Feedback controller design for linear and a class of nonlinear optimal control problems

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SUMMARY

This paper presents a feedback controller designing approach for a large class of finite-time optimal control problems. This approach involves a piecewise truncated variational iteration method (PTVIM) for solving the nonlinear Hamilton–Jacobi–Bellman equation. By using the finite iterations of PTVIM, an analytic approximate solution for value function and suboptimal feedback control law is obtained. Some illustrative examples are employed to demonstrate the accuracy and efficiency of the proposed approach. Copyright © 2013 John Wiley & Sons, Ltd.

Received 8 August 2011; Revised 6 September 2012; Accepted 18 January 2013

KEY WORDS: optimal control problem; variational iteration method; Hamilton–Jacobi–Bellman equation; suboptimal feedback control

1. INTRODUCTION

The optimal control of linear and nonlinear systems is one of the most challenging subjects in control theory. For the general optimal control problem (OCP) for nonlinear systems, however, an analytical solution does not exist. This has inspired researchers to propose approaches to obtain an approximate solution for it. There exists two major approaches. One of them is to directly face the problem and attempt to find a minimum of the objective (or Lagrangian) functional. The other one is using the optimality necessary conditions and attempt to solve them [1]. In the past two decades, the indirect methods have been extensively developed. It is well known that the OCP leads to a two-point boundary value problem (TPBVP) or a Hamilton–Jacobi–Bellman (HJB) partial differential equation. Many recent researches have been devoted to solve these two problems.

The TPBVP can be solved by using different methods such as successive approximation approach (SAA) [2] and modal series approach [3], resulting in an open-loop candidate optimal control history. The open-loop control solution has to be subjected to the control history and the second order sufficient conditions before it can be declared to be locally optimal. In the ‘real-world environment’, with uncertainties in the initial conditions and other unmodeled effects, a locally optimal feedback solution is more useful than an open-loop solution, which needs to be recomputed for a new initial condition. However, if the system or constraints, or both, are nonlinear, then solving the OCP in a feedback setting can be extremely challenging to solve.

Unfortunately, even for relatively low-dimensional nonlinear systems, solving the HJB equation is a formidable task, and even for linear systems, closed-form solutions are available only for a few special cases [4]. However, there are various methods to solve this equation approximately. One of
these approaches is power series approximation (PSA). The PSA needs an iterative solution of a series of HJB equations to obtain the approximate optimal control law in a series form [5, 6].

Another technique that systematically solves the nonlinear regulator problem is the state-dependent Riccati equation (SDRE) method [7]. By turning the equations of motion into a linear-like structure, this approach permits the designer to employ linear optimal control methods such as the linear-quadratic regulator methodology and the $H_\infty$ design technique for the synthesis of nonlinear control systems. The major problem with SDRE is the time-consuming online computation of the Riccati equation.

The other approach is successive Galerkin approximation (SGA), where an iterative process is used to find a sequence of approximations approaching the solution of the HJB equation [8, 9], which is carried out by solving a sequence of generalized HJB equations. However, being an iterative method, SGA is dependent on the iterative initial value. If it is not well chosen, the method may converge very slowly or even diverge. Other approaches for solving HJB equation have been presented in [4] and [10].

On the other hand, in the context of numerical analysis, the variational iteration method (VIM), which was proposed originally by He [11–13], has been proved by many authors to be a powerful mathematical tool for various kinds of linear and nonlinear ODEs or PDEs. Unlike the traditional numerical methods, VIM needs no discretization, linearization, transformation, or perturbation. The method has been widely applied to solve nonlinear problems, and different modifications are suggested to overcome the demerits arising in the solution procedure (see, e.g., [14, 15]). Among a various number of VIM applications, the use of this method in solving Riccati equations [16, 17] made it a powerful tool in the context of control theory.

In [18], the authors used the original or basic VIM for linear quadratic OCP’s. They transfer the linear TPBVP obtained from PMP to an initial value problem and then implement the basic VIM to have a feedback control. In another recent paper [19], the authors solved HJB equations using Adomian decomposition method.

This paper concerns with designing a feedback controller for linear and a class of nonlinear OCP’s. Applying the piecewise truncated variational iteration method (PTVIM), we solve the HJB equation to have an approximate solution for value function and obtain a suboptimal feedback control law. Some illustrative examples are given to demonstrate the accuracy and efficiency of the proposed method.

This paper is organized as follows. Section 2 describes the OCP formulation and its necessary and sufficient extreme conditions. Solving procedure based on PTVIM is given in Section 3. Section 4 illustrates some examples to show the validity and efficiency of the proposed approach.

