A Linearly Implicit Conservative Scheme for the Coupled Nonlinear Schrödinger Equation

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Abstract

The coupled nonlinear Schrödinger equation models several interesting physical phenomena. It presents a model equation for optical fiber with linear birefringence. In this paper we present a finite difference method to solve this equation. This method is second order in space and conserves the energy exactly. Many numerical tests have been conducted and the experiments have shown that this method is quite accurate and describe the interaction picture clearly.

Keywords: Coupled nonlinear Schrödinger equation, linearly implicit scheme, finite difference method, solitons interaction.

1 Introduction

The propagation of pulses with equal mean frequencies in birefringent nonlinear fiber is governed by the coupled nonlinear Schrödinger equation\([1]\),

\[
i \frac{\partial \Psi_1}{\partial t} + \frac{\partial^2 \Psi_1}{\partial^2 x} + (|\Psi_1|^2 + e|\Psi_2|^2) \Psi_1 = 0
\]
\[
\frac{\partial \Psi_2}{\partial t} + \frac{\partial^2 \Psi_2}{\partial^2 x} + (e|\Psi_1|^2 + |\Psi_2|^2)\Psi_2 = 0
\]  
(1)

where \( \Psi_1 \) and \( \Psi_2 \) are the wave amplitudes in two polarizations.

Following the discussion of Wadati[1], we derive the solution of the system in (1) which has the following form

\[
\Psi_1(x, t) = \sqrt{\frac{2\alpha}{1 + e}} \sec h(\sqrt{\alpha}(x - 2vt)) \\
\exp i\{vx - [v^2 - \alpha]t\} \\
= \pm \Psi_2(x, t)
\]  
(2)

Two conserved quantity of equation will be considered

\[
\int_{-\infty}^{\infty} |\Psi_1|^2 dx
\]  
(3)

and

\[
\int_{-\infty}^{\infty} \sum_{j=1}^{2} \bar{\Psi}_j \Psi_j - \frac{1}{2}|\Psi|^4 - e|\Psi|^2|\Psi|^2 - \frac{1}{2}|\Psi|^4]dx
\]  
(4)

To prove Eq.(3), we multiply Eq(1) by \( \bar{\Psi}_1 \) and its complex conjugate by \( \Psi_1 \) and subtract the latter from the former to obtain

\[
i \frac{\partial}{\partial t}(\Psi_1 \bar{\Psi}_1) + \frac{\partial}{\partial x}(\Psi_{1x} \bar{\Psi}_1 - \Psi_{1x} \Psi_1) = 0
\]  
(5)

Integration with respect to \( x \) gives

\[
i \frac{\partial}{\partial t} \int_{-\infty}^{\infty} |\Psi_1|^2 dx = 0
\]  
(6)

and this completes the proof.

If \( e = 1 \), equation (1) is just a version of Manakov’s equation, which is integrable using spectral transform method. In particular solitons of one polarization should pass through pulses of the opposite polarization without creating shadows, i.e. the collision is elastic. Many numerical methods have been developed for solving the nonlinear Schrödinger equation, but there is little numerical work on the coupled nonlinear Schrödinger equation [2,3].
2 Numerical Method

We assume that, in the time interval \(0 \leq t \leq T_{\text{max}}\) under consideration, the solution of the system given in (1) is negligibly small outside the interval \(x_L \leq x \leq x_R\). Therefore in our numerical study of this system we replace the system in (1) by

\[
\begin{align*}
  i \frac{\partial \Psi_1}{\partial t} + \frac{\partial^2 \Psi_1}{\partial x^2} + (|\Psi_1|^2 + e |\Psi_2|^2) \Psi_1 &= 0 \\
  \frac{\partial \Psi_2}{\partial t} + \frac{\partial^2 \Psi_2}{\partial x^2} + (e |\Psi_1|^2 + |\Psi_2|^2) \Psi_2 &= 0
\end{align*}
\]

with the following initial conditions

\[
\Psi_1(x,0) = g_1(x), \quad \Psi_2(x,0) = g_2(x)
\]

and boundary conditions

\[
\frac{\partial \Psi_1(x,t)}{\partial x} = \frac{\partial \Psi_2(x,t)}{\partial x} = 0 \quad \text{at } x = x_L, x_R
\]

The proposed extension of the scheme given in [4] can be written as follows:

\[
\begin{align*}
  i \frac{\Psi_{1,m}^{n+1} - \Psi_{1,m}^{n-1}}{2k} + \delta_x^2 \left( \frac{\Psi_{1,m}^{n+1} + \Psi_{1,m}^{n-1}}{2h^2} \right) + f_1(\Psi_{1,m}^n, \Psi_{2,m}^n) \left( \frac{\Psi_{1,m}^{n+1} + \Psi_{1,m}^{n-1}}{2} \right) &= 0 \quad (8) \\
  i \frac{\Psi_{2,m}^{n+1} - \Psi_{2,m}^{n-1}}{2k} + \delta_x^2 \left( \frac{\Psi_{2,m}^{n+1} + \Psi_{2,m}^{n-1}}{2h^2} \right) + f_2(\Psi_{1,m}^n, \Psi_{2,m}^n) \left( \frac{\Psi_{2,m}^{n+1} + \Psi_{2,m}^{n-1}}{2} \right) &= 0 \quad (9)
\end{align*}
\]

where \(\delta_x^2 \Psi_m^m = \frac{1}{h^2}(\Psi_{m+1}^n - 2\Psi_m^n + \Psi_{m-1}^n)\).

