Almost lacunary statistical and strongly almost lacunary convergence of sequences of fuzzy numbers

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The purpose of this paper is to introduce the concepts of almost lacunary statistical convergence and strongly almost lacunary convergence of sequences of fuzzy numbers. We give some relations related to these concepts. We establish some connections between strongly almost lacunary convergence and almost lacunary statistical convergence of sequences of fuzzy numbers. It is also shown that if a sequence of fuzzy numbers is strongly almost lacunary convergent with respect to an Orlicz function then it is almost lacunary statistical convergent.

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1. Introduction

The concepts of fuzzy sets and fuzzy set operations were first introduced by Zadeh [33]. Subsequently several authors have discussed various aspects of the theory and applications of fuzzy sets, such as fuzzy topological spaces, similarity relations and fuzzy orderings, fuzzy measures of fuzzy events and fuzzy mathematical programming. In particular, the concept of fuzzy topology has very important applications in quantum particle physics, especially in the connections with both string and $\varepsilon(\infty)$ theory which were given and studied by El Naschie [7]. Recently, Saadati and Park [24] have introduced the idea of intuitionistic fuzzy normed space. Matloka [20] introduced bounded and convergent sequences of fuzzy numbers, studied some of their properties, and showed that every convergent sequence of fuzzy numbers is bounded. In addition, sequences of fuzzy numbers have been discussed by Aytar and Pehlivan [1], Başarır and Mursaleen [22], Nuray [23], Talo and Basar [29] and many others.

The idea of statistical convergence was introduced independently by Fast [8] and Schoenberg [28]. Over the years, and under different names, statistical convergence has been discussed in the theory of Fourier analysis, ergodic theory and number theory. Later on it was further investigated from the sequence space point of view and linked with summability theory by Connor [3], Fridy [10], Šalát [26], Tripathy [30] and many others. In recent years, generalizations of statistical convergence have appeared in the study of strong integral summability and the structure of ideals of bounded continuous functions on locally compact spaces. Statistical convergence and its generalizations are also connected with subsets of the Stone-Čech compactification of the natural numbers. Moreover, statistical convergence is closely related to the concept of convergence in probability [4,5].

The study of Orlicz sequence spaces was initiated with a certain specific purpose in Banach space theory. Indeed, Lindberg [15] got interested in Orlicz spaces in connection with finding Banach spaces with symmetric Schauder bases having complementary subspaces isomorphic to $c_0$ or $\ell_p$ ($1 \leq p < \infty$). Subsequently Lindenstrauss and Tzafriri [16] investigated...
Orlicz sequence spaces in more detail, and they proved that every Orlicz sequence space \( \ell_M \) contains a subspace isomorphic to \( \ell_p \) \((1 \leq p < \infty)\).

Recently, Parashar and Choudhary [25] have introduced and discussed some properties of the four sequence spaces defined by using an Orlicz function \( M \), which generalized the well-known Orlicz sequence space \( \ell_M \) and strongly summable sequence spaces \([c, 1, p], [c, 1, p_0] \) and \([c, 1, p_{\infty}] \). Later on, Mursaleen et al. [22], Tripathy and Mahanta [31] used the idea of an Orlicz function to construct some spaces of complex sequences. The Orlicz sequence spaces are the special cases of Orlicz spaces studied in [13]. Orlicz spaces find a number of useful applications in the theory of nonlinear integral equations. Whereas the Orlicz sequence spaces are the generalizations of \( \ell_p \)-spaces, the \( \ell_p \)-spaces find themselves enveloped in Orlicz spaces [12].

The existing literature on statistical convergence appears to have been restricted to real or complex sequences, but Ayar and Pehlivan [1], Tripathy and Dutta [32], Başarır and Mursaleen [22,21] and Nuray [23] extended the idea to apply to sequences of fuzzy numbers. In the present paper, using an Orlicz function, we introduce and examine the concepts of almost lacunary statistical convergence and strongly almost lacunary convergence of sequences of fuzzy numbers. In Section 2 we give a brief overview about statistical convergence, fuzzy numbers, Orlicz function and, using the sequence \( u \). Then from (i)–(iv), it follows that

\[
\text{if } u \in \ell_M, \text{ then } u \}\]

\[
\text{provided the limit exists, where } \chi_k \text{ is the characteristic function of } E. \text{ It is clear that any finite subset of } \mathbb{N} \text{ has zero natural density and } \delta(E^c) = 1 - \delta(E). \]

A sequence \((x_n)\) is said to be statistically convergent to \( L \) if for every \( \varepsilon > 0 \), \( \delta \{ k \in \mathbb{N} : |x_k - L| \geq \varepsilon \} = 0 \). In this case we write \( S - \lim x_n = L \).