2. FIXED-FINAL-TIME HJB OPTIMAL CONTROL PROBLEM

Consider an affine in the control dynamical system of the form

$$\dot{x}(t) = f(x(t)) + g(x(t))u(t),$$

$$x(t_0) = x_0.\tag{1}$$

where $x \in \mathbb{R}^n$ denotes the state variable, $u \in \mathbb{R}^m$ the control variable for $t \in [t_0, t_f]$, and $x_0$ the given initial state at $t_0$. Moreover, $f(x) \in \mathbb{R}^n$ and $g(x) \in \mathbb{R}^{n \times m}$ are two continuously differentiable functions in all arguments. It is desired to find the optimal control $u^*(t)$, which minimizes the objective functional

$$J(x, u) = \phi(x(t_f), t_f) + \int_{t_0}^{t_f} (Q(x) + u^T R u) dt \tag{2}$$

subject to the dynamical system (1), for $Q(x) \in \mathbb{R}$, positive semi-definite on a set $\Omega \subseteq \mathbb{R}^n$ containing the origin and $R \in \mathbb{R}^{m \times m}$ a positive definite matrix.
Under regularity assumptions, that is, \( V(x,t) \in C^1(\Omega) \), an infinitesimal equivalent to (2) is
\[
- \frac{\partial V(x,t)}{\partial t} = Q(x) + u^T R u + \left( \frac{\partial V(x,t)}{\partial x} \right)^T (f(x) + g(x)u) .
\] (3)
This is a time-varying partial differential equation with
\[
V(x(t),t) = \phi(x(t_f),t_f) + \int_t^{t_f} (Q(x) + u^T R u) dt ,
\] (4)
the cost function for any given \( u(t) \) and is solved backward in time from \( t = t_f \). By setting \( t = t_f \) in (2), its boundary condition is seen to be
\[
V(x(t_f),t_f) = \phi(x(t_f),t_f) .
\] (5)
According to Bellman’s optimality principle, the optimal cost is given by
\[
\frac{\partial V^*(x,t)}{\partial t} = \min_{u(t)} \left\{ Q(x) + u^T R u + \left( \frac{\partial V^*(x,t)}{\partial x} \right)^T (f(x) + g(x)u) \right\} .
\] (6)
which yields the optimal control law as follows:
\[
u^* = -\frac{1}{2} R^{-1} g^T(x) \frac{\partial V^*(x,t)}{\partial x},
\] (7)
where \( V^*(x,t) \) is the optimal value function. Substituting (7) into (6) yields the well-known time-varying HJB equation [4]:
\[
\frac{\partial V^*(x,t)}{\partial t} + \left( \frac{\partial V^*(x,t)}{\partial x} \right)^T (f(x) + Q(x) - \frac{1}{4} \frac{\partial V^*(x,t)^T}{\partial x} g(x) R^{-1} g(x)^T \frac{\partial V^*(x,t)}{\partial x}) = 0 .
\] (8)
The HJB equation in (8) and (7) provide the solution to fixed-final time optimal control for general nonlinear systems. However, a closed-form solution for HJB equation is impossible to find, in general. In the next section, we show how to approximately solve this equation using the approximate analytic method, VIM and PTVIM.

**Remark 2.1**
The HJB equation requires that \( V(x,t) \) is a continuously differentiable function. Usually, this requirement is not satisfied in constrained optimization because the control function is piecewise continuous. But control problems do not necessarily have smooth or even continuous value functions. In this paper, all derivations are performed under the assumption of smooth solutions to (8). A similar assumption was made by Van der Schaft [20], Isidori [21], and Cheng [4].

### 3. SOLVING HJB BASED ON PTVIM

In this section, we first describe the VIM for solving HJB equation in (8). VIM gives rapidly convergent successive approximations of the exact solution if such a solution exists; otherwise, approximations can be used for numerical purposes. Define the following linear and nonlinear operators:
\[
\mathcal{L}[V(x,t)] = \frac{\partial V(x,t)}{\partial t} ,
\] (9)
\[
\mathcal{N}[V(x,t)] = \frac{\partial V(x,t)}{\partial x} f(x) + Q(x) - \frac{1}{4} \frac{\partial V(x,t)^T}{\partial x} g(x) R^{-1} g(x)^T \frac{\partial V(x,t)}{\partial x} .
\] (10)
Thus, the HJB equation in (8) with the boundary condition (5) becomes
\[
\mathcal{L}[V(x,t)] + \mathcal{N}[V(x,t)] = 0 , \quad V(x(t_f),t_f) = \phi(x(t_f),t_f) .
\] (11)
According to the VIM [11, 12], we can construct a correction functional as follows:

\[ V_{n+1}(x, t) = V_n(x, t) + \int_{t_f}^{t} \lambda(s) \{ \mathcal{L}[V_n(x, s)] + \mathcal{N}[V_n(x, s)] \} \, ds, \quad (12) \]

where \( V_0(x, t) = \phi(x, t) \) is the initial guess, \( \lambda(s) \neq 0 \) is a general multiplier, and the subscript \( n \) denotes the \( n \)th approximation. By taking variations with respect to \( V_n \) and considering the restricted variation, \( \delta \mathcal{N}[V_n(x, t)] = 0 \), we have

\[ \delta V_{n+1}(x, t) = \delta V_n(x, t) + \lambda(s)\delta V_n(x, t)|_{s=t} - \int_{t_f}^{t} \dot{\lambda}(s)\delta V_n(x, s) \, ds, \]

which gives the following stationary conditions:

\[ 1 + \lambda(s)|_{s=t} = 0, \quad \dot{\lambda}(s) = 0. \]

