex minimization problem

$$\min_{w \in \mathbb{R}^N} f(w), \quad (1)$$

where $f : \mathbb{R}^N \to \mathbb{R}$ is a convex function. We increase the dimension of the problem by one to define the epigraph set in $\mathbb{R}^{N+1}$ corresponding to the cost function $f(w)$ as follows:

$$C_f = \{w = [w^T y]^T : y \geq f(w)\}, \quad (2)$$

which is the set of $N + 1$ dimensional vectors, whose $(N + 1)^{th}$ component $y$ is greater than $f(w)$. We use bold face letters for $N$ dimensional vectors and underlined bold face letters for $N + 1$ dimensional vectors, respectively. The second set that is related with the cost function $f(w)$ is the level set:

$$C_s = \{w = [w^T y]^T : y \leq \alpha, \ w \in \mathbb{R}^{N+1}\}, \quad (3)$$

where $\alpha$ is a real number. Here it is assumed that $f(w) \geq \alpha$ for all $f(w) \in \mathbb{R}$ such that the sets $C_f$ and $C_s$ do not intersect or the intersection contains a single vector. They are both closed and convex sets in $\mathbb{R}^{N+1}$. Sets $C_f$ and $C_s$ are graphically illustrated in Fig. 1.
An important component of the PESC approach is to perform an orthogonal projection onto the epigraph set. Let \( \mathbf{w}_0 \) be an arbitrary vector in \( \mathbb{R}^{N+1} \). The projection \( \mathbf{w}_2 \) is determined by minimizing the distance between \( \mathbf{w}_1 \) and \( C_f \), i.e.,

\[
\mathbf{w}_2 = \arg \min_{\mathbf{w} \in C_f} \| \mathbf{w}_1 - \mathbf{w} \|^2. \tag{4}
\]

Equation (4) is the ordinary orthogonal projection operation onto the set \( C_f \in \mathbb{R}^{N+1} \). In order to solve the problem in Eq. (4) we do not need to compute the Bregman’s so-called D-projection or Bregman projection. Projection onto the set \( C_s \) is trivial. We simply force the last component of the \( N + 1 \) dimensional vector to zero. In the PESC algorithm, iterates eventually oscillate between the two nearest vectors of the sets \( C_s \) and \( C_f \) as shown in Fig. 1. As a result, we obtain

\[
\lim_{n \to \infty} \mathbf{w}_{2n} = [w^* f(w^*)]^T, \tag{5}
\]

where \( w^* \) is the \( N \) dimensional vector minimizing \( f(w) \). The proof of Eq. (5) follows from Bregman’s POCs theorem [11]. It was generalized to non-intersection case by Gubin et al. [30]. Since the two closed and convex sets \( C_s \) and \( C_f \) are closest to each other at the optimal solution case, iterations oscillate between the two sets \( C_s \) and \( C_f \). As \( n \) tends to infinity, it is possible to increase the speed of convergence by non-orthogonal projections [21].

If the cost function \( f \) is not convex and have more than one local minimum then the corresponding set \( C_f \) is not convex in \( \mathbb{R}^{N+1} \). In this case iterates may converge to one of the local minima.

In current TV based denoising methods [32, 34] the following cost function is used:

\[
\min \| \mathbf{v} - \mathbf{w} \|^2 + \lambda \text{TV}(\mathbf{w}), \tag{6}
\]

where \( \mathbf{v} \) is the observed signal. The solution of this problem can be obtained using the method in an iterative manner, by performing successive orthogonal projections onto \( C_f \) and \( C_s \), as discussed above. In this case the cost function is \( f(w) = \| \mathbf{v} - \mathbf{w} \|^2 + \lambda \text{TV}(\mathbf{w}) \). Therefore,

\[
C_f = \{ \| \mathbf{v} - \mathbf{w} \|^2 + \lambda \text{TV}(\mathbf{w}) \leq y \}. \tag{7}
\]

The denoising solutions that we obtained are very similar to the ones found by Chambolle’s in [32] as both methods use the same cost function. One problem in [32] is the estimation of the regularization parameter \( \lambda \). One has to determine the \( \lambda \) in an ad-hoc manner or by visual inspection. In the next section, a new denoising method with a different TV based cost function is described. The new method does not require a regularization parameter.