Fuzzy sets are considered with respect to a nonempty base set \( X \) of elements of interest. The essential idea is that each element \( x \in X \) is assigned a membership grade \( u(x) \) taking values in \([0, 1]\), with \( u(x) = 0 \) corresponding to nonmembership, \( 0 < u(x) < 1 \) to partial membership, and \( u(x) = 1 \) to full membership. According to Zadeh [33] a fuzzy subset of \( X \) is a nonempty subset \( \{(x, u(x)) : x \in X\} \) of \( X \times [0, 1] \) for some function \( u : X \rightarrow [0, 1] \). The function \( u \) itself is often used for the fuzzy set.

Let \( C(\mathbb{R}^n) \) denote the family of all nonempty, compact, convex subsets of \( \mathbb{R}^n \). If \( \alpha, \beta \in \mathbb{R} \) and \( A, B \in C(\mathbb{R}^n) \), then

\[
\alpha(A + B) = \alpha A + \alpha B, \quad (\alpha \beta)A = \alpha (\beta A), \quad 1A = A
\]

and if \( \alpha, \beta \geq 0 \), then \( (\alpha + \beta)A = \alpha A + \beta A \). The distance between \( A \) and \( B \) is defined by the Hausdorff metric

\[
\delta(A, B) = \max \left\{ \sup_{a \in A} \inf_{b \in B} \| a - b \|, \sup_{b \in B} \inf_{a \in A} \| a - b \| \right\},
\]

where \( \| . \| \) denotes the usual Euclidean norm in \( \mathbb{R}^n \). It is well known that \((C(\mathbb{R}^n), \delta_\infty)\) is a complete metric space.

Denote

\[
L(\mathbb{R}^n) = \{ u : \mathbb{R}^n \rightarrow [0, 1] \mid u \text{ satisfies (i)–(iv) below}\},
\]

where

(i) \( u \) is normal, that is, there exists an \( x_0 \in \mathbb{R}^n \) such that \( u(x_0) = 1 \);

(ii) \( u \) is fuzzy convex, that is, for \( x, y \in \mathbb{R}^n \) and \( 0 \leq \lambda \leq 1 \), \( u(\lambda x + (1 - \lambda)y) \geq \min[u(x), u(y)] \);

(iii) \( u \) is upper semicontinuous;

(iv) \( u \) is upper \( \alpha \)-semicontinuous.

If \( u \in L(\mathbb{R}^n) \), then \( u \) is called a fuzzy number, and \( L(\mathbb{R}^n) \) is said to be a fuzzy number space. For \( 0 < \alpha \leq 1 \), the \( \alpha \)-level set \([u]_\alpha \) is defined by

\[
[u]_\alpha = \{ x \in \mathbb{R}^n : u(x) \geq \alpha \}.
\]

Then from (i)–(iv), it follows that \([u]_\alpha \in C(\mathbb{R}^n) \). For the addition and scalar multiplication in \( L(\mathbb{R}^n) \), we have

\[
[u + v]_\alpha = [u]_\alpha + [v]_\alpha, \quad [ku]_\alpha = k[u]_\alpha
\]

where \( u, v \in L(\mathbb{R}^n), k \in \mathbb{R} \).
Define, for each \(1 \leq q < \infty\),
\[
d_q(u, v) = \left( \int_0^1 \left[ \delta_\infty([u]^q, [v]^q) \right]^q \, d\alpha \right)^{1/q}
\]
and \(d_\infty(u, v) = \sup_{0 \leq \alpha \leq 1} \delta_\infty([u]^q, [v]^q)\), where \(\delta_\infty\) is the Hausdorff metric. Clearly \(d_\infty(u, v) = \lim_{q \to \infty} d_q(u, v)\) with \(d_q \leq d\), if \(q \leq s\).