Thus, \( \lambda(s) = -1 \), and the VIM formula becomes

\[ V_{n+1}(x, t) = V_n(x, t) - \int_{t_f}^{t} \{ \mathcal{L}[V_n(x, s)] + \mathcal{N}[V_n(x, s)] \} \, ds. \quad (13) \]

where \( V_0(x, t) = \phi(x, t) \) and \( n \geq 0 \).

**Theorem 3.1**

Assume that \( \{V_n(x, t)\} \) is the solution sequence produced by VIM formula (13), which converges to \( \hat{V}(x, t) \), as \( n \to \infty \). Then, \( \hat{V}(x, t) \) is the exact solution of (8). Accordingly, the sequence \( \{u_n(x, t)\} \) defined by

\[ u_n(x, t) = -\frac{1}{2} R^{-1} g^T(x) \frac{\partial V_n(x, t)}{\partial x} \quad (14) \]

converges to the optimal feedback control law.

**Proof**

Taking limits of both sides of (13) as \( n \to \infty \), because \( \lim_{n \to \infty} V_n(x, t) = \lim_{n \to \infty} V_{n+1}(x, t) = \hat{V}(x, t) \), results in

\[ \int_{t_f}^{t} \{ \mathcal{L}[\hat{V}(x, s)] + \mathcal{N}[\hat{V}(x, s)] \} \, ds = 0. \]

Differentiating from both sides with respect to \( t \) yields

\[ \mathcal{L}[\hat{V}(x, t)] + \mathcal{N}[\hat{V}(x, t)] = 0. \]

In the light of (13), if \( t = t_f \), then \( V_{n+1}(x, t_f) = V_n(x, t_f) \), for every \( n \geq 0 \). Thus, \( V_n(x, t_f) = V_0(x, t_f) = \phi(x, t_f) \) or \( \hat{V}(x, t_f) = \lim_{n \to \infty} V_n(x, t_f) = \phi(x, t_f) \). Hence, \( \hat{V}(x, t) \) is the exact solution of (11) or equivalently (8). Moreover, taking limit from (14), as \( n \to \infty \), and imposing the regularity assumptions of \( V_n(x, t) \), described in Section 2, completes the proof.

The VIM formula (13) is directly dependent on the integration. In our optimal control problem, \( f(x) \) and \( g(x) \) are two nonlinear functions, in general, which cause the right-hand-side integration of (13) to be very time consuming and complicated, even at the first few iterations of VIM. To overcome this undesirable case and to eliminate the unnecessary and repeated calculations that arise in basic VIM (13), we use the following modified VIM proposed in [17], which is called the truncated VIM:

\[ V_{n+1}(x, t) = V_n(x, t) - \int_{t_f}^{t} \{ T_n(s) - T_{n-1}(s) \} \, ds, \quad n \geq 0, \quad (15) \]
where $T_n(s)$ is the $n$th order Taylor expansion of $\mathcal{L}[V_n(x,s)] + \mathcal{N}[V_n(x,s)]$ around $t_f$, $T_{-1}(s) = 0$, and $V_0(x,t) = \phi(x,t)$.

By using the truncated VIM formula (15), we obtain a truncated series solution. Unfortunately, this series solution gives a good approximation to the exact solution only in a small region of $t$. An easy and reliable way to ensure the validity of the approximations (15) for larger interval of $t$ is to determine the solution in a sequence of equal subintervals $t$, that is, $I_i = [t_i, t_{i+1}]$, where $\Delta t = t_{i+1} - t_i$, $i = N - 1, \ldots, 2, 1, 0$, with $t_N = t_f$. Therefore, we obtain the PTVIM as follows:

$$V_{i+1,n+1}(x,t) = V_{i+1,n}(x,t) - \int_{t_{i+1}}^{t} \{T_{i+1,n}(s) - T_{i+1,n-1}(s)\} \, ds, \, t \in I_i$$