### 3. Denoising using PESC

In this section, we present a new denoising method, based on the epigraph set of the convex cost function. It is possible to use TV, FV and \( \ell_1 \) norm as the convex cost function. Let the original signal or image be \( \mathbf{w}_{\text{orig}} \) and its noisy version be \( \mathbf{v} \). Suppose that the observation model is the additive noise model:

\[
\mathbf{v} = \mathbf{w}_{\text{orig}} + \mathbf{\eta}, \tag{8}
\]

where \( \mathbf{\eta} \) is the additive noise. In this approach we solve the following problem for denoising:

\[
\mathbf{w}^* = \arg \min_{\mathbf{w} \in C_f} \| \mathbf{v} - \mathbf{w} \|^2, \tag{9}
\]

where \( \mathbf{v} = [\mathbf{v}^T \mathbf{0}] \) and \( C_f \) is the epigraph set of TV or FV in \( \mathbb{R}^{N+1} \). The TV function, which we used for an \( M \times M \) discrete image \( \mathbf{w} = [w_{i,j}] \). 0 \( \leq i, j \leq M - 1 \in \mathbb{R}^{M \times M} \) is as follows:

\[
TV(\mathbf{w}) = \sum_{i,j} \left( |w_{i+1,j} - w_{i,j}| + |w_{i,j+1} - w_{i,j}| \right). \tag{10}
\]

The minimization problem Eq. (9) is essentially the orthogonal projection onto the set \( C_f = \{ TV(\mathbf{w}) \leq y \} \). Notice that, this \( C_f \) is different from Eq. (7). This means that we select the nearest vector \( \mathbf{w}^* \) on the set \( C_f \) to \( \mathbf{v} \). This is graphically illustrated in Fig. 2. During this orthogonal projection operations, we do not require any parameter adjustment as in [32].
Implementation: The projection operation described in Eq. (9) can not be obtained in one step when the cost function is TV. The solution is determined by performing successive orthogonal projections onto supporting hyperplanes of the epigraph set $C_f$. In the first step, $TV(v_0)$ and the surface normal at $v_1 = [v_0^T, TV(v_0)]$ in $\mathbb{R}^{N+1}$ are calculated. In this way, the equation of the supporting hyperplane at $v_1$ is obtained. The vector $v_0 = [v_0^T, 0]$ is projected onto this hyperplane and $w_1$ is obtained as our first estimate as shown in Fig. 2. In the second step, $w_1$ is projected onto the set $C_s$ by simply making its last component zero. The TV of this vector and the surface normal, and the supporting hyperplane are calculated as in the previous step. Next, $v_0$ is projected onto the new supporting hyperplane, and $w_2$ is obtained. In Fig. 2 $w_2$ is very close to the denoising solution $w^\star$. In general iterations continue until $\|w_i - w_{i-1}\| \leq \epsilon$, where $\epsilon$ is a prescribed number, or iterations can be stopped after a certain number of iterations.

We calculate the distance between $v_0$ and $w_i$ at each step of the iterative algorithm described in the previous paragraph. The distance $\|v_0 - w_i\|^2$ does not always decrease for high $i$ values. This happens around the optimal denoising solution $w^\star$. Once we detect an increase in $\|v_0 - w_i\|^2$, we perform a refinement step to obtain the final solution of the denoising problem. In refinement step, the supporting hyperplane at $w_i + \frac{w_i - w_{i-1}}{2}$ is used in the next iteration. A typical convergence graph is shown in Fig. 3 for the “note” image.