Throughout the paper, \(d\) will denote \(d_q\) with \(1 \leq q \leq \infty\) [6,14].

A sequence \(X = (X_k)\) of fuzzy numbers is a function \(X\) from the set \(\mathbb{N}\) of all positive integers into \(L(\mathbb{R}^n)\). Thus, a sequence of fuzzy numbers \((X_k)\) is a correspondence from the set of positive integers to a set of fuzzy numbers, i.e., to each positive integer \(k\) there corresponds a fuzzy number \(X(k)\). It is more common to write \(X_k\) rather than \(X(k)\) and to denote the sequence by \((X_k)\) rather than \(X\). The fuzzy number \(X_0\) is called the \(k\)-th term of the sequence.

Let \(X = (X_k)\) be a sequence of fuzzy numbers. The sequence \(X = (X_k)\) of fuzzy numbers is said to be bounded if the set \([X_k : k \in \mathbb{N}\})\) of fuzzy numbers is bounded and convergent to the fuzzy number \(X_0\), written as \(\lim k \to X_0\), if for every \(\varepsilon > 0\) there exists a positive integer \(k_0\) such that \(d(X_k, X_0) < \varepsilon\) for \(k > k_0\). By \(w^r\), \(\ell^r_\infty\), and \(c^r\) we denote the set of all, bounded and convergent sequences of fuzzy numbers, respectively [20].

Recall [12,13] that an Orlicz function is a function \(M: [0, \infty) \to [0, \infty)\), which is continuous, non decreasing and convex with \(M(0) = 0\), \(M(x) > 0\) for \(x > 0\) and \(M(x) \to \infty\) as \(x \to \infty\).

Lindenstrauss and Tzafriri \[16\] used the idea of Orlicz function to construct the sequence space
\[
\ell_M = \left\{ x \in w : \sum_{k=1}^\infty M\left( \frac{|x_k|}{\rho} \right) < \infty, \text{ for some } \rho > 0 \right\}.
\]
The space \(\ell_M\) is a Banach space with the norm
\[
\|x\| = \inf \left\{ \rho > 0 : \sum_{k=1}^\infty M\left( \frac{|x_k|}{\rho} \right) \leq 1 \right\}
\]
and this space is called an Orlicz sequence space. For \(M(t) = t^p, 1 \leq p < \infty\), the space \(\ell_M\) coincides with the classical sequence space \(\ell_p\).

By a lacunary sequence \(\theta = (\theta_k) : r = 0, 1, 2, \ldots, \) where \(\theta_0 = 0\), we mean an increasing sequence of non-negative integers with \(\theta_r = (\theta_{k_r} - \theta_{k_{r-1}}) \to \infty\) as \(r \to \infty\). The intervals determined by \(\theta\) will be denoted by \(I_r = (\theta_{k_{r-1}}, \theta_{k_r})\) and \(q_r = \frac{\theta_{k_r}}{\theta_{k_{r-1}}}\). Lacunary sequences have been discussed in [9,11,23,27].

The famous space \(\ell\) of all almost convergent sequences was introduced by Lorentz [17] and Maddox [18,19] has defined \(x\) to be strongly almost convergent to a number \(\ell\) if
\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^n |x_{k+m} - \ell| = 0, \text{ uniformly in } m.
\]

Now we introduce strongly almost lacunary convergence and almost lacunary statistical convergence of sequences of fuzzy numbers with respect to the Orlicz function \(M\).