$$V_{i,0}(x,t) = V_{i+1,n_i+1}(x,t),$$

$$\mathcal{L}[V_n(x,s)] + \mathcal{N}[V_n(x,s)] = T_{i+1,n}(s) + O \left((s - t_{i+1})^{n+1}\right), \quad n = 0, 1, \ldots, n_{i+1} - 1, \quad i = N - 1, \ldots, 2, 1, 0,$$

where $V_{0,n_0}(x,t_f) = \phi(x,t)$, $T_{i+1,n}(s)$ is the Taylor expansion of $\mathcal{L}[V_n(x,s)] + \mathcal{N}[V_n(x,s)]$ around $t_{i+1}$ and $T_{i+1,n-1}(s) = 0$, $0 \leq i \leq N - 1$. Thus, in the light of (16), the approximation of (11) on the entire interval $[t_0, t_f]$ can easily be obtained. Following the present section, the $n_{i+1}$th-order PTVIM approximate analytical solution via the PTVIM for (11) can be written as

$$V_{i+1,n_{i+1}}(x,t) = \sum_{m=0}^{n_{i+1}} \frac{\gamma_{i,m}(x,t_{i+1},c_i)}{m!} (t - t_{i+1})^m + O \left(\left(t - t_{i+1}\right)^{n_{i+1}+1}\right), \quad t \in I_i,$$

where $\gamma_{i,m}(x,t_{i+1},c_i)$ is a coefficient dependent on $x$, $t_{i+1}$, and $c_i = V_{i,0}(x,t)$. The expression (18) demonstrates that the $n_{i+1}$th-order PTVIM has an error of order $(\Delta t)^{n_{i+1}+1}$ per step, whereas the total accumulated error is of order $(\Delta t)^{n_{i+1}+1}$.

**Remark 3.1**

For analyzing the accuracy of VIM (13), truncated VIM (15), or PTVIM (16), if the exact value function or consequently, the exact control law, $u^*(x,t)$, is available, then it is straightforward to calculate the absolute error function as follows:

$$E_n(t) = \|u_n(x,t) - u^*(x,t)\|_1, \quad E_{n,i}(t) = \|u_{n,i}(x,t) - u^*(x,t)\|_1,$$

for some $x \in \mathbb{R}^n$. Otherwise, the following error function for HJB equation (11) can be calculated:

$$E_n(t) = |\mathcal{L}[V_n(x,t)] + \mathcal{N}[V_n(x,t)]|, \quad E_{n,i}(t) = |\mathcal{L}[V_{n,i}(x,t)] + \mathcal{N}[V_{n,i}(x,t)]|,$$

where $n$ and $i$ are sufficiently large number of iterations and subintervals, respectively, $\mathcal{L}$ and $\mathcal{N}$ are defined as (9) and (10), and $x(.)$ is approximated by solving the ODE system (1), when $u(x,t) = u_n(x,t)$ or $u(x,t) = u_{n,i}(x,t)$, for $t \in [t_i, t_{i+1}]$, $i = 0, 1, \ldots, N - 1$.

### 4. ILLUSTRATIVE EXAMPLES

The following five examples are given to illustrate the validity and efficiency of the proposed method. The codes are developed using symbolic computation software MAPLE 13 (Waterloo Maple Inc., Waterloo, Canada), and the calculations are implemented on a machine with Intel Core 2 Duo Processor 2.53 Ghz and 4 GB RAM.

**Example 4.1**

Consider the following purely mathematical optimal control problem:

minimize $J = x^2(t_f) + \int_0^{t_f} u^2(t) \, dt$

subject to: $\dot{x} = x + u$

$x(0) = x_0$. 

From (8) and (5), the corresponding HJB equation with the boundary conditions are given by

\[-V_t = xV_x - \frac{1}{4}V_x^2, \quad V(x(t_f), t_f) = x^2(t_f).\]

where \(V_x = \frac{\partial V}{\partial x}\) and \(V_t = \frac{\partial V}{\partial t}\), and the optimal control law is easily obtained by

\[u^* = -\frac{1}{2}V_x.\]  \hfill (19)

The exact solution of the foregoing HJB equation is

\[V(x, t) = \frac{2x^2}{1 + e^{2(t-t_f)}},\]  \hfill (20)

which implies the optimal feedback control law to be \(u^* = -\frac{2x}{1 + e^{2(t-t_f)}}\) [23, Chapter 19]. We choose the linear and nonlinear operators \(L[V(x, t)] = V_t\) and \(N[V(x, t)] = xV_x - \frac{1}{4}V_x^2\), respectively. Suppose in addition that \(t_f = 1\) and \(x_0 = 1\). According to VIM (13), taking \(V_0(x, t) = x^2\), the approximate solutions are calculated successively as follows:

\[V_1(x, t) = [1 - (t - 1)]x^2,\]
\[V_2(x, t) = \left[1 - (t - 1) + \frac{1}{3}(t - 1)^3\right]x^2,\]
\[V_3(x, t) = \left[1 - (t - 1) + \frac{1}{3}(t - 1)^3 - \frac{2}{15}(t - 1)^5 + O((t - 1)^7)\right]x^2,\]
\[V_4(x, t) = \left[1 - (t - 1) + \frac{1}{3}(t - 1)^3 - \frac{2}{15}(t - 1)^5 + \frac{17}{315}(t - 1)^7 + O((t - 1)^9)\right]x^2,\]

\[\vdots\]

or by truncated VIM (15), we have

\[V_1(x, t) = V_2(x, t) = [1 - (t - 1)]x^2,\]
\[V_3(x, t) = V_4(x, t) = \left[1 - (t - 1) + \frac{1}{3}(t - 1)^3\right]x^2,\]
\[V_5(x, t) = V_6(x, t) = \left[1 - (t - 1) + \frac{1}{3}(t - 1)^3 - \frac{2}{15}(t - 1)^5\right]x^2,\]
\[V_7(x, t) = V_8(x, t) = \left[1 - (t - 1) + \frac{1}{3}(t - 1)^3 - \frac{2}{15}(t - 1)^5 + \frac{17}{315}(t - 1)^7\right]x^2,\]

\[\vdots\]

which both are the Taylor expansions of the exact value function (20) around \(t = t_f = 1\). In addition, the optimal control law can be easily obtained by (19). The comparison of the truncated VIM approximations and the exact solution is depicted in Figure 1, when \(x = 1\). Clearly, the larger number of iterations lead to the higher accuracy of obtained approximations. Besides the good accuracy of VIM approximations in the first few steps, because of the repeated and time-consuming calculations, the CPU time grows as iterations increase. The CPU time and the absolute error of VIM and truncated VIM for this example are given in Table I. Also, Figure 2 shows the absolute error function in Remark 3.1, for \(n = 10, 40\) iterations of truncated VIM. It is noteworthy that because of the small region of \(t\), the use of PTVIM is not reasonable. Nevertheless, one can use PTVIM to vanish the absolute error of approximations as we did in the next example.
Table I. Comparison of the basic and truncated variational iteration method (VIM) for Example 4.1, based on CPU time and absolute error function.

<table>
<thead>
<tr>
<th>N (Iteration)</th>
<th>VIM</th>
<th>Truncated VIM</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>max $E_N(t)$</td>
<td>CPU time (s)</td>
</tr>
<tr>
<td>2</td>
<td>9.49275E−02</td>
<td>0.015</td>
</tr>
<tr>
<td>4</td>
<td>4.42789E−03</td>
<td>0.016</td>
</tr>
<tr>
<td>6</td>
<td>9.35700E−05</td>
<td>0.031</td>
</tr>
<tr>
<td>8</td>
<td>1.10700E−06</td>
<td>0.067</td>
</tr>
<tr>
<td>10</td>
<td>8.36677E−09</td>
<td>87.033</td>
</tr>
<tr>
<td>20</td>
<td>–</td>
<td>&gt;1800</td>
</tr>
<tr>
<td>40</td>
<td>–</td>
<td>–</td>
</tr>
<tr>
<td>60</td>
<td>–</td>
<td>–</td>
</tr>
<tr>
<td>80</td>
<td>–</td>
<td>–</td>
</tr>
</tbody>
</table>

Figure 1. Comparison of different iterations for control function $u(t)$.