4. SIMULATION RESULTS

The PESC algorithm is tested with a wide range of images. Let us start with the “Note” image shown in Fig. 4(a). This is corrupted by a zero mean Gaussian noise with $\sigma = 45$ in Fig. 4(b). The image is restored using PESC and Chambolle’s algorithm [32] and the denoised images are shown in Fig. 4(c) and 4(d) with SNR values equal to 15.08 and 13.20 dB, respectively. Chambolle’s algorithm produces some patches of gray pixels at the background. The regularization parameter $\lambda$ in Eq. (5) is manually adjusted to get the best possible results for each image and each noise type and level. PESC algorithm not only produces a higher SNR, but also provides a visually better looking image. Denoising results for other noise levels are presented in Table 2. We also tested the PESC algorithm against $\epsilon$-contaminated Gaussian noise (salt-and-pepper noise) with the PDF of

$$f(x) = \epsilon \phi\left(\frac{x}{\sigma_1}\right) + (1 - \epsilon)\phi\left(\frac{x}{\sigma_2}\right),$$  

where $\phi(x)$ is the standard Gaussian distribution with mean zero and unit standard deviation. The results of the tests are presented in Table 1. The performance of the reconstruction is measured using the SNR criterion, which is defined as follows

$$SNR = 20 \times \log_{10}\left(\frac{\|w_{orig}\|}{\|w_{orig} - w_{rec}\|}\right),$$

where $w_{orig}$ is the original signal and $w_{rec}$ is the reconstructed signal. All the SNR values in Tables are in dB.

![Figure 3](image3.png)

Figure 3: Euclidian distance from $v$ to the epigraph of TV at each iteration ($\|v - w_i\|$) with noise standard deviation of $\sigma = 30$.

It is possible to obtain a smoother version of $w^\star$ by simply projecting $v$ inside the set $C_f$ instead of the boundary of $C_f$.

![Figure 4](image4.png)

Figure 4: (a) A portion of original “Note” image, (b) image corrupted with Gaussian noise with $\sigma = 45$, (c) denoised image, using PESC algorithm; SNR = 15.08 dB and, (d) denoised image, using Chambolle’s algorithm; SNR = 13.20 dB. Chambolle’s algorithm produces some patches of gray pixels at the background.
Figure 5: NRMSE vs. iteration number for denoising the “Note” image with Gaussian noise with standard deviation of $\sigma = 30$.

Table 1: Comparison of the results for denoising algorithms for $\epsilon$-Contaminated Gaussian noise for “note” image.

<table>
<thead>
<tr>
<th>$\epsilon$</th>
<th>$\sigma_1$</th>
<th>$\sigma_2$</th>
<th>Input SNR</th>
<th>PESC</th>
<th>Chambolle [32]</th>
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<tr>
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It is also possible to use Normalized Root Mean Square Error metric as

$$NRMSE(i) = \frac{\|w_i - w_{avg}\|}{\|w_{avg}\|} \quad i = 1, \ldots, N,$$  (13)

which N is the number of the iterations, in [20] to illustrate the convergence of the PESC based denoising algorithm. As shown in Fig. [3] $NRMSE$ value decreases as the iterations proceeds while denoising the “Note” image corrupted with Gaussian noise ($\sigma = 25$). For the same image another convergence metric called Normalized Total Variation metric which defined as $NTV(i) = TV(w_i)/TV(w_{avg})$ in [20], also converges to 1 in almost 100 iterations. In Table 3 denoising results for 34 images including 10 well-known test images from image processing literature and 24 images from Kodak Database [35], with different noise levels are presented. In almost all cases PESC method produces higher SNR results than the denoising results obtained using [32].

5. CONCLUSION

A new denoising method based on the epigraph of the TV function is developed. Epigraph sets of other convex cost functions can be also used in the new denoising approach. The denoised signal is obtained by making an orthogonal projection onto the epigraph set from the corrupted signal in $\mathbb{R}^{N+1}$. The new algorithm does not need the optimization of the regularization parameter as in standard TV denoising methods. Experimental results indicate that better SNR results are obtained compared to standard TV based denoising in a large range of images.
References