**Definition 2.1.** Let \(\theta = (\theta_k)\) be a lacunary sequence, \(M\) be an Orlicz function and \(p = (p_k)\) be any sequence of strictly positive real numbers. A sequence \(X = (X_k)\) of fuzzy numbers is said to be almost lacunary statistically convergent to the fuzzy number \(X_0\), with respect to the Orlicz function \(M\), if for every \(\varepsilon > 0\)
\[
\lim_{r \to \infty} \theta^{-1}_r \left\{ k \in I_r : \left[ M\left( \frac{d(X_k, X_0)}{\rho} \right) \right]^{p_k} \geq \varepsilon \right\} = 0, \text{ uniformly in } m
\]
where
\[
t_{\theta} = \frac{x_{m+1} + x_{m+2} + \cdots + x_{m+k}}{k+1} = \frac{1}{k+1} \sum_{i=0}^k x_{m+i}.
\]
The set of all almost lacunary statistically convergent sequences is denoted by \(S^\theta (M, p, \theta)\). In this case we write \(X_k \to X_0 \left( S^\theta (M, p, \theta)\right)\). In the special cases \(\theta = (2^r)\) and \(M(x) = x, p_k = 1\) for all \(k \in \mathbb{N}\), we shall write \(S^\theta (M, p)\) and \(S^\theta (\theta)\) instead of \(S^\theta (M, p, \theta)\), respectively.

**Definition 2.2.** Let \(\theta = (\theta_k)\) be a lacunary sequence, \(M\) be an Orlicz function and \(p = (p_k)\) be any sequence of strictly positive real numbers. We define the following sets
\[
\hat{S}^\theta (M, p, \theta) = \left\{ X = (X_k) : \lim_{r \to \infty} \theta^{-1}_r \sum_{k \in I_r} \left[ M\left( \frac{d(X_k, X_0)}{\rho} \right) \right]^{p_k} = 0 \right\},
\]
uniformly in \(m\), for some \(\rho > 0\).
Let the sequence
\[
\hat{w}^e_0(M, p, \theta) = \left\{ X = (X_k) : \lim_{r \to \infty} \frac{1}{h_r} \sum_{k \in \mathbb{N}} \left[ M \left( \frac{d(t_m(X_k), \hat{0})}{\rho} \right) \right]^{\hat{p}_k} = 0, \right. \\
\text{uniformly in } m, \text{ for some } \rho > 0
\]
and
\[
\hat{w}^e_\infty(M, p, \theta) = \left\{ X = (X_k) : \sup_{r \in \mathbb{N}} \frac{1}{h_r} \sum_{k \in \mathbb{N}} \left[ M \left( \frac{d(t_m(X_k), \hat{0})}{\rho} \right) \right]^{\hat{p}_k} < \infty, \right.
\]
for some \( \rho > 0 \)

where
\[
\hat{0}(t) = \begin{cases} 
1, & t = (0, 0, 0, \ldots, 0) \\
0, & \text{otherwise.}
\end{cases}
\]

If \( X \in \hat{w}^e_\infty(M, p, \theta) \), we say that \( X \) is strongly almost lacunary convergent with respect to the Orlicz function \( M \). In this case we write \( X_k \to X_0 (\hat{w}^e(M, p, \theta)) \).

We get the following sequence sets from the above sequence sets by giving particular values to \( M, \theta \) and \( p \).

(i) \( \hat{w}^e(M, p, \theta) = \hat{w}^e_0(M, p) \), \( \hat{w}^e_0(M, p, \theta) = \hat{w}^e_0(M, p) \) and \( \hat{w}^e_\infty(M, p, \theta) = \hat{w}^e_\infty(M, p) \) when \( \theta = (2') \).

(ii) If \( M(x) = x \) then \( \hat{w}^e(M, p, \theta) = \hat{w}^e(p, \theta) \), \( \hat{w}^e_0(M, p, \theta) = \hat{w}^e_0(p, \theta) \) and \( \hat{w}^e_\infty(M, p, \theta) = \hat{w}^e_\infty(p, \theta) \).

(iii) If \( p_k = 1 \) for all \( k \in \mathbb{N} \) then \( \hat{w}^e(M, p, \theta) = \hat{w}^e_0(M, p) \), \( \hat{w}^e_0(M, p, \theta) = \hat{w}^e_0(M, \theta) \) and \( \hat{w}^e_\infty(M, p, \theta) = \hat{w}^e_\infty(M, \theta) \).

(iv) If \( M(x) = x \) and \( p_k = 1 \) for all \( k \in \mathbb{N} \), then \( \hat{w}^e(M, p, \theta) = \hat{w}^e_{\theta}(\theta) \), \( \hat{w}^e_0(M, p, \theta) = \hat{w}^e_{\theta}(\theta) \) and \( \hat{w}^e_\infty(M, p, \theta) = \hat{w}^e_\infty(\theta) \).