Figure 2. Absolute error of control function $u(t)$ for $n = 10, 40$ iterations.
Example 4.2
Consider a second-order dynamical system as follows [18]:
\[
\begin{align*}
\dot{x}(t) &= \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} x(t) + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u(t), \\
x(0) &= [1, 1]^T,
\end{align*}
\]
with the cost functional:
\[
J = \frac{1}{2} \int_0^{\frac{\pi}{2}} (x^T(t) \begin{pmatrix} 0 & 0 \\ 0 & 4 \end{pmatrix} x(t) + u^2(t))dt.
\]
The HJB equation and the optimal control law are as follows:
\[
V_t + x_1 V_{x_2} + 2x_2^2 - \frac{1}{2} V_{x_1}^2 = 0,
\]
\[
u^* = -V_{x_1}, \tag{21}
\]
where \( V_{x_i} = \frac{\partial V}{\partial x_i}, i = 1, 2 \). The exact solution of the optimal control is \( u^*(t) = -k(t)x(t) \), where
\[
k(t) = \left( \frac{\sinh(\pi - 2t) - \sin(\pi - 2t)}{\cosh^2(\pi/2 - t) + \cos^2(\pi/2 - t)}, \frac{\cosh(\pi - 2t) - \cos(\pi - 2t)}{\cosh^2(\pi/2 - t) + \cos^2(\pi/2 - t)} \right).
\]
Applying the formula (15) with \( V_0(x,t) = 0 \) results in
\[
\begin{align*}
V_1(x,t) &= (-2t + \pi)x_2^2, \\
V_2(x,t) &= \left(2t^2 - 2\pi t + \frac{1}{2} \pi^2\right)x_1 x_2 + (-2t + \pi)x_2^2, \\
V_3(x,t) &= \left(-\frac{2}{3} t^3 + \pi t^2 - \frac{1}{2} \pi^2 t + \frac{1}{12} \pi^3\right)x_1^2 + \left(2t^2 - 2\pi t + \frac{1}{2} \pi^2\right)x_1 x_2 + (-2t + \pi)x_2^2 \\
&\quad \vdots
\end{align*}
\]
Also, by (16) for \( V_{0,i}(x,t) = 0, N = 2 \) subintervals and different \( n_i \)'s, \( i = 1, 2 \), we obtain the following:
\begin{itemize}
\item \( n_i = 1, i = 1, 2 \)
\[
V_{1,1}(x,t) = (-2t + \pi)x_2^2, \ t \in \left[0, \frac{\pi}{4}\right]
\]
\[
V_{2,1}(x,t) = \left(-\pi t + \frac{1}{4} \pi^2\right)x_1 x_2 + (-2t + \pi)x_2^2, \ t \in \left[\frac{\pi}{4}, \frac{\pi}{2}\right]
\]
\item \( n_i = 2, i = 1, 2 \)
\[
V_{1,2}(x,t) = \left(2t^2 - 2\pi t + \frac{1}{2} \pi^2\right)x_1 x_2 + (-2t + \pi)x_2^2, \ t \in \left[0, \frac{\pi}{4}\right]
\]
\[
V_{2,2}(x,t) = \left(\frac{1}{2} \pi t^2 - \frac{3}{8} \pi^2 t + \frac{1}{16} \pi^3\right)x_1^2 \\
&\quad + \left(-\frac{3}{128} \pi^4 t^2 + 2t^2 - 2\pi t + \frac{3}{256} \pi^5 t - \frac{3}{2048} \pi^6 + \frac{1}{2} \pi^2\right)x_1 x_2 \\
&\quad + \left(-\frac{1}{16} \pi^3 t^2 - 2t + \frac{5}{128} \pi^4 t - \frac{3}{512} \pi^5 + \pi\right)x_2^2, \ t \in \left[\frac{\pi}{4}, \frac{\pi}{2}\right]
\]
\end{itemize}
and so on. From (21), the optimal control law can be easily determined. The absolute error function of approximated $k(t)$ coordinates for $N = 5$ and $n_i = 10, 20, i = 1, 2, \ldots, N$, are plotted in Figure 3. Furthermore, Figure 4 depicts this absolute error function for $N = 10$ and $n_i = 20, i = 1, 2, \ldots, N$, for which the error is vanishing for larger subintervals $N$. This problem was also solved by Yousefi et al. [18], where the basic VIM was implemented to solve the PMP and achieve a feedback optimal control law by some transformations. In contrast to their method, Table II shows the better accuracy of our proposed method.

**Example 4.3**

Consider the following nonlinear example with a quadratic cost functional,

\[
\begin{align*}
\text{minimize} \quad J &= \int_0^1 \left( x^2 + u^2 \right) dt \\
\text{subject to:} \quad \dot{x} &= -x - 2x^2 - 0.5x^3 + u \\
x(0) &= 0.5.
\end{align*}
\]
The HJB equation and the optimal control law become

\[ V_t + (-x - 2x^2 - 0.5x^3) V_x + x^2 - \frac{1}{4} V_x^2 = 0, \]
\[ u^* = -\frac{1}{2} V_x. \]

Applying the PTVIM formula (16), for \( V_{0,0}(x, t) = 0, N = 4 \) and \( n_i = 1, i = 0, 1, 2, 3 \), yields the following piecewise optimal value function:

\[
V_{1,1}(x, t) = (0.12889 \times 10^{-3} t - 0.16111 \times 10^{-4}) x^{10} + (0.15106 \times 10^{-2} t - 0.18883 \times 10^{-3}) x^9 \\
+ (0.016350 t - 0.20437 \times 10^{-2}) x^8 + (0.11044 t - 0.013804)x^7 \\
+ (0.29256 t - 0.032785)x^6 + (0.046735 t + 0.020770)x^5 \\
+ (-0.46251 t + 0.070372)x^4 + (0.57944 t - 0.21480)x^3 \\
+ (-0.35719 t + 0.32637)x^2, \quad t \in \left[0, \frac{1}{8}\right]
\]

\[
V_{2,1}(x, t) = (-0.030273 t + 0.756841 \times 10^{-2}) x^6 + (-0.21289 t + 0.053223)x^5 \\
+ (-0.22547 t + 0.040743)x^4 + (0.63901 t - 0.22225)x^3 \\
+ (-0.51942 t + 0.34665)x^2, \quad t \in \left[\frac{1}{8}, \frac{1}{4}\right]
\]

\[
V_{3,1}(x, t) = (0.12500 t - 0.046875) x^4 + (0.50000 t - 0.18750) x^3 \\
+ (-0.73438 t + 0.40039)x^2, \quad t \in \left[\frac{1}{4}, \frac{3}{8}\right]
\]

\[
V_{4,1}(x, t) = (-1.00000 t + 0.50000)x^2, \quad t \in \left[\frac{3}{8}, \frac{7}{8}\right].
\]