The most striking property of the scheme is that it has a discrete analogous of (4) that is conserved exactly and this can be proved in the following manner.

We multiply Eq. (8) by \((\bar{\Psi}_{1,m}^{n+1} + \bar{\Psi}_{1,m}^{n-1})\) and summing over \(m\) and keeping only the imaginary terms, one gets

\[
\sum |\Psi_{1,m}^{n+1}|^2 + |\Psi_{1,m}^n|^2 = \text{constant} \quad (10)
\]
For our numerical work we write the proposed scheme in Eqs. (8) and (9) in a matrix vector form by assuming

\[ \Psi_1(x,t) = u_1 + iu_2 \]
\[ \Psi_2(x,t) = u_3 + iu_4 \]  

where \( (u_i, i = 1, \ldots, 4) \) are real functions [3]. To get a finite difference scheme

\[
\frac{(U_{m+1}^{n+1} - U_m^{n-1})}{2k} + B\delta_x^2 \left( \frac{(U_m^{n+1} + U_m^{n-1})}{2h^2} \right) + F(U_m^n) \left( \frac{U_m^{n+1} + U_m^{n-1}}{2} \right) = 0
\]

where

\[ u = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}, \quad \quad B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}, \]

\[ F(u) = \begin{bmatrix} 0 & z_1 & 0 & 0 \\ -z_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & z_2 \\ 0 & 0 & -z_2 & 0 \end{bmatrix} \]

and \( z_1 = u_1^2 + u_2^2 + e(u_3^2 + u_4^2) \) and \( z_2 = e(u_1^2 + u_2^2) + u_3^2 + u_4^2 \).

The scheme in (12) is a linear block tridiagonal system and it can be easily solved by a Crout’s method.

Using Taylor series expansion, the scheme can be easily shown to be of second order in space and time directions and the principal part of the truncation error is

\[
\frac{k^2}{6} \frac{\partial^3 \Psi_j}{\partial t^3} + \frac{k^2}{12} \frac{\partial^4 \Psi}{\partial x^4} + \frac{k^2}{4} \frac{\partial^4 \Psi_j}{\partial x^2 \partial t^2} + \frac{k^2}{2} f_j(\Psi, \Psi) \frac{\partial^2 \Psi}{\partial t^2} \quad \text{for } j = 1, 2
\]

and hence the scheme is consistent since the principal part of the truncation error vanishes as \( h, k \to 0 \).

Using von Neumann stability analysis the scheme is unconditionally stable. The proposed scheme (12) is a finite difference three level global linearly
implicit scheme. This means that at each discrete time level we only need to solve a linear block tridiagonal system only, and as a consequence the proposed scheme is faster and simpler than the nonlinearly implicit ones previously proposed in [2, 3]. The scheme is not self-starting, in the sense that the function values \(u_m^1\) have to provided by other schemes such as second order like implicit Crank-Nicolson scheme.

3 Numerical Results

To test the proposed scheme for the efficiency and robustness we use the infinity norm to calculate the error and the conserved quantities so we consider the following tests

3.1 Single soliton

We pick the initial condition from the solution given where we have select the following parameters \(h = 0.1, k = 0.01, c = 1, v = 1.0, \alpha = 1\)

the results shows that the scheme conserved the first quantity exactly and the second quantity is nearly conserved (see Table 1 and Figure 1).

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3.2 Interaction of two solitons

To study the interaction of two solitons we will take the initial condition [5]

\[
\Psi_1(x, t) = \sqrt{2}r_1 \sec h(r_2x + x_{10}) \exp\{iv_1x\}
\]

\[
\Psi_2(x, t) = \sqrt{2}r_2 \sec h(r_2x + x_{10}) \exp\{iv_2x\}
\]

we choose \(v_1 = -v_2 = \frac{v}{4}\).
In this case we take the parameters $r_1 = 1.2, r_2 = 1, x_{10} = -x_{20} = 20, e = 1, v = 1, h = 0.1, k = 0.1, t = 0, 4, 8, ..., 80$. The interaction scenario is given in figure 2 which shows the interaction is elastic and this is not strange since $e = 1$, gives Manakov equation which is completely integrable, see Figure 2.

As a second test we select the parameters $h = 0.1, k = 0.1, v = 0.4, e = \frac{2}{3}$, we observe that collision takes place at $t \approx 40$. During collision, the velocity of right moving soliton steadily decreases, and becomes negative when it emerges from the collision. This means that this soliton is reflected back by collision. The same happens to the other soliton it initially moves to the left, but turns around after the collision with daughter wave created in the direction of $\Psi_1$, see Figs. 3 and 4.

4 Concluding Remarks

In this paper we have developed a numerical linearly implicit scheme for solving the coupled nonlinear Schrödinger equation, this scheme is very easy to apply we need only one solution of a block tridiagonal system and need for any iteration like the nonlinear methods like nonlinear Crank and Douglas methods [2,3]. The scheme conserves the mass exactly ruling out any possibility of blowing up of the numerical solution.
Figure 2: Interaction of two solitons with parameters $h = 0.1, k = 0.1, e = 1, v = 1, r = 1.2, r = 1, x_{10} = -x_{20} = 20$

Figure 3: The modulus of $\Psi_2$ using parameters $h = 0.1, k = 0.1, e = \frac{2}{3}, v = 0.4$
Figure 4: The modulus of $\Psi_1$ with parameters $h = 0.1, k = 0.1, e = \frac{2}{3}, v = 0.4$

References