3. Main results

In this section we give some inclusion relations between \( \hat{w}^e(M, p, \theta) \) and \( \hat{w}^{\theta'}(\theta) \), between \( \hat{w}^e_\infty(\theta) \) and \( \hat{w}^{\theta'}(\theta) \).

**Theorem 3.1.** Let the sequence \((p_k)\) be bounded, then \( \hat{w}^e_0(M, p, \theta) \subset \hat{w}^e(M, p, \theta) \subset \hat{w}^e_\infty(M, p, \theta) \).

**Proof.** Let \( X \in \hat{w}^e(M, p, \theta) \). Then we have
\[
\frac{1}{h_r} \sum_{k \in \mathbb{N}} \left[ M \left( \frac{d(t_m(X_k), \hat{0})}{2\rho} \right) \right]^{\hat{p}_k} \leq \frac{D}{h_r} \sum_{k \in \mathbb{N}} \left[ M \left( \frac{d(t_m(X_k), 0)}{\rho} \right) \right]^{\hat{p}_k} + \frac{D}{h_r} \sum_{k \in \mathbb{N}} \left[ M \left( \frac{d(X_0, \hat{0})}{\rho} \right) \right]^{\hat{p}_k}
\]
\[
\leq \frac{D}{h_r} \sum_{k \in \mathbb{N}} \left[ \frac{d(t_m(X_k), 0)}{\rho} \right]^{\hat{p}_k} + D \max \left\{ 1, \left[ M \left( \frac{d(X_0, \hat{0})}{\rho} \right) \right]^{\hat{p}} \right\},
\]
where \( \sup \hat{p}_k = H \) and \( D = \max(1, 2^{\alpha-1}) \). Thus we get \( X \in \hat{w}^e_\infty(M, p, \theta) \). The inclusion \( \hat{w}^e_0(M, p, \theta) \subset \hat{w}^e(M, p, \theta) \) is obvious. \( \square \)

We give the following theorems without proof.

**Theorem 3.2.** Let the sequence \( (p_k) \) be bounded, then \( \hat{w}^e(M, p, \theta) \subset \hat{w}^e(M, p, \theta) \), \( \hat{w}^e_\infty(M, p, \theta) \) and \( \hat{w}^e_\infty(M, p, \theta) \) are closed under the operations of addition and scalar multiplication.

**Theorem 3.3.** Let \( M_1, M_2 \) be Orlicz functions, then we have

(i) \( \hat{w}^e_\infty(M_1, p, \theta) \cap \hat{w}^e_\infty(M_2, p, \theta) \subset \hat{w}^e_\infty(M_1 + M_2, p, \theta) \),

(ii) \( \hat{w}^e_\infty(M_1, p, \theta) \cap \hat{w}^e_\infty(M_2, p, \theta) \subset \hat{w}^e_\infty(M_1 + M_2, p, \theta) \),

(iii) \( \hat{w}^e_\infty(M_1, p, \theta) \cap \hat{w}^e_\infty(M_2, p, \theta) \subset \hat{w}^e_\infty(M_1 + M_2, p, \theta) \).

**Theorem 3.4.** Let \( 0 < p_k \leq q_k \) for all \( k \in \mathbb{N} \) and \( \left( \frac{q_k}{p_k} \right) \) be bounded, then we have \( \hat{w}^e(M, q, \theta) \subset \hat{w}^e(M, p, \theta) \).

**Theorem 3.5.** Let \( \theta = (k_r) \) be a lacunary sequence with \( 1 < \lim \inf q_r \leq \lim \sup q_r < \infty \). Then for any Orlicz function \( M \), \( \hat{w}^e(M, p) \subset \hat{w}^e(M, p, \theta) \).