Continuing the PTVIM successive formula, one can obtain more accurate results. The results are shown for \( N = 4 \) and \( n_i = 3, i = 0, 1, 2, 3 \), in Figure 5. Precisely, Figure 5 shows the approximated state function, \( x(t) \), compared with the collocation method [24], and the absolute error function of HJB equation as in Remark 3.1.
Example 4.4
Consider the following optimal control problem for the Van Der Pol oscillator [25]:

\[
\begin{align*}
\text{minimize} & \quad J = \frac{1}{2} \int_0^2 (x_1^2 + x_2^2 + u^2) \, dt \\
\text{subject to:} & \quad \dot{x}_1 = x_2 \\
& \quad \dot{x}_2 = -x_1 + x_2 (1 - x_1^2) + u \\
& \quad x_1(0) = 1, \quad x_2(0) = 0.
\end{align*}
\]

The HJB equation and the optimal control law become

\[
\begin{align*}
V_t + x_2 V_{x_1} + (-x_1 + x_2 (1 - x_1^2)) V_{x_2} + \frac{1}{2} (x_1^2 + x_2^2) - \frac{1}{2} V_{x_2}^2 &= 0, \quad (22) \\
\textit{u}^* &= -V_{x_2}. \quad (23)
\end{align*}
\]

By using the PTVIM formula (16), for \( V_{0,0}(x,t) = 0 \), \( N = 4 \) and \( n_i = 1, i = 1, 2, 3, 4 \), the following results are calculated:

\[
\begin{align*}
V_{1,1}(x,t) &= \left( \frac{9}{128} t - \frac{9}{256} \right) x_1^8 x_2^2 + \left( \frac{3}{32} t - \frac{3}{64} \right) x_1^7 x_2 + \left( \frac{33}{128} t + \frac{33}{256} \right) x_1^6 + \left( \frac{13}{64} t - \frac{13}{128} \right) x_1^5 x_2^2 \\
&+ \left( \frac{1}{4} t - \frac{1}{8} \right) x_2^4 + \left( \frac{15}{64} t + \frac{15}{128} \right) x_2^3 + \left( \frac{631}{512} t + \frac{887}{1024} \right) x_2^2 + \left( \frac{343}{1024} t + \frac{1111}{2048} \right) x_2^1 x_1 \\
&+ \left( \frac{9}{32} t - \frac{9}{64} \right) x_2^4 + \left( \frac{-2573}{2048} t - \frac{883}{4096} \right) x_2^3 + \left( \frac{-2573}{1024} t + \frac{1111}{2048} \right) x_2^2 + \left( \frac{1265}{32768} t + \frac{33376}{29457} \right) x_2^1 x_1, \quad t \in \left[ 0, \frac{1}{2} \right]
\end{align*}
\]
By continuing the successive approximations, more accurate results can be obtained. Figure 6 demonstrates the accuracy of results for $N = 4$ and $n_1 = n_4 = 2$ and $n_2 = n_3 = 3$ iterations, in comparison with differential transformation (DT) method [25] and collocation method used by the dsolve toolbox of MAPLE. The optimal feedback control law can be obtained by (23).

**Example 4.5**

Consider the two-dimensional nonlinear composite system described by

\[
\begin{align*}
\dot{x}_1 &= x_1 - x_1^3 + x_2^2 + u_1 \\
\dot{x}_2 &= -x_2 + x_2 (x_1 + x_2^2) + u_2 \\
x_1(0) &= 0, \quad x_2(0) = 0.8.
\end{align*}
\]

The quadratic cost functional to be minimized is given by

\[
J = \frac{1}{2} \int_0^1 (x_1^2 + x_2^2 + u_1^2 + u_2^2) \, dt.
\]

The HJB equation and the optimal control law become

\[
\begin{align*}
V_t + x_2 V_{x_1} + (x_1 - x_1^3 + x_2^2) V_{x_1} + (-x_2 + x_1 x_2 + x_2^3) V_{x_2} + & \frac{1}{2} (x_1^2 + x_2^2 - V_{x_1}^2 - V_{x_2}^2) = 0, \\
u_1^* &= -V_{x_1}, \quad u_2^* = -V_{x_2}.
\end{align*}
\]