**Proof.** Suppose \( \lim \inf q_r > 1 \), then there exists \( \delta > 0 \) such that \( q_r = \frac{k_r}{k_r-1} \geq 1 + \delta \) for all \( r \geq 1 \). Then for \( X \in \hat{w}^e(M, p) \) we write
\[
A_r = \frac{1}{h_r} \sum_{k \in \mathbb{N}} \left[ M \left( \frac{d(t_m(X_k), 0)}{\rho} \right) \right]^{\hat{p}_k}
\]
\[
= \frac{1}{h_r} \sum_{k \in \mathbb{N}} \left[ M \left( \frac{d(t_m(X_k), 0)}{\rho} \right) \right]^{\hat{p}_k} - \frac{1}{h_r} \sum_{k \in \mathbb{N}} \left[ M \left( \frac{d(t_m(X_k), 0)}{\rho} \right) \right]^{\hat{p}_k}
\]
\[
= \frac{k_r}{h_r} \left( k_r^{-1} \sum_{k \in \mathbb{N}} \left[ M \left( \frac{d(t_m(X_k), 0)}{\rho} \right) \right]^{\hat{p}_k} \right) - \frac{k_r-1}{h_r} \left( k_r^{-1} \sum_{k \in \mathbb{N}} \left[ M \left( \frac{d(t_m(X_k), 0)}{\rho} \right) \right]^{\hat{p}_k} \right).
Since \( h_r = k_r - k_{r-1} \), we have \( \frac{k_r}{h_r} \leq 1 + \frac{\delta}{2} \) and \( \frac{k_{r-1}}{h_r} \leq \frac{1}{2} \). The terms \( k_r^{-1} \sum_{k=1}^{k_r} \left[ M \left( \frac{d(t(X), X_0)}{\rho} \right) \right]^{\gamma} \) and \( k_{r-1}^{-1} \sum_{k=1}^{k_{r-1}} \left[ M \left( \frac{d(t(X), X_0)}{\rho} \right) \right]^{\gamma} \) both converge to zero uniformly in \( m \), and hence it follows that \( A_r \to 0 \) as \( r \to \infty \), uniformly in \( m \). Hence \( \hat{w}^F (M, p, \theta) \subset \hat{w}^F (M, p, \theta) \).

Now suppose that \( \limsup_q q_r < \infty \). Then, there exists \( \beta > 0 \) such that \( q_r < \beta \) for all \( r \geq 1 \). Let \( X \in \hat{w}^F (M, p, \theta) \) and \( \varepsilon > 0 \). Then \( \exists R > 0 \) such that for every \( j \geq R \)

\[
A_j = \frac{1}{h_j} \sum_{k=1}^{k_j} \left[ M \left( \frac{d(t(X), X_0)}{\rho} \right) \right]^{\gamma}
< \varepsilon.
\]

We can also find \( K > 0 \) such that \( A_j \leq K \) for all \( j = 1, 2, \ldots \). Now let \( n \) be any integer with \( k_{r-1} < n \leq k_r \), where \( r > R \). Then

\[
\frac{1}{n} \sum_{k=1}^{n} \left[ M \left( \frac{d(t(X), X_0)}{\rho} \right) \right]^{\gamma} \leq \frac{1}{k_{r-1}} \sum_{k=1}^{k_{r-1}} \left[ M \left( \frac{d(t(X), X_0)}{\rho} \right) \right]^{\gamma}
\]

\[
= \frac{k_1}{k_{r-1}} A_1 + \frac{k_2 - k_1}{k_{r-1}} A_2 + \ldots + \frac{k_{r-1} - k_{r-2}}{k_{r-1}} A_{r-1} + \frac{k_{r-1}}{k_{r-1}} A_r
\]

\[
= \left( \sup_{j \geq 1} A_j \right) \frac{k_8}{k_{r-1}} + \left( \sup_{j \geq k_r} A_j \right) \frac{k_r - k_{r-1}}{k_{r-1}}
< K \frac{k_{r-1}}{k_{r-1}} + \varepsilon \beta.
\]

Since \( k_{r-1} \to \infty \) as \( r \to \infty \), it follows that \( \frac{1}{n} \sum_{k=1}^{n} \left[ M \left( \frac{d(t(X), X_0)}{\rho} \right) \right]^{\gamma} \to 0 \) and consequently \( \hat{w}^F (M, p, \theta) \subset \hat{w}^F (M, p) \).

**Theorem 3.6.** If \( \lim p_k \to 0 \) and \( X \) is strongly almost lacunary convergent to \( X_0 \), with respect to the Orlicz function \( M \), that is \( X_k \to X_0 (\hat{w}^F (M, p, \theta)) \), then \( X_0 \) is unique.

**Proof.** Let \( \lim p_k = s > 0 \) and suppose that \( X_k \to X_1 (\hat{w}^F (M, p, \theta)) \), \( X_k \to X_0 (\hat{w}^F (M, p, \theta)) \). Then there exist \( \rho_1 \) and \( \rho_2 \) such that

\[
\lim_{r \to \infty} h_r^{-1} \sum_{k=1}^{k_r} \left[ M \left( \frac{d(t(X), X_0)}{\rho_1} \right) \right]^{\gamma} = 0
\]

and

\[
\lim_{r \to \infty} h_r^{-1} \sum_{k=1}^{k_r} \left[ M \left( \frac{d(t(X), X_1)}{\rho_2} \right) \right]^{\gamma} = 0,
\]

uniformly in \( m \).

Let \( \rho = \max (2 \rho_1, 2 \rho_2) \). Then we have

\[
h_r^{-1} \sum_{k=1}^{k_r} \left[ M \left( \frac{d(X, X_0)}{\rho} \right) \right]^{\gamma} \leq \frac{D}{h_r} \sum_{k=1}^{k_r} \left[ M \left( \frac{d(t(X), X_0)}{\rho_1} \right) \right]^{\gamma} + \frac{D}{h_r} \sum_{k=1}^{k_r} \left[ M \left( \frac{d(t(X), X_1)}{\rho_2} \right) \right]^{\gamma} \to 0, \quad (r \to \infty)
\]

where \( \sup_k p_k = H \) and \( D = \max (1, 2^{H-1}) \). Thus

\[
\lim_{r \to \infty} h_r^{-1} \sum_{k=1}^{k_r} \left[ M \left( \frac{d(X, X_1)}{\rho} \right) \right]^{\gamma} = 0.
\]

Also, since clearly

\[
\lim_{r \to \infty} \left[ M \left( \frac{d(X, X_1)}{\rho} \right) \right]^{\gamma} = \left[ M \left( \frac{d(X, X_1)}{\rho} \right) \right]^{\gamma},
\]

we have, from the last two equality \( \left[ M \left( \frac{d(X, X_1)}{\rho} \right) \right]^{\gamma} = 0 \). Hence \( X_0 = X_1 \).

**Theorem 3.7.** Let \( \theta = (1) \) be a lacunary sequence and \( M \) be an Orlicz function, then \( \hat{w}^F (M, \theta) = \hat{w}^F (M) \) and \( \hat{w}^F (\theta) = \ell^F \), where

\[
\hat{w}^F (M) = \left\{ X = (X_k) : \sup_{k, m} \left[ M \left( \frac{d(t(X), \theta)}{\rho} \right) \right] < \infty, \text{ for some } \rho > 0 \right\}.
\]
Proof. Let $X \in \hat{\omega}_\infty^F (M, \theta)$. Then there exists a constant $K_1 > 0$ such that

$$\frac{1}{h_1} \left[ M \left( \frac{d(\mathfrak{t}_{km}(X), \hat{0})}{\rho} \right) \right] \leq \frac{1}{h_1} \sum_{k \in k_0} \left[ M \left( \frac{d(\mathfrak{t}_{km}(X), \hat{0})}{\rho} \right) \right] \leq K_1 \quad \text{for each } m, r$$

and so we have $X \in \hat{m}_\infty^F (M)$.

Conversely, let $X \in \hat{m}_\infty^F (M)$. Then there exists a constant $K_2 > 0$ such that $M \left( \frac{d(\mathfrak{t}_{km}(X), \hat{0})}{\rho} \right) \leq K_2$ for each $k, m$ and so

$$\frac{1}{h_r} \sum_{k \in k_0} M \left( \frac{d(\mathfrak{t}_{km}(X), \hat{0})}{\rho} \right) \leq \frac{1}{h_r} \sum_{k \in k_0} 1 \leq K_2 \quad \text{for each } k, m.$$ 

Thus $X \in \hat{\omega}_\infty^F (M, \theta)$. The other can be treated similarly. □

Theorem 3.8. Let $\theta = (k_0)$ be a lacunary sequence. Then

(i) If $X_k \to X_0 (\hat{\omega}_F (\theta)) \Rightarrow X_k \to X_0 (\hat{S}_F (\theta))$,

(ii) If $X \in \ell_\infty^F$ and $X_k \to X_0 (\hat{S}_F (\theta))$ then $X_k \to X_0 (\hat{\omega}_F (\theta))$.