Figure 6. State functions $x_1(t)$ and $x_2(t)$ of Example 4.4.
By using the PTVIM formula (16), for \( V_{0,0}(x, t) = 0, N = 2 \) and different \( n_i \)'s, \( i = 1, 2 \), we obtain the following:

- \( n_i = 1, i = 1, 2 \)

\[
V_{1,1}(x, t) = \left( \frac{1}{2} t - \frac{1}{4} \right) x_1^4 + \left( -\frac{1}{2} t + \frac{1}{4} \right) x_2^4 + \left( -t + \frac{1}{2} \right) x_1 x_2^2 + \left( \frac{7}{8} t - \frac{11}{16} \right) x_1^2
\]
\[
+ \left( \frac{1}{8} t + \frac{3}{16} \right) x_2^2, \quad t \in \left[ 0, \frac{1}{2} \right),
\]

\[
V_{2,1}(x, t) = \left( -\frac{1}{2} t + \frac{1}{2} \right) x_1^2 + \left( \frac{1}{2} t + \frac{1}{2} \right) x_2^2, \quad t \in \left[ \frac{1}{2}, 1 \right]
\]

- \( n_i = 2, i = 1, 2 \), we have

\[
V_{1,2}(x, t) = -\frac{9}{16} \left( t - \frac{1}{2} \right)^2 x_1^8 + \frac{9}{16} \left( t - \frac{1}{2} \right)^2 x_2^8 + \frac{21}{16} \left( t - \frac{1}{2} \right)^2 x_1^5 x_2^3
\]
\[
+ \frac{37}{32} \left( t - \frac{1}{2} \right) \left( t - \frac{67}{74} \right) x_1^6 - \frac{41}{32} \left( t - \frac{17}{82} \right) \left( t - \frac{1}{2} \right) x_2^6
\]
\[
+ \left( -\frac{2}{3} \left( t - \frac{1}{2} \right) x_1^2 - \frac{59}{64} \left( t - \frac{46159}{68364} \right) \left( t - \frac{27478}{21569} \right) \right) x_1^4
\]
\[
+ \left( \frac{85}{128} t^2 - \frac{65}{128} t + \frac{109}{512} \right) x_2^4 + \left( -\frac{2}{3} \left( t - \frac{1}{2} \right)^2 x_1^4 - \frac{11}{16} \left( t - \frac{1}{2} \right) \left( t - \frac{31}{22} x_2^2 \right) \right) x_1^3
\]
\[
+ \left( \frac{15}{64} \left( t - \frac{1}{2} \right)^2 x_2^4 - \frac{5}{32} \left( t + \frac{19}{10} \right) \left( t - \frac{1}{2} \right) x_2^2
\]
\[
+ \frac{31}{128} \left( t - \frac{58359}{62467} \right) \left( t - \frac{241982}{59517} \right) \right) x_1^2
\]
\[
- \frac{35}{128} \left( \frac{24583}{35857} + \frac{11494}{12979} \right) x_1^2
\]
\[
+ \left( \frac{21}{16} \left( t - \frac{1}{2} \right)^2 x_1^6 - \frac{23}{16} \left( t + \frac{1}{46} \right) \left( t - \frac{1}{2} \right) x_2^4
\]
\[
+ \left( \frac{11}{32} t^2 - \frac{25}{32} t + \frac{71}{128} x_2^2 \right) \right) x_1, \quad t \in \left[ 0, \frac{1}{2} \right)
\]

\[
V_{2,2}(x, t) = -\frac{1}{2} (t - 1)^2 x_1^4 + \frac{1}{2} (t - 1)^2 x_2^4 - \frac{1}{2} (t - 1)^2 x_1 x_2^2 + \frac{1}{2} (t - 1)(t - 2) x_1 x_2
\]
\[
+ \frac{1}{2} (t - 1)(t - 2) x_2^2, \quad t \in \left[ \frac{1}{2}, 1 \right]
\]

Figure 7 gives a comparison between the present PVIM results for \( N = 2 \) with \( n_i = 2, i = 1, 2 \) and the results of collocation method generated by the dsolve toolbox of MAPLE for \( x_1(t), x_2(t) \) in time interval \( [0, 1] \). Therefore, in view of the results, the present method is quite effective.
5. CONCLUSIONS

In this paper, a successive approximation approach has been used to generate the suboptimal control law for a wide class of optimal control problems. The proposed method consists of solving the nonlinear HJB equation employing a recent modification of VIM, called PTVIM. The simulation results show the validity, accuracy, and efficiency of the proposed method. Future works are focused on introducing an optimal VIM to increase the accuracy with a few number of iterations.

REFERENCES