Proof. Omitted. □

Theorem 3.9. Let $\theta = (k_0)$ be a lacunary sequence with $1 < \lim \inf, q_r < \lim \sup, q_r < \infty$, then we have $\hat{S}_F = \hat{\omega}_F (\theta)$.

Proof. If $\lim \inf, q_r > 1$, then there exists a constant $\delta > 0$ such that $1 + \delta 

\leq \frac{1}{h_r} \sum_{k \in k_0} M \left( \frac{d(\mathfrak{t}_{km}(X), \hat{0})}{\rho} \right) \leq K_2 \quad \text{for each } k, m.$$
Theorem 3.11. Let \( \theta = (k_r) \) be a lacunary sequence, \( M \) be an Orlicz function, \( X = (X_k) \) be a bounded sequence of fuzzy numbers and 0 < \( h = \inf_p p_k \leq p_k \leq \sup_p p_k = H < \infty \). Then \( \tilde{S}^\varphi (\theta) \subset \tilde{w}^\varphi (M, p, \theta) \).

Proof. Suppose that \( X \in \ell^p_\varphi \) and \( X_k \rightarrow X_0 \) \( \left( \tilde{S}^\varphi (\theta) \right) \). Since \( X \in \ell^p_\varphi \), there is a constant \( T > 0 \) such that \( d(t_{\varphi, m}(X), X_0) \leq T \) for all \( k, m \). Given \( \varepsilon > 0 \) we have

\[
\begin{align*}
\frac{1}{h_r} \sum_{k=0}^{h_r} \left[ M \left( \frac{d(t_{\varphi, m}(X), X_0)}{\rho} \right) \right]^k & = \frac{1}{h_r} \sum_{d_{\varphi}(X_k, X_0) \leq \varepsilon} \left[ M \left( \frac{d(t_{\varphi, m}(X), X_0)}{\rho} \right) \right]^k + \frac{1}{h_r} \sum_{d_{\varphi}(X_k, X_0) > \varepsilon} \left[ M \left( \frac{d(t_{\varphi, m}(X), X_0)}{\rho} \right) \right]^k \\
& \leq \frac{1}{h_r} \sum_{d_{\varphi}(X_k, X_0) \leq \varepsilon} \max \left[ M \left( \frac{T}{\rho} \right)^k \right] + \frac{1}{h_r} \sum_{d_{\varphi}(X_k, X_0) > \varepsilon} \left[ M \left( \frac{\varepsilon}{\rho} \right) \right]^k \\
& \leq \max \left[ M \left( K \right)^k, M \left( M \left( \frac{T}{\rho} \right) \right)^k \right] + \frac{1}{h_r} \sum_{d_{\varphi}(X_k, X_0) \leq \varepsilon} \left[ M \left( \frac{\varepsilon}{\rho} \right) \right]^k \\
& = M \left( K \right)^k, M \left( M \left( \frac{T}{\rho} \right) \right)^k \\
& \leq M \left( \frac{T}{\rho} \right)^k = K, \frac{\varepsilon}{\rho} = \varepsilon.
\end{align*}
\]

Hence \( X \in \tilde{w}^\varphi (M, p, \theta) \).

4. Conclusion

The concepts of statistical convergence and strongly Cesàro convergence of sequences of real numbers have been studied by various mathematicians. In this paper we have introduced some fairly wide classes of sequences of fuzzy numbers using an Orlicz function and an increasing sequence \( \theta = (k_r) \) of non-negative integers such that \( h_r = (k_r - k_{r-1}) \rightarrow \infty \) as \( r \rightarrow \infty \) and \( k_0 = 0 \). Giving particular values to the sequence \( \theta = (k_r) \) and \( M \) we obtain some sequence spaces which are the special cases of the sequence spaces that we have defined. The majority of the results proved in the previous sections will be true for these spaces.

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